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On δ-Scattered Spaces

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Abstract

The object of the present paper is to introduce the concept of δ-scattered spaces as a natural generalization of the concept of scattered spaces. We prove that the concept of δ-scatteredness of the space coincides with scatteredness. It is noted that scattered need not be δ-scattered in general, also I-space are comparable with δ-scattered space. We start out by giving a characterization of δ-scattered spaces. We study relationships between δ-scatteredness and with scattered, semi-scattered, α-scattered, sub maximal, irresolvable and N-scattered.

1. Introduction

In 1974 Kannan and Rajayopalan introduced some results on scattered space [14], and in 1977 they studies and investigated new results about this space. Later in 1989 Hdeib and Pareek introduced a new generalization of scattered space [5]. In 1996 Dontchev and Rose introduced N-scattered [6], and in 1997 Dontchev et al. introduced α-scattered [7] and in 1998 Nour introduced applications of semi-open sets and he refers in this search to semi-scattered space but not directly [12], the concept of irresolvable space used in [1], [2], [4] and [11]. The reason that these spaces appear in so many places is that this concept arises naturally in many parts of general topology. The aim of this paper is to continue the study of scattered and δ-scattered spaces. Throughout this paper, (X, T_X) is simply X, represents topological space. A point $x \in X$ is called the δ-cluster [13] (resp. δ-limit [10], limit [14]) points of a subset A of a space X if $A \cap U \neq \emptyset\ \forall U \ni x$, where U is regular open set (resp.if $U \cap (A - \{x\}) \neq \emptyset\ \forall U \ni x$, where U is δ-open, if $A \cap U \neq \emptyset\ \forall U \ni x$, where U is open). A point $x \in A$ is called the δ-isolated [16] (resp. isolated [14]) point of a subset A of a space X if there is a δ-open (resp. open) subset

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$U \ni x$ such that $A \cap U = \{x\}$. The set of all δ-cluster [13] (resp. δ-limit, δ-isolated, limit, isolated) points of A is denoted by $cl_\delta(A)$ (resp. $D_\delta(A), \delta I(A)$, $D(A)$,$I(A)$), so we note that every δ-isolated point of a set is not δ-limit point of this set. A subset A of a space X is called δ-closed [13] (resp. δ-nowhere dense [16], δ-crowded [16], discrete [9], regular open [13], δ-open [13], δ-dense [16], δ-scattered [16], δ-perfect [16]) if $cl_\delta(A) = A$ (resp. $int_\delta(cl_\delta(A)) = \emptyset$, if it does not have any δ-isolated point, if it does not have any limit point, $int(cl(A)) = A$, $A = \bigcup U_i$, where U_i regular open $\forall i \in I, cl_\delta(A) = X$, if A have any δ-isolated point, if A is δ-closed and δ-crowded). A space X is called discrete (resp. nodec [1], α-scattered [7], scattered [3], semi-scattered [12], N-scattered [6], sub maximal [1]) if and only if each point $\{x\}$ is open in X (resp. if all nowhere dense subsets of X are closed, the space is T_α-space and nodec space, if every subset of X has an isolated point, every subset of X has semi-isolated point, if every nowhere dense subset of X is scattered, if every dense set is open). The α-topology (resp. δ-topology) on a topological space X is the collection of all subset of the form $U \backslash V$, where $U \in T$ and V is a nowhere dense subset (resp. is the collection of all δ-open subset) of X. Finally in this search we take $X \in R$ be a nonempty finite set, and T is T_X on X is a subset of a power set $P(X)$ to a set X.

2. Some Characterization of δ-Scattered Space

In this section we introduce the concept of δ-scattered space and some results about this space.

Definition 2.1. A topological space (X, T_X) is called *δ-scattered* if there is no nonempty δ-closed (resp. δ-open) subset of X is δ-crowded. Or equivalently every subset A of X has at least δ-isolated point.

Lemma 2.1 [16]**.** *For subsets A, B of a space X, the following properties are true*:

(1) $\delta I(A) \subset A$.

(2) *If $A \subset B$, then $\delta I(A) \subset \delta I(B)$.*

(3) *If A is δ-dense, then A has δ-isolated point.*

(4) $\delta I(A) \subseteq SK_\delta(A) \subseteq cl_\delta(A)$.

(5) *A is δ-dense if and only if $cl_\delta(A) = X$.*

Lemma 2.2 [15]**.** *For subset A of a space X, the following properties are true*:

(1) *If A is regular open, then A is δ-open (resp. open).*

(2) *If A is an open or dense (resp. δ-closed) subset of a space X, and $B \in \delta o(X)$ (resp. $B \in g\delta c(X)$), then $A \cap B \in \delta o(A)$ (resp. $A \cap B \in g\delta c(A)$).*

(3) *A is clopen if and only if A is δ-open and δ-closed.*

Remark 2.2. Every subset of δ-scattered space is either δ-open and regular open (δ-closed and regular closed) or δ-open not regular open (δ-closed not regular closed).

Example 2.3. Let $X = \{a, b, c\}$, $T_X = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$ is δ-scattered, since every subset of X has an δ-isolated point.

Definitions 2.4 [2], [4], [11]**.**

(1) A topological space (X, T_X) is called *resolvable* if there is a subset D of X such that D and X/D are both dense in X.

(2) *Irresolvable* if it is not resolvable.

(3) A subset A of a space X is called *resolvable* if the subspace (A, T_A) is resolvable, and called *irresolvable* if it is not resolvable. Or equivalently if for any two dense subset D_1 and D_2 of X such that $D_1 \cap A \neq \emptyset$ and $D_2 \cap A \neq \emptyset$ we have $(D_1 \cap D_2) \cap A \neq \emptyset$.

(4) The *scattered δ-kernel* of A is the set $SK_\delta(A) = A - PK_\delta(A)$, where $PK_\delta(A)$ is the perfect δ-kernel of A which is largest possible δ-crowded subset contained in A.

Proposition 2.5. *If X is δ-scattered, then:*

(1) *$\delta I(X)$ is dense set in X.*

(2) *$D_\delta(X)$ is a closed δ-nowhere dense set in X.*

Proof.

(1) Suppose that $A \cap \delta I(X) \neq \emptyset$ for every nonempty $A\delta$-open subset in X, let $x \in X$ but $x \notin cl(\delta I(X))$, so $x \in X - cl(\delta I(X))$ and $\delta I(X) \cap (X - cl_\delta(\delta I(X)) = \emptyset$, a contradiction to the fact that $X - cl_\delta(\delta I(X))$ is δ-open, hence $x \in cl_\delta(\delta I(X))$, since X is δ-scattered, we have $cl_\delta(\delta I(X)) = cl(\delta I(X))$. Therefore $cl(\delta I(X)) = X$.

(2) Suppose that $X - D_\delta(X)$ contain dense open subset, then $A \subset X - D_\delta(X)$, $int_\delta(A) \subset int_\delta(X - D_\delta(X)), A \subset int_\delta(X - D_\delta(X))$. So $A \cap int_\delta(X - D_\delta(X)) \neq \emptyset$, since $int_\delta(X - D_\delta(X))$ is dense, thus

$$cl_\delta(int_\delta(X - D_\delta(X)) = cl_\delta(X - cl_\delta(D_\delta(X)) = X - cl_\delta\left(int_\delta(D_\delta(X))\right) = X.$$

Thus $int_\delta\left(cl_\delta(D_\delta(X))\right) = \emptyset$.

Proposition 2.6. *For a space X the followings hold*:

(1) *X is δ-scattered if and only if every subset of X is δ-scattered.*

(2) *If X is δ-scattered space, then every δ-open subspace is δ-scattered space.*

(3) *If every point of space X has an δ-scattered neighbourhood, then X is δ-scattered.*

(4) *A space X is δ-scattered if for every δ-open subset A of X there is an element $x \in A$ which is δ-isolated relative to X.*

(5) *Every δ-scattered subset is irresolvable set in X.*

Proof.

(1) Clearly.

(2) Let $A \subseteq X$ be non empty δ-open subset of X, since X is δ-scattered, and $\delta I(X)$ is dense set in X, so by Lemma 2.1(1) $B = \delta I(A) \cap \delta I(X) \neq \emptyset$ and $B = \delta I(A) \neq \emptyset$. If $\emptyset \neq U \subseteq A$, then U is δ-open in X, thus $U \cap B = B$. Hence B is dense in A and $\delta I(A)$ being its superset. Therefore A is δ-scattered.

(3) Suppose that X is not δ-scattered. So there is $\emptyset \neq U \subset X$ δ-open, let $x_0 \in U$ and there is an δ-scattered subset V and an δ-open set M not δ-perfect, such that $x_0 \in M \subset V$, by (2) M is δ-scattered in V, then M is δ-scattered and $K = U \cap M$, $(U \cap V) \cap K - \{x_0\} = \emptyset$ is nonempty δ-open subset not δ-perfect, so K has x_0 as a δ-isolated point. Thus x_0 must be δ-open set in X and $x_0 \notin \delta I(X)$, but this is contradiction, since x_0 is a point in X.Thus X is δ-scattered.

(4) Clearly.

(5) Let A be δ-open and δ-scattered subset of X, let B be δ-scattered subset, thus by Lemma 2.2(2) $B = u \cap A$, where u open subset of X, since X is δ-scattered then B is δ-open and clearly it is not regular open, since X is δ-scattered, so B dense set, but $X - B$ is δ-closed not dense subset, that is there is not two dense subsets in T_A, therefore T_A is not resolvable mean A not resolvable. Thus A is irresolvable.

Proposition 2.7. *If X is δ-scattered, then*:

(1) *The scattered δ-kernal of X is dense set.*

(2) *Every δ-scattered closed nowhere dense subset is nowhere.*

(3) *Every δ-scattered dense subset is dense.*

(4) *$A \subset X$ is δ-open not regular open if and only if A is dense subset.*

Proof.

(1) Clearly $SK_\delta(X) \subset cl_\delta(\delta I(X)$, so $cl_\delta(SK_\delta(X)) \subset cl_\delta(cl_\delta(\delta I(X)\big) = X$ ……(1), also by Proposition 2.5(1) and Lemma 2.1(4) $\delta I(X) \subset SK_\delta(X)$, hence $cl_\delta(\delta I(X)) \subset cl_\delta(SK_\delta(X))$, so $X \subset cl_\delta(SK_\delta(X))$ ……(2), from (1) and (2) $cl_\delta\big(SK_\delta(X)\big) = X$.

(2) By Proposition 2.5(1), Proposition 3.10 [16].

(3) Let A be δ-dense, by Lemma 2.1(4) A has δ-isolated point. Therefore A is δ-scattered.

(4) Let A be δ-dense subset of X, so A is dense and δ-scattered subset, clearly that A is not δ-closed since $cl_\delta(A) = X$ it must be δ-open and since every δ-open is not regular open this by Lemma 2.2 (1).

Conversely, let A be δ-open not regular open, that is, $int\big(cl(A)\big) \neq A = int_\delta(A)$, so $cl(A) \neq cl_\delta(A) \neq A$, thus $cl_\delta(int_\delta(A)) = cl_\delta(A) = X$ and $cl_\delta(A) = X$ by Lemma 2.1(5) A is δ-dense.

Lemma 2.3 [16]**.** *If A be δ-closed subset of a space X and has no δ-isolated point, then $int_\delta\,(A)$ has no δ-isolated point.*

Proposition 2.8. *If A be δ-open subset of T_δ-space and $int_\delta\,(A)$ is δ-scattered, then T_A is δ-scattered.*

Proof. Let A be δ-open subset of space X, since $int_\delta\,(A)$ is δ-scattered, so $int_\delta\,(A) = A$ is δ-open scattered set, by Lemma 2.3 $\delta I(int_\delta\,(A)) = \delta I(A) \neq \emptyset$, also since A is δ-open then there is regular open subsets V_i of A $\forall i$, where $V_i \subseteq X$, since A is δ-scattered thus there is singleton regular open (that is singleton δ-open) $V_j \subseteq V_i \cap A$ such that $V_j \cap ((V_i \cap A) - \{x\}) = V_j \cap (B - \{x\}) = \emptyset$, where B any subset of T_A contains x as δ-isolated point. Thus T_A is δ-scattered.

Remark 2.9. δ-scattered is not δ-closed hereditary Property. See Example 2.3, note that $A = \{a, c\} \subseteq X$ is δ-scattered set and $\delta I(A) = \{c\}$, but $T_A = \{\emptyset, A, \{c\}\}$ not δ-scattered space, since A does not have any δ-isolated point in T_A.

Proposition 2.10. *Let X be a space and $A \subseteq X$ be an open subset, if $U \subseteq X - A$ and $V \subseteq A$ are δ-scattered subsets of X, then $U \cup V$ is δ-scattered.*

Proof. Let $B \subseteq U \cup V$, if $B \cap V \neq \emptyset$, then the subspace B has an δ-isolated point, and B being δ-scattered, otherwise, the subspace $B \cap V = B \cap U$ has an δ-isolated point p, so there exist an δ-open subset M of Xsuch that $B \cap (A \cap M) = \{p\}$, so p is an δ-isolated point of the subspace B. Thus $U \cup V$ is δ-scattered.

