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Some Novel Aspects of Quasi Variational Inequalities

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Abstract

Quasi variational inequalities can be viewed as novel generalizations of the variational inequalities and variational principles, the origin of which can be traced back to Euler, Lagrange, Newton and Bernoulli's brothers. It is well known that quasi-variational inequalities are equivalent to the implicit fixed point problems. We consider this alternative equivalent fixed point formulation to suggest some new iterative methods for solving quasi-variational inequalities and related optimization problems using projection methods, Wiener-Hopf equations, dynamical systems, merit function and nonexpansive mappings. Convergence analysis of these methods is investigated under suitable conditions. Our results present a significant improvement of previously known methods for solving quasi variational inequalities and related optimization problems. Since the quasi variational inequalities include variational inequalities and complementarity problems as special cases. Results obtained in this paper continue to hold for these problems. Some special cases are discussed as applications of the main results. The implementation of these algorithms and comparison with other methods need further efforts.

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1 Introduction

Variational inequality theory contains a wealth of new ideas and techniques, which can be viewed as a novel extension and generalization of the variational principles. Variational inequalities were introduced and considered in early sixties by Stampacchia [64]. It is amazing that a wide class of unrelated problems, which arise in various different branches of pure and applied sciences, can be studied in the general and unified framework of variational inequalities. For the applications, motivation, numerical results, sensitivity analysis, dynamical systems and other aspects of variational inequalities, see [6, 8, 13–15, 20, 24, 29, 31, 34, 35, 37, 40, 41, 48, 55–57, 60–62, 64] and the references therein.

If the set involved in the variational inequality depends upon the solution explicitly or implicity, then the variational inequalities are called the quasi variational inequality. The quasi variational inequalities were introduced by Bensoussan and Lions [6] in the field of impulse control. Noor [35] proved that the quasi variational inequalities are equivalent to the implicit fixed point problem. This equivalent formulation played an important role in developing numerical methods [35, 44, 54, 62], sensitivity analysis [39, 42], dynamical systems [43], merit functions [44] and other aspects of quasi variational inequalities [36, 41, 48, 49, 51]. The Wiener-Hopf equations were introduced and studied by Shi [62] and Robinson [61]. The technique of Wiener-Hopf equations is quite general and unifying one. This technique has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inequalities. Noor [39] have proved that quasi variational inequalities are equivalent to the Wiener-Hopf equations. The alternative fixed point technique is used to establish the equivalence between the variational inequalities and dynamical systems by Noor [43]. This equivalence has been used to study the existence and stability of the solution of variational inequalities. Noor et al [56] have been shown that the dynamical system can be used to suggest some implicit iterative method for solving variational inequalities. For the applications and numerical methods of the dynamical systems, see [41, 43, 55, 56] and the references therein. We use the dynamical systems to suggest the iterative methods for solving the quasi variational inequalities, which is considered in Section 6.

It is known that the sensitivity analysis of variational inequalities can provide new insight concerning the problem being studied and can stimulate ideas for problem solving. Dafermos [13] explored the sensitivity analysis of the variational inequalities. The techniques suggested so far vary with the problem setting being studied. Noor [39] applied the Wiener-Hopf equations technique to study the sensitivity analysis of quasi variational inequalities. In Section 6, we study the sensitivity analysis of quasi variational inequalities, essentially considering the technique is mainly due to Noor [39].

The variational inequalities can be reformulated as optimization problem, which is used to define the concept of merit functions. Various merit (gap) functions for variational inequalities and complementarity problems have been suggested in recent years. Using the merit functions, error bounds are obtained. Error bounds of the variational inequalities provide a measure of the distance between a solution set and an arbitrary point. Therefore, error bounds play an important role in the analysis of global or local convergence analysis of algorithms for solving variational inequalities. Noor [44] obtained some error bounds for the quasi variational inequalities. To the best of our knowledge, very few merit functions have been considered for quasi variational inequalities. These results are discussed in Section 7.

One of the most difficult and important problems in variational inequalities is the development of efficient numerical methods. Several numerical methods have been developed for solving the variational inclusions and their variant forms. These methods have been extended and modified in numerous ways for solving the variational inclusions and their variant forms. Noor [36, 37, 39] suggested and analyzed some three-step forward-backward splitting algorithms for solving variational inequalities and quasi variational inclusions by using the updating technique. These three-step methods are also known as Noor's iterations. It is noted that these forward-backward splitting algorithms are similar to those of Glowinski et al. [16], which they have suggested by using the Lagrangian technique. Haubruge et al. [17] discussed the convergence analysis and applications of the Glowinski-Le Tallec splitting method. It is known that three-step schemes are versatile and efficient. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations. For applications of the splitting techniques to partial differential equations, see Ames [3] and the references therein. For novel applications of the three-step methods, see Ashish et al. [7]. These methods include the Mann iteration [28] and Ishikawa iteration [19] and modified forward-backward splitting methods of Tseng [66], Noor [36, 37, 39] and Noor et al. [48] as special cases. Related to the variational inequalities is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis, see Noor and Huang [48]. Motivated by the research going on these fields, Noor and Huang [49] have analyzed and suggested some three-step iterative method for finding the common solution of these problems along with convergence criteria. This is the subject of Section 8.

Noor [37] introduced and considered general variational inequalities, which are being used to study the odd-order, nonsymmetric and nonpositive obstacle boundary vale problems [40, 41]. In Section 9, it is shown that quasi variational inequalities are equivalent to general variational inequalities for a special case of convex-valued set. For more details, see [43, 56, 57] and the references therein. Several important special cases are also discussed as applications of our results.

Quasi variational inequalities theory is quite broad, so we shall content ourselves here to give the flavour of the main ideas and techniques involved. The techniques used to analysis the iterