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# Some Iterative Schemes for Solving Mixed Equilibrium Variational-like Inequalities

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**Abstract**

Some new types of equilibrium variational-like inequalities are considered, which is called the bifunction mixed equilibrium variational-like inequalities. The auxiliary principle technique is used to construct some iterative schemes to solve these new equilibrium variational-like inequalities. Convergence of the suggested schemes is discussed under relaxed conditions. Several special cases are discussed as applications of the main results. The ideas and techniques may be starting point for future research.

## 1 Introduction

Variational inequality theory, which was introduced by Stamapcchia [19], has played an important and significant part in the development of several areas

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of applied and pure mathematics. This theory provides us natural framework to solve many complex and simple problems arise in the fields of finance, economics, structural analysis, image reconstruction, transportation, elasticity, network analysis and linear and nonlinear optimization. In 1964, Stampacchia [19] provided the initial concept of variational inequalities containing non-symmetric bilinear form. For recent developments in the field of variational inequalities and related optimizations, see [1, 8, 14, 15, 18] and the references therein. Due to the importance of convexity theory, it has been extended and generalized in several directions using new techniques and innovative ideas. A important generalization of convex functions is known as preinvex functions, which was introduced by Ben-Israel and Mond [2] and Hanson and Mond [6]. The preinvex functions enjoys some nice properties of convex functions have. In some cases, preinvex functions can be decomposed as the sum of directional differentiable and non-differentiable preinvex functions. This decomposition enables us to discuss the problem as minimization problem. Noor [10] has shown that the optimality condition of the differentiable preinvex function can be characterized by a class of variational inequalities, called variational-like inequalities. Weir and Mond [20] and Noor [9] studied the fundamental properties of the preinvex functions and their significance in optimizations and engineering sciences. For recent applications, motivation, generalizations and numerical aspects of the preinvex functions, see [16, 17]. Recently several numerical methods have been developed to solve variational-like inequalities using projections, its variant forms and auxiliary principle technique.

Inspired and motivated by the research activities, we introduce and consider a new class of variational-like inequalities, which is called the bifunction mixed equilibrium variational-like inequalities. Several important class of equilibrium variational-like inequalities are discussed. It is worth mentioning that the projection, resolvent and their invariant forms can not used to suggest iterative methods for fining the approximate solutions of the equilibrium variational-like inequalities. This motivated us to consider the auxiliary principle technique, which is mainly due to Glowinski et al. [4], as developed by Noor [10–13]. We again use this auxiliary principle technique to suggest some implicit and explicit iterative schemes for soling these new classes of equilibrium variational-like inequalities. The convergence analysis of the proposed methods is discussed under suitable weak conditions. We have discussed the theoretical aspects only. Comparison and numerical implementation of these methods need further efforts.

## 2 Preliminaries

Let $\mathcal{H}$ be Hilbert space and $\langle\cdot,\cdot\rangle$ be inner product and $\|\cdot\|$ be norm space of $\mathcal{H}$. Let $\Omega_\eta$ be invex set in $\mathcal{H}$ and $\eta(\cdot,\cdot):\Omega_\eta\times\Omega_\eta\longrightarrow\mathbb{R}$ be bifunction. Let $T:\mathcal{H}\longrightarrow\mathcal{H}$ be non-linear operator, $F:\mathcal{H}\times\mathcal{H}\longrightarrow\mathbb{R}$ be continuous bifunction and $\phi(.):\mathcal{H}\longrightarrow\mathbb{R}^\infty$ be a lower semi-continuous convex function.

We also need following definitions and concepts.

**Definition 2.1.** [5] A set $\Omega_\eta\subseteq\mathbb{R}^n$ is said to be *invex set* with respect to the bifunction $\eta(\cdot,\cdot)$, if and only if,

$$\varrho+\sigma\eta(\varsigma,\varrho)\in\Omega_\eta,\quad\forall\quad\varrho,\varsigma\in\Omega_\eta,\quad\sigma\in[0,1].$$

The invex set $\Omega_\eta$ is also called *$\eta$-connected set.*

**Definition 2.2.** [7] A function $\psi$ on $\Omega_\eta$ is said to be *preinvex function* with respect to $\eta(\cdot,\cdot)$, if and only if,

$$\psi(\varrho+\sigma\eta(\varsigma,\varrho))\leq(1-\sigma)\psi(\varrho)+\sigma\psi(\varsigma),\quad\forall\quad\varrho,\varsigma\in\Omega_\eta,\quad\sigma\in[0,1].\tag{2.1}$$

**Definition 2.3.** A function $\psi$ is *strongly preinvex function* on $\Omega_\eta$, w.r.t. bifunction $\eta(\cdot,\cdot)$ with $|\theta|\geq 0$ if,

$$\psi(\varrho+\sigma\eta(\varsigma,\varrho))\leq(1-\sigma)\psi(\varrho)+\sigma\psi(\varsigma)-\sigma\theta(1-\sigma)\|\eta(\varsigma,\varrho)\|^2,\quad\forall\varrho,\varsigma\in\Omega_\eta,\quad\sigma\in[0,1].$$

Differentiable preinvex function satisfies a very nice property like the differentiable convex function which can be seen as.

**Theorem 2.1.** *If $\psi$ is strongly preinvex differentiable function on $\Omega_\eta$, then*

$$\psi(\varsigma) - \psi(\varrho) \geq \langle \psi'(\varrho), \eta(\varsigma, \varrho)\rangle + \theta\|\eta(\varsigma, \varrho)\|^2, \quad \forall \quad \varrho, \varsigma \in \Omega_\eta. \tag{2.2}$$

**Lemma 2.1.** *We have a result that will be used in convergence analysis in our work, as follows*

$$2\langle \varrho, \varsigma\rangle = \| \ \varrho + \varsigma\|^2 - \| \ \varrho\|^2 - \| \ \varsigma\|^2, \quad \forall \quad \varsigma, \varrho \in \mathcal{H}. \tag{2.3}$$

**Definition 2.4.** The bifunction $\eta(\cdot, \cdot)$ is said to be *lipschitz continuous*, if $\exists \quad \vartheta > 0$, such that,

$$\|\eta(\varsigma, \varrho)\| \leq \vartheta\|\varsigma - \varrho\|, \quad \forall \quad \varrho, \varsigma \in \Omega_\eta.$$

We also need a condition on $\eta(\cdot, \cdot)$, such that

$$\eta(\varsigma, \varrho) = \eta(\varsigma, \xi) + \eta(\xi, \varrho), \quad \forall \quad \varrho, \varsigma, \xi \in \mathcal{H}, \tag{2.4}$$

which is called **condition-$\eta$**.

Note that $\eta(\varsigma, \varrho) = 0$, if and only if, $\varsigma = \varrho, \quad \forall \varsigma, \varrho \in \mathcal{H}$.

And

$$\eta(\varsigma, \varrho) + \eta(\varrho, \varsigma) = 0, \quad \forall \varsigma, \varrho \in \mathcal{H}.$$

That is $\eta(., .)$ is skew symmetric.

**Definition 2.5.** A bifunction $F : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ is said to be *$\eta$-monotone*, if and only if,

$$F(\varrho, \eta(\varsigma, \varrho)) + F(\varsigma, \eta(\varrho, \varsigma)) \leq 0, \quad \forall \quad \varrho, \varsigma \in \mathcal{H}$$

and *partially relaxed strongly* (PRS) $\eta$-monotone with $\gamma > 0$, if and only if,

$$F(\varrho, \eta(\varsigma, z)) + F(\varsigma, \eta(z, \varsigma)) \leq \gamma\|\eta(z, \varrho)\|^2, \quad \forall \quad \varrho, \varsigma, z \in \mathcal{H}.$$

**Definition 2.6.** An operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is known as *$\eta$-monotone*, if and only if,

$$\langle T\varrho - T\varsigma, \eta(\varrho, \varsigma)\rangle \geq 0, \quad \forall \quad \varrho, \varsigma \in \mathcal{H}$$

and *PRS $\eta$-monotone* with $\gamma > 0$, if and only if,

$$\langle T\varrho - T\varsigma, \eta(z, \varsigma)\rangle \geq -\gamma\|\eta(z, \varrho)\|^2, \quad \forall \quad \varrho, \varsigma, z \in \mathcal{H}.$$

# 3 Main Results

We have sum of directional differentiable, differentiable and non-differential preinvex functions can be characterized by the form of an inequality. Consider

$$I[\varsigma] = F(\varsigma) + T[\varsigma] + \phi(\varsigma), \quad \forall \quad \varsigma \in \mathcal{H} \tag{3.1}$$

where $F$ is directional differentiable, $T$ is differentiable and $\phi$ is non-differentiable preinvex functions on $\mathcal{H}$. We now show that $\varrho \in \mathcal{H}$ is minimum of $I[\varsigma]$, if and only if $\varrho \in \mathcal{H}$, can be characterized by the form of inequality.

**Theorem 3.1.** *If $F$ is directional differentiable, $T$ is differentiable and $\phi$ is non-differentiable, then $\varrho \in \mathcal{H}$ is minimum of $I[\varsigma]$, if and only if,*

$$F'(\varrho, \eta(\varsigma, \varrho)) + \langle T'(\varrho), \eta(\varsigma, \varrho)\rangle + \phi(\varsigma) + \phi(\varrho) \geq 0, \quad \forall \quad \varsigma, \varrho \in \mathcal{H}. \tag{3.2}$$

*Proof.* If $\varrho \in \mathcal{H}$, is a minimum of $I[\varsigma]$, then we have

$$I[\varrho] \leq I[\varsigma], \quad \forall \quad \varsigma \in \mathcal{H}. \tag{3.3}$$

Taking $\varsigma$ by $\varsigma_\sigma = \varrho + \sigma\eta(\varsigma, \varrho) \in \mathcal{H}$ in inequality (3.3), we have

$$F(\varrho) + T(\varrho) + \phi(\varrho) \leq F(\varrho + \sigma\eta(\varsigma, \varrho)) + T(\varrho + \sigma\eta(\varsigma, \varrho)) + \phi(\varrho + \sigma\eta(\varsigma, \varrho)), \tag{3.4}$$

which implies that

$$[F(\varrho + \sigma\eta(\varsigma, \varrho)) - F(\varrho)] + [T(\varrho + \sigma\eta(\varsigma, \varrho)) - T(\varrho)] + \sigma(\phi(\varsigma) - \phi(\varrho)) \geq 0. \tag{3.5}$$

Dividing the inequality (3.5) by $\sigma$ and apply $\lim_{\sigma \to 0}$, from inequality (3.5) we obtain

$$F'(\varrho, \eta(\varsigma, \varrho)) + \langle T'(\varrho), \eta(\varsigma, \varrho)\rangle + \phi(\varsigma) - \phi(\varrho) \geq 0, \quad \forall \quad \varrho, \varsigma \in \mathcal{H},$$

which is required result 3.2.