Remark 2.11. Let $\{X_i : i \in I\}$ be a family of δ-scattered spaces then the sum of these topological spaces is not necessarily δ-scattered space, see the following Example.

Example 2.12. Let $X = \{a, b, c, d\}$, $T_{x_1} = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ is δ-scattered, $T_{x_2} = \{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$ is δ-scattered, but $T_x = T_{x_1} \cup T_{x_2} = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}\}$ is not δ-scattered space.

Proposition 2.13. *Let $\{X_i : i \in I\}$ be a family of δ-scattered spaces and let X denote their topological, where X is topology, then the following conditions are equivalent:*

(1) *X is δ-scattered.*

(2) *X_i is δ-open scattered for each $i \in I$.*

Proof.

(1) $\Rightarrow$ (2) : Clearly by Proposition 2.6(2).

(2) $\Rightarrow$ (1): let U be a nonempty δ −open subset of X_i, since $X_i \delta$-open scattered spaces in X and U_i nonempty δ-open subsets in X_i , thus for every δ-open subset V of X_i we have $V \cap (U_i - \{x\}) = \emptyset$, thus $V \cap ((\cup U_i) - \{x\}) = \emptyset$ since $\cup U_i \subseteq \cup X_i = X$.$X$ is δ-scattered.

The following table shows the case of sets that construct δ-scattered and discrete spaces.

No. of elements x	No. of sets in δ-Scattered Space	No. of sets in $P(X) = T_D$	min (A)	No. of min(A) as a sets	max (B)	No. of max(B) as a sets	C
1	2	2	1	1	1	1	0
2	4	4	1	2	2	1	0
3	5	8	1	2	2	1	0
4	9	16	1	3	3	1	3
5	17	32	1	4	4	1	10
6	33	64	1	5	5	1	25
7	65	128	1	6	6	1	56
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
n	$2^{n-1}+1$	2^n	1	n-1, $n>2$	n -1, n>2	1	$2^{n-1}+1-(n+2)$, $n \geq 3$

Where A = singleton regular open, B = δ-open not regular open, C = δ-open and regular open not singleton and not X, $\min(A)$ (resp. max (A)) means the smallest (resp. largest) set in terms of containing the elements.

3. The Relationships between δ-scattered Space and other Topological Spaces

In this section we introduce the relation between types of scattered spaces and δ-scattered. First we remember the following lemma.

Lemma 3.1 [16]**.**

(1) *Every δ-isolated point is isolated (resp. α-isolated, semi-isolated, pre-isolated) point.*

(2) *Every α-scattered set is semi-scattered (resp. pre-scattered) set.*

Proposition 3.1. *Let X be a space. Then the following statements hold*:

(1) *If X is discrete space, then X is T_δ-space and scattered.*

(2) *If X is T_δ-space and discrete, then X is δ-scattered space.*

(3) *If X is T_δ-space, then X is δ-scattered if and only if every subset of X has an δ-isolated point.*

(4) *If X is δ-scattered, then X is T_δ-space.*

Proof.

(1) Since every subset of discrete space is open and closed, so every subset is clopen thus by Lemma 2.2(3) every subset is δ-clopen and regular-clopen, thus every subset has isolated point that is every subset is scattered. Thus X is T_δ and scattered.

(2) By (1) and by Proposition 2.6(1) X is δ-scattered.

(3) Clearly.

(4) Clearly.

Remark 3.2. Every T_δ-space is not necessarily δ-scattered space. See the following Example.

Example 3.3. Let $X = \{a, b, c\}, T_X = \{\emptyset, X, \{a\}, \{b, c\}\}$ is T_δ-space but not δ-scattered space, since $\delta I\{b, c\} = \emptyset$.

Definitions 3.4.

(1) A space X is called *quasi-maximal* [2] if for every dense set D in X with $int(D) \neq \emptyset$, $int(D)$ is also dense in X.

(2) A space X is called *I-space* [7] if $D(X)$ is closed and discrete.

Proposition 3.5.

(1) *If X is δ-scattered, then X is irresolvable and quasi-maximal.*

(2) *If X is δ-scattered, then X is sub-maximal.*

Proof.

(1) Let B be δ-open not regular open. Then by Proposition 2.7(4), B is dense, $X - B = C$ is δ-closed not dense that is X is not resolvable. Thus X irresolvable, also since B is δ-open, then $int_\delta(B) = B \neq \emptyset$ also $int_\delta(B)$ is dense set. Thus X is quasi-maximal.