Conversely, consider the inequality (3.2) holds for $\varrho \in \mathcal{H}$, then we have to show that $\varrho \in \mathcal{H}$ is the minimum of $I[\varsigma]$.

$$I[\varsigma] - I[\varrho] = F(\varsigma) + T(\varsigma) + \phi(\varsigma) - F(\varrho) - T(\varrho) - \phi(\varrho). \tag{3.6}$$

Since $F$, $T$ and $\phi$ all are preinvex functions, replacing $\varsigma$ by $\varsigma_\sigma = \varrho + \sigma\eta(\varsigma, \varrho) \in \mathcal{H}$, in inequality (3.6), we have

$$\begin{aligned} I[\varsigma] - I[\varrho] = [F(\varrho + \sigma\eta(\varsigma,\varrho)) - F(\varrho)] + [T(\varrho + \sigma\eta(\varsigma,\varrho)) - T(\varrho)] \\ + \sigma(\phi(\varsigma) - \phi(\varrho)). \end{aligned} \tag{3.7}$$

Dividing the inequality (3.7) by $\sigma$ and then apply $\lim_{\sigma \to 0}$, then inequality (3.7) becomes

$$I[\varsigma] - I[\varrho] \leq 0 \quad \Longrightarrow \quad I[\varsigma] \leq I[\varrho], \quad \forall \quad \varrho, \varsigma \in \mathcal{H}. \tag{3.8}$$

This shows that $\varsigma \in \mathcal{H}$, is the minimum of $I[\varsigma]$. □

We consider the more generalized form of the inequality (3.2), To be more precise, consider the problem of finding $\varrho \in \mathcal{H}$, such that

$$F(\varrho, \eta(\varsigma, \varrho)) + \langle T\varrho, \eta(\varsigma, \varrho)\rangle + \phi(\varsigma) - \phi(\varrho) \geq 0, \quad \forall \quad \varsigma \in \mathcal{H}. \tag{3.9}$$

The following inequality (3.9) is called mixed equilibrium variational-like inequality.

We discuss some known and results results as special cases of our main problem (3.9).

## Special cases

We will discuss some special cases of our main problem by applying condition on functions and operators involve in the problem.

(i). If $\eta(\varsigma, \varrho) = \varsigma - \varrho$, then (3.9) is reduced to find $\varrho \in \mathcal{H}$, given as

$$F(\varrho, \varsigma - \varrho) + \langle T\varrho, \varsigma - \varrho\rangle + \phi(\varsigma) - \phi(\varrho) \geq 0, \quad \forall \quad \varsigma \in \mathcal{H}. \tag{3.10}$$

This is known as the mixed bifunction variational inequality, which appears to be a new one.

(ii). If $F(\varrho, \eta(\varsigma, \varrho)) = 0$, then (3.9) is reduced to find $\varrho \in \mathcal{H}$, given as

$$\langle T\varrho, \eta(\varsigma, \varrho)\rangle + \phi(\varsigma) - \phi(\varrho) \geq 0, \quad \forall \quad \varsigma \in \mathcal{H}, \tag{3.11}$$

which is called the mixed variational-like inequality, see Noor [10].

(iii). If $\langle T\varrho, \eta(\varsigma, \varrho)\rangle = 0$, then (3.9) is reduced to find $\varrho \in \mathcal{H}$, given as

$$F(\varrho, \eta(\varsigma, \varrho)) + \phi(\varsigma) - \phi(\varrho) \geq 0, \quad \forall \quad \varsigma \in \mathcal{H}, \tag{3.12}$$

following is the mixed equilibriun-like inequality and seems to be a new one.

(iv). If $\phi$ is the indicator function, then (3.9) is reduced to find $\varrho \in \Omega_\eta$, given as

$$F(\varrho, \eta(\varsigma, \varrho)) + \langle T\varrho, \eta(\varsigma, \varrho)\rangle \geq 0, \quad \forall \quad \varsigma \in \Omega_\eta, \tag{3.13}$$

which is called equilibrium variational-like inequality.

(v). If $\phi$ is the indicator function, then (3.10) is reduced to find $\varrho \in \Omega$, given as

$$F(\varrho, \varsigma - \varrho) + \langle T\varrho, \varsigma - \varrho\rangle \geq 0, \quad \forall \quad \varsigma, \varrho \in \Omega, \tag{3.14}$$

which is called bifunction variational inequality.

(vi). If $\phi$ is the indicator function, then (3.11) is reduced to find $\varrho \in \Omega_\eta$, given as

$$\langle T\varrho, \eta(\varsigma, \varrho)\rangle \geq 0, \quad \forall \quad \varsigma \in \Omega_\eta, \tag{3.15}$$

which is called variational-like inequality.

(vii). If $\phi$ is the indicator functio, then (3.12) is reduced to find $\varrho \in \Omega_\eta$, given as

$$F(\varrho, \eta(\varsigma, \varrho)) \geq 0, \quad \forall \quad \varsigma \in \Omega_\eta, \tag{3.16}$$

which is called bifunction-like inequality.

For suitable and appropriate choices of the operators and spaces. one can obtain a wide class of equilibrium-like, variational inequalities and related optimization probelms. This show that the problem (3.9) is quite general and unified one.

# 4 Approximate Methods

In this section, we use the auxiliary principle technique to suggest and investigate some iterative schemes for solving the mixed equilibrium variational-like inequalities (3.9). In this technique, one consider an auxiliary problem associated with the original problems. This way, one defines the mapping associated with both problems. It is shown that this arbitrary function has a fixed point, which is the solution of the original problem. Consequently, this observation allows us to suggest some numerical methods for solving the original problem (3.9).

We propose and suggest some implicit and explicit iterative methods for solving the mixed equilibrium variational inequalities.

We now consider some implicit methods.

**(I).** For given $\varrho \in \mathcal{H}$ satisfying (3.9) and constant $\rho > 0$, consider the problem of finding $z \in \mathcal{H}$, such that

$$\rho F(z, \eta(\varsigma, z)) + \langle \rho T z, \eta(\varsigma, z)\rangle + \langle z - \varrho, \varsigma - z\rangle \geq \rho\phi(z) - \rho\phi(\varsigma), \quad \forall \quad \varsigma \in \mathcal{H}. \tag{4.1}$$

It can easily be seen that $\varrho \in \mathcal{H}$ is solution of (3.9), if and only if, $z = \varrho$. This observation allows to construct following iterative scheme to solve the problem (3.9).

**Algorithm 4.1.** *Compute $\varrho_{n+1} \in \mathcal{H}$, for given $\varrho_0 \in \mathcal{H}$, by the iterative scheme*

$$\begin{aligned} \rho F(\varrho_{n+1}, \eta(\varsigma, \varrho_{n+1})) + \langle \rho T \varrho_{n+1}, \eta(\varsigma, \varrho_{n+1})\rangle + \langle \varrho_{n+1} - \varrho_n, \varsigma - \varrho_{n+1}\rangle \\ \geq \rho\phi(\varrho_{n+1}) - \rho\phi(\varsigma), \quad \forall \quad \varsigma \in \mathcal{H}. \end{aligned} \tag{4.2}$$

We now study the convergence of iterative scheme in algorithm 4.1.

**Theorem 4.1.** *Let $\varrho_{n+1}$ be computed from (4.2) and $\varrho \in \mathcal{H}$ be solution of (3.9). If $F(\cdot,\cdot)$ and $T$ both are $\eta$-monotone, then following inequality satisfies*

$$\| \varrho - \varrho_{n+1}\|^2 \leq \| \varrho - \varrho_n\|^2 - \| \varrho_{n+1} - \varrho_n\|^2. \tag{4.3}$$

*Proof.* Let $\varrho$ be solution of (3.9). Replacing $\varsigma$ by $\varrho_{n+1}$ in (3.9), we have

$$\rho F(\varrho, \eta(\varrho_{n+1}, \varrho)) + \langle \rho T\varrho, \eta(\varrho_{n+1}, \varrho)\rangle + \rho\phi(\varrho_{n+1}) - \rho\phi(\varrho) \geq 0. \tag{4.4}$$

Let $\varrho_{n+1}$ be computed from (4.2). Taking $\varsigma = \varrho$, in (4.2), we have

$$\begin{aligned}\rho F(\varrho_{n+1}, \eta(\varrho, \varrho_{n+1})) + \langle \rho T\varrho_{n+1}, \eta(\varrho, \varrho_{n+1})\rangle + \langle \varrho_{n+1} - \varrho_n, \varrho - \varrho_{n+1}\rangle \\ \geq \rho\phi(\varrho_{n+1}) - \rho\phi(\varrho).\end{aligned} \tag{4.5}$$

Adding (4.4) and (4.5), we have

$$\begin{aligned}\langle \varrho_{n+1} - \varrho_n, \varrho - \varrho_{n+1}\rangle \geq & -\rho[F(\varrho_{n+1}, \eta(\varrho, \varrho_{n+1})) + F(\varrho, \eta(\varrho_{n+1}, \varrho))] \\ & + \rho\langle T\varrho_{n+1} - T\varrho, \eta(\varrho_{n+1}, \varrho)\rangle \geq 0.\end{aligned}$$

Since $F(\cdot,\cdot)$ and $T$ are $\eta$-monotone. Now from (2.3) we have

$$\| \varrho - \varrho_n\|^2 - \| \varrho_{n+1} - \varrho_n\|^2 - \| \varrho - \varrho_{n+1}\|^2 \geq 0,$$

which is the required result (4.3). □

**Theorem 4.2.** *Let $\varrho_{n+1}$ be computed from* (4.2) *and $\varrho$ be a solution of* (3.9)*. If $\mathcal{H}$ is finite dimensional, then* $\lim_{n\to\infty} \varrho_n = \varrho$.