(2) By Definition 2.1 and Proposition 2.7(4).

The converse of the above proposition is not true in general, see the following Example.

Example 3.6. Let $X = \{a, b, c\}, T_X = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ is irresolvable and quasi-maximal and sub-maximal but not δ-scattered, since $\delta I\{a, c\} = \emptyset$.

Proposition 3.7. *Let X be a space. Then the following statements hold*:

(1) *If X is δ-scattered and discrete, then X is I-space.*

(2) *If X is discrete, then X is I-space if and only if it is δ-scattered and the set of all isolated points is dense set.*

(3) *Let X be T_δ-space. If X is δ-scattered, then X is an I-space.*

Proof.

(1) By Proposition 2.5(2), $D_\delta(X)$ is closed δ-nowhere dense, since X is δ-scattered discrete and $D_\delta(X) \subset X$ is δ-closed and discrete, so by Lemma 2.2(1), $D_\delta(X)$ is closed and discrete.

(2) Suppose that X is a discrete, δ-scattered and the set of all isolated points is dense set by (1) $D(X) \subseteq D_\delta(X) \subset X$ is closed and discrete. Thus X is I-space.

Conversely, suppose that X is a discrete and I-space, then X is T_δ-space and scattered that is δ-scattered this by Proposition 3.1(1) and (2) also by Proposition 2.5(1) $\delta I(X)$ is dense set.

(3) Clearly.

Proposition 3.8. *Let X be a space. Then the following statements hold*:

(1) *If X is I-space, then X is scattered* (*resp. sub-maximal space*) [1].

(2) *If X is α-space, then X is N-scattered space* [8].

(3) *X is scattered space if and only if X is N-scattered and α-scattered* [6].

(4) *If X is scattered and I-space, then X is α-scattered* [7].

(5) *If X is scattered space, then X is semi-scattered* [12].

Proposition 3.9.

(1) *Every δ-scattered space is N-scattered.*

(2) *Every δ-scattered space is scattered.*

(3) *Every δ-scattered space is α-scattered.*

(4) *Every δ-scattered space is semi-scattered.*

Proof.

(1) By Proposition 2.7(2), Proposition 2.6(1) and Lemma 3.1(1).

(2) By Definition 2.1 and Lemma 3.1(1).

(3) By Definition 2.1 and Lemma 3.1(1).

(4) By Definition 2.1 and Lemma 3.1(1).

The converse of Proposition 3.9 is not true, in general, see the following examples.

Examples 3.10.

(1) Let $X = \{a, b, c\}$, $T_X = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ is N-scattered and scattered and semi-scattered since $A = \{c\}$ is the only nowhere dense subset which is scattered, but not δ-scattered, since $\delta I\{a, c\} = \emptyset$.

(2) Let $X = \{a, b, c\}, T_X = \{\emptyset, X, \{a\}\}$ is α-scattered but not δ-scattered, since $\alpha I(X) = \{a\}$ but $\delta I(X) = \emptyset$.

Proposition 3.11. *If X is α-scattered space, then X is semi-scattered.*

Proof. By Definition 2.1 and Lemma 3.1(2).

The following diagram shows the relation among certain type of scattered spaces.

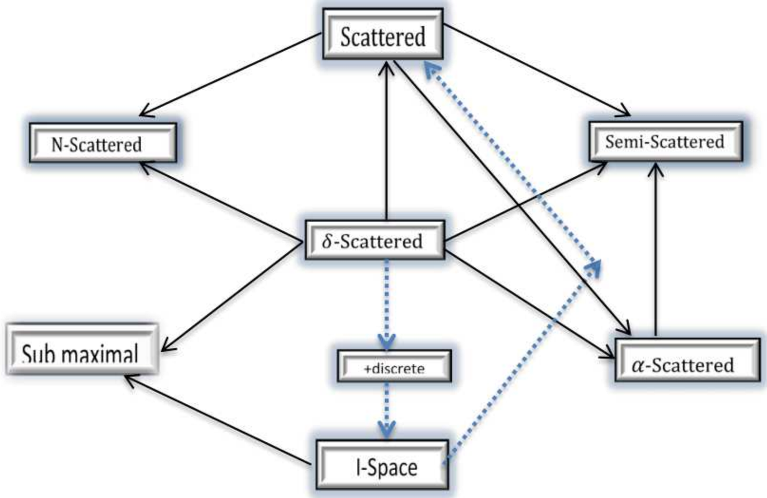

Diagram (1)

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