*Proof.* Let $\varrho$ be a solution of (3.9). Then the sequence $\{\| \varrho - \varrho_n\|\}$ is nonincreasing and therefore $\{\varrho_n\}$ is bounded. Now from (4.3), we can analyze

$$\sum_{n=0}^{\infty} \| \varrho_{n+1} - \varrho_n\|^2 \leq \| \varrho - \varrho_0\|^2$$

implies that

$$\lim_{n\to\infty} \| \varrho_{n+1} - \varrho_n\| = 0. \tag{4.6}$$

Let $\hat{\varrho}$ be the limit point of $\{\varrho_n\}$ and the subsequence $\{\varrho_{n_j}\}$ of the sequence converges to $\hat{\varrho} \in \mathcal{H}$. Replace $\varrho_n$ by $\varrho_{n_j}$ in (4.2) and take the $\lim_{n_j\to\infty}$ and by using (4.6), we have

$$F(\hat{\varrho}, \eta(\varsigma, \hat{\varrho})) + \langle T\hat{\varrho}, \eta(\varsigma, \hat{\varrho})\rangle + \phi(\varsigma) - \phi(\hat{\varrho}) \geq 0, \quad \forall \quad \varsigma \in \mathcal{H}. \tag{4.7}$$

Clearly (4.7), This shows $\hat{\varrho}$ is the solution of mixed equilibrium variational-like inequalities (3.9) and

$$\|\varrho_{n+1} - \hat{\varrho}\|^2 \leq \| \varrho_n - \hat{\varrho}\|^2.$$

This shows that the sequence $\{\varrho_n\}$ has only one limit point and $\lim_{n\to\infty} \varrho_n = \hat{\varrho}$, which is the required result. □

By using the auxiliary principle technique in conjunction with Bregman function, we propose implicit iterative schemes for solving the mixed equilibrium variational-like inequality (3.9).

**(ii).** Consider the problem of finding a $z \in \mathcal{H}$, for given $\varrho \in \mathcal{H}$, and a constant $\rho > 0$, such that

$$\begin{aligned} \rho F(z, \eta(\varsigma, z)) + \langle \rho T z, \eta(\varsigma, z)\rangle + \langle B'(z) - B'(\varrho), \eta(\varsigma, z)\rangle \\ \geq \rho\phi(z) - \rho\phi(\varsigma), \quad \forall \quad \varsigma \in \mathcal{H} \end{aligned} \tag{4.8}$$

where $B$ is strongly preinvex differentiable function with $|\theta| > 0$.

It can easily be seen that $\varrho \in \mathcal{H}$, is solution of (3.9), if $z = \varrho$. The preceding observation allows to construct and study following iterative scheme to solve bifunction mixed variational-like inequality (3.9).

**Algorithm 4.2.** *Compute $\varrho_{n+1} \in \mathcal{H}$, for given $\varrho_0 \in \mathcal{H}$, by the iterative scheme*

$$\begin{aligned} \rho F(\varrho_{n+1}, \eta(\varsigma, \varrho_{n+1})) + \langle \rho T \varrho_{n+1}, \eta(\varsigma, \varrho_{n+1})\rangle + \langle B'(\varrho_{n+1}) - B'(\varrho_n), \eta(\varsigma, \varrho_{n+1})\rangle \\ \geq \rho\phi(\varrho_{n+1}) - \rho\phi(\varsigma), \quad \forall \varsigma \in \mathcal{H}. \end{aligned} \tag{4.9}$$

We analyze the convergence of iterative scheme in algorithm 4.2.

**Theorem 4.3.** *Let $\varrho_n$ be computed from* (4.9) *and $\varrho \in \mathcal{H}$ be solution of* (3.9)*. If $T$ and $F(\cdot, \cdot)$ both are $\eta$-monotone and condition-$\eta$* (2.4) *satisfies, then* $\lim_{n\to\infty} \varrho_n = \varrho$.

*Proof.* Let $\varrho$ be solution of (3.9). Replace $\varsigma$ by $\varrho_{n+1}$ in (3.9), we have

$$\rho F(\varrho, \eta(\varrho_{n+1}, \varrho)) + \langle \rho T\varrho, \eta(\varrho_{n+1}, \varrho)\rangle + \rho\phi(\varrho_{n+1}) - \rho\phi(\varrho) \geq 0. \tag{4.10}$$

Let $\varrho_{n+1}$ be computed from (4.9). Take $\varsigma = \varrho$ in (4.9), we have

$$\begin{aligned}&\rho F(\varrho_{n+1}, \eta(\varrho, \varrho_{n+1})) + \langle \rho T\varrho_{n+1}, \eta(\varrho, \varrho_{n+1})\rangle \\ &+\langle B'(\varrho_{n+1}) - B'(\varrho_n), \eta(\varrho, \varrho_{n+1})\rangle \geq \rho\phi(\varrho_{n+1}) - \rho\phi(\varrho).\end{aligned} \tag{4.11}$$

Adding (4.10) and (4.11), we have

$$\begin{aligned}\langle B'(\varrho_{n+1}) - B'(\varrho_n), \eta(\varrho, \varrho_{n+1})\rangle \geq &- \rho[F(\varrho_{n+1}, \eta(\varrho, \varrho_{n+1})) + F(\varrho, \eta(\varrho_{n+1}, \varrho))] \\ &+ \rho\langle T\varrho_{n+1} - T\varrho, \eta(\varrho_{n+1}, \varrho)\rangle \geq 0,\end{aligned} \tag{4.12}$$

since $F(\cdot,\cdot)$ and $T$ are $\eta$-monotone.

Consider the Bregman function,

$$\begin{aligned}\mathcal{G}(\varrho, \xi) &= B(\varrho) - B(\xi) - \langle B'(\xi), \eta(\varrho, \xi)\rangle \quad \forall \quad \varrho, \varsigma, \xi \in \mathcal{H}. \\ &\geq \theta\| \ \eta(\varrho, \xi)\|^2,\end{aligned} \tag{4.13}$$

where we have used the fact that $B$ is a differentiable strongly preinvex function. Now from (4.13), (2.2) and (2.4), we have

$$\begin{aligned}\mathcal{G}(\varrho, \varrho_n) - \mathcal{G}(\varrho, \varrho_{n+1}) = B(\varrho_{n+1}) - B(\varrho_n) &- \langle B'(\varrho_n), \eta(\varrho_{n+1}, \varrho_n)\rangle \\ &+\langle B'(\varrho_n) - B'(\varrho_{n+1}), \eta(\varrho, \varrho_{n+1})\rangle.\end{aligned}$$

From (4.12) and (4.13), we have

$$\mathcal{G}(\varrho, \varrho_n) - \mathcal{G}(\varrho, \varrho_{n+1}) \geq \theta\| \ \eta(\varrho_{n+1}, \varrho_n)\|^2. \tag{4.14}$$

Clearly if $\varrho_{n+1} = \varrho_n$, then $\varrho_n$ is solution of bifunction mixed variational-like inequalities (3.9). Otherwise, the sequence $\{\mathcal{G}(\varrho, \varrho_n) - \mathcal{G}(\varrho, \varrho_{n+1})\}$, is non-negative, so we must have

$$\lim_{n\to\infty} \| \ \eta(\varrho_{n+1}, \varrho_n)\| = 0.$$

From the technique of Zhu and Marcotte [21], it follows that the sequence $\{\varrho_n\}$ converges to the limit point $\varrho$ which is solution of mixed equilibrium variational-like inequalities (3.9). □

**(II).** We proposed and discussed some explicit iterative schemes for solving the problem (3.9) using the auxiliary principle technique.

Consider the problem of finding $z \in \mathcal{H}$, For given $\varrho \in \mathcal{H}$, and constat $\rho > 0$, such that

$$\rho F(\varrho, \eta(\varsigma, z)) + \langle \rho T\varrho, \eta(\varsigma, z)\rangle + \langle z - \varrho, \varsigma - z\rangle \geq \rho\phi(z) - \rho\phi(\varsigma), \quad \forall \varsigma \in \mathcal{H}. \tag{4.15}$$

It can easily be seen that $\varrho \in \mathcal{H}$, is solution of (3.9), if $z = \varrho$. The preceding observation allows to construct and study following iterative scheme to solve the defined problem (3.9).

**Algorithm 4.3.** *Compute $\varrho_{n+1} \in \mathcal{H}$, for given $\varrho_0 \in \mathcal{H}$, by the iterative scheme.*

$$\begin{aligned} \rho F(\varrho_n, \eta(\varsigma, \varrho_{n+1})) + \langle \rho T\varrho_n, \eta(\varsigma, \varrho_{n+1})\rangle + \langle \varrho_{n+1} - \varrho_n, \varsigma - \varrho_{n+1}\rangle \\ \geq \rho\phi(\varrho_{n+1}) - \rho\phi(\varsigma), \quad \forall \quad \varsigma \in \mathcal{H}. \end{aligned} \tag{4.16}$$

We now discuss the convergence of the Algorithm 4.3.

**Theorem 4.4.** *Let $\varrho \in \mathcal{H}$ be solution of* (3.9) *and $\varrho_{n+1}$ be computed from* (4.4)*. If $F(\cdot,\cdot)$ and $T$ both are PRS $\eta$-monotones with $\mu > 0$ and $\gamma > 0$, respectively. Also $\eta(,.,)$ is Lipschitz continuous with $\vartheta > 0$, then following inequality satisfies*

$$\| \varrho - \varrho_{n+1}\|^2 \leq \| \varrho - \varrho_n\|^2 - (1 - 2\rho\vartheta(\gamma + \mu))\| \varrho_{n+1} - \varrho_n\|^2. \tag{4.17}$$

*Proof.* Let $\varrho$ be a solution of (3.9). Replace $\varsigma$ by $\varrho_{n+1}$ in (3.9), we have

$$\rho F(\varrho, \eta(\varrho_{n+1}, \varrho)) + \langle \rho T\varrho, \eta(\varrho_{n+1}, \varrho)\rangle + \rho\phi(\varrho_{n+1}) - \rho\phi(\varrho) \geq 0. \tag{4.18}$$

Let $\varrho_{n+1}$ be computed from (4.4). Take $\varsigma = \varrho$, in (4.4), we have

$$\begin{aligned} \rho F(\varrho_n, \eta(\varrho, \varrho_{n+1})) + \langle \rho T\varrho_n, \eta(\varrho, \varrho_{n+1})\rangle + \langle \varrho_{n+1} - \varrho_n, \varrho - \varrho_{n+1}\rangle \\ \geq \rho\phi(\varrho_{n+1}) - \rho\phi(\varrho). \end{aligned} \tag{4.19}$$

Adding (4.18) and (4.19), we have

$$\begin{aligned} \langle \varrho_{n+1} - \varrho_n, \varrho - \varrho_{n+1}\rangle \geq & - \rho[F(\varrho_n, \eta(\varrho, \varrho_{n+1})) + F(\varrho, \eta(\varrho_{n+1}, \varrho))] \\ & + \rho\langle T\varrho_n - T\varrho, \eta(\varrho_{n+1}, \varrho)\rangle. \end{aligned}$$

As $F(\cdot,\cdot)$ and $T$ are PRS $\eta$-monotone, with $\mu > 0$, and $\gamma > 0$, respectively, implies that

$$\langle \varrho_{n+1} - \varrho_n, \varrho - \varrho_{n+1} \rangle \geq -\rho(\mu + \gamma) \| \ \eta(\varrho_{n+1}, \varrho_n) \|^2. \tag{4.20}$$

Here $\eta(\cdot,\cdot)$ is Lipschitz continuous and from (2.3), we have

$$\| \ \varrho - \varrho_n \|^2 - \| \ \varrho_{n+1} - \varrho_n \|^2 - \| \ \varrho - \varrho_{n+1} \|^2 \geq -\rho\vartheta(\mu + \gamma) \| \ \varrho_{n+1} - \varrho_n \|^2.$$

From which inequality (4.17) follows. □

**Theorem 4.5.** *Let $\varrho_{n+1}$ be computed from* (4.4) *and $\varrho \in \mathcal{H}$ be a solution* (3.9)*. If $\mathcal{H}$ is finite dimensional, then $\lim_{n\to\infty} \varrho_n = \varrho$.*

*Proof.* Let $\varrho$ be the solution of (3.9), for $0 < \rho < \frac{1}{2\vartheta(\mu+\gamma)}$, then sequence $\{\| \ \varrho - \varrho_n \|\}$, is non-increasing and therefore $\{\varrho_n\}$ is bounded. Now from (4.17), we can analyze

$$\sum_{n=0}^{\infty} (1 - 2\rho\vartheta(\mu + \gamma)) \| \ \varrho_{n+1} - \varrho_n \|^2 \leq \| \ \varrho - \varrho_0 \|^2, \tag{4.21}$$

implies that

$$\lim_{n\to\infty} \| \ \varrho_{n+1} - \varrho_n \| = 0. \tag{4.22}$$

Consider $\hat{\varrho}$ is the limit point of $\{\varrho_n\}$ and subsequence $\{\varrho_{n_j}\}$ of the sequence converges to $\hat{\varrho} \in H$. Replace $\varrho_n$ by $\varrho_{n_j}$ in (4.4) and take the $\lim_{n_j\to\infty}$ and use (4.22), we have

$$F(\hat{\varrho}, \eta(\varsigma, \hat{\varrho})) + \langle T\hat{\varrho}, \eta(\varsigma, \hat{\varrho}) \rangle + \phi(\varsigma) - \phi(\hat{\varrho}) \geq 0, \quad \forall \quad \varsigma \in \mathcal{H}. \tag{4.23}$$

clearly (4.23) shows that $\hat{\varrho}$ is solution of the mixed equilibrium variational-like inequalities(3.9) and

$$\| \ \varrho_{n+1} - \hat{\varrho} \|^2 \leq \| \ \varrho_n - \hat{\varrho} \|^2.$$

This shows that the sequence $\{\varrho_n\}$ has only one limit point and $\lim_{n\to\infty} \varrho_n = \hat{\varrho}$, which is required result. □

Now we use auxiliary principle technique in conjunction with Bregman function to propose explicit iterative scheme for solving the problem (3.9), as follows:

Consider the problem of finding a $z \in \mathcal{H}$, for given $\varrho \in \mathcal{H}$ and a constant $\rho > 0$, such that

$$\begin{aligned} \rho F(\varrho, \eta(\varsigma, z)) + \langle \rho T\varrho, \eta(\varsigma, z)\rangle + \langle B'(z) - B'(\varrho), \eta(\varsigma, z)\rangle \\ \geq \rho\phi(z) - \rho\phi(\varsigma), \quad \forall \quad \varsigma \in \mathcal{H}. \end{aligned} \tag{4.24}$$

Following algorithm can be constructed by the observation that, if $z = \varrho$, then $z \in \mathcal{H}$, is the solution of the problem (3.9).

**Algorithm 4.4.** *Compute $\varrho_{n+1} \in \mathcal{H}$, for given $\varrho_0 \in \mathcal{H}$, by the iterative scheme.*

$$\begin{aligned} \rho F(\varrho_n, \eta(\varsigma, \varrho_{n+1})) + \langle \rho T\varrho_n, \eta(\varsigma, \varrho_{n+1})\rangle + \langle B'(\varrho_{n+1}) - B'(\varrho_n), \eta(\varsigma, \varrho_{n+1})\rangle \\ \geq \rho\phi(\varrho_{n+1}) - \rho\phi(\varsigma), \quad \forall \quad \varsigma \in \mathcal{H}. \end{aligned} \tag{4.25}$$

**Theorem 4.6.** *Let $\varrho_n$ be computed from* (4.25) *and $\varrho \in \mathcal{H}$ be solution of* (3.9). *If $F(\cdot,\cdot)$ and $T$ is PRS $\eta$-monotone with $\mu > 0$ and $\gamma > 0$, respectively, for $0 < \rho < \frac{\theta}{\mu+\gamma}$ and condition-$\eta$* (2.4), *satisfies then $\lim_{n\to\infty} \varrho_n = \varrho$.*

*Proof.* Let $\varrho$ be solution of (3.9). Replacing $\varsigma$ by $\varrho_{n+1}$ in (3.9), we have

$$\rho F(\varrho, \eta(\varrho_{n+1}, \varrho)) + \langle \rho T\varrho, \eta(\varrho_{n+1}, \varrho)\rangle + \rho\phi(\varrho_{n+1}) - \rho\phi(\varrho) \geq 0. \tag{4.26}$$

Let $\varrho_{n+1}$ be computed from (4.25). Take $\varsigma = \varrho$ in (4.25), we have

$$\begin{aligned} \rho F(\varrho_n, \eta(\varrho, \varrho_{n+1})) + \langle \rho T\varrho_n, \eta(\varrho, \varrho_{n+1})\rangle + \langle B'(\varrho_{n+1}) - B'(\varrho_n), \eta(\varrho, \varrho_{n+1})\rangle \\ \geq \rho\phi(\varrho_{n+1}) - \rho\phi(\varrho). \end{aligned} \tag{4.27}$$

Adding (4.26) and (4.27), we have

$$\begin{aligned} \langle B'(\varrho_{n+1}) - B'(\varrho_n), \eta(\varrho, \varrho_{n+1})\rangle \geq & -\rho[F(\varrho, \eta(\varrho_{n+1}, \varrho)) + F(\varrho_n, \eta(\varrho, \varrho_{n+1}))] \\ & + \rho\langle T\varrho_n - T\varrho, \eta(\varrho_{n+1}, \varrho)\rangle, \end{aligned}$$

we have $F(\cdot,\cdot)$ and $T$ are PRS $\eta$-monotone with $\mu > 0$ and $\gamma > 0$, implies that

$$\langle B'(\varrho_{n+1}) - B'(\varrho_n), \eta(\varrho, \varrho_{n+1})\rangle \geq -\rho(\mu+\gamma)\| \eta(\varrho_{n+1}, \varrho_n)\|^2. \tag{4.28}$$

Consider the Bregman function,

$$\begin{aligned} \mathcal{G}(\varrho, \xi) &= B(\varrho) - B(\xi) - \langle B'(\xi), \eta(\varrho, \xi)\rangle, \quad \forall \quad \varrho, \varsigma, \xi \in \mathcal{H}. \\ &\geq \theta \| \eta(\varrho, \xi)\|^2. \end{aligned} \tag{4.29}$$

We have used the fact that is a $B$ differentiable strongly preinvex function. Now from (2.2) and (2.4), we have

$$\begin{aligned} \mathcal{G}(\varrho, \varrho_n) - \mathcal{G}(\varrho, \varrho_{n+1}) = B(\varrho_{n+1}) - B(\varrho_n) - \langle B'(\varrho_n), \eta(\varrho_{n+1}, \varrho_n)\rangle \\ + \langle B'(\varrho_n) - B'(\varrho_{n+1}), \eta(\varrho, \varrho_{n+1})\rangle. \end{aligned}$$

From (4.28) and (4.29), we have

$$\mathcal{G}(\varrho, \varrho_n) - \mathcal{G}(\varrho, \varrho_{n+1}) \geq \vartheta \| \eta(\varrho_{n+1}, \varrho_n)\|^2 - \rho(\mu+\gamma)\| \eta(\varrho_{n+1}, \varrho_n)\|^2,$$

implies that

$$\mathcal{G}(\varrho, \varrho_n) - \mathcal{G}(\varrho, \varrho_{n+1}) \geq (\theta - \rho(\mu+\gamma))\| \eta(\varrho_{n+1}, \varrho_n)\|^2.$$

Clearly, if $\varrho_{n+1} = \varrho_n$, then $\varrho_n$ is a solution of the bifunction mixed variational-like inequalities (3.9). Otherwise, the sequence $\{\mathcal{G}(\varrho, \varrho_n) - \mathcal{G}(\varrho, \varrho_{n+1})\}$ is non-negative so, we must have

$$\lim_{n\to\infty} \| \eta(\varrho_{n+1}, \varrho_n)\| = 0.$$

Using the technique of Zhu and Marcotte [21], we see that the sequence $\{\varrho_n\}$ converges to the limit point $\varrho$ which is solution of mixed equilibrium variational-like inequalities (3.9). □

# Conclusion

Some new classes of mixed equilibrium ariational-like inequalities are introduced, which contains known and new classes of variational inequalities and their forms as special cases. The auxiliary principle techniques are used to suggest a wide class of implicit and explicit methods for solving the equilibrium variational-like inequalities. Convergence analysis of the proposed new methods is investigated under some suitable weaker conditions. Our method of proofs is very simple as compared with techniques. It is an interesting problem to consider the numerical implementation of the proposed iterative methods and comparison with other methods.

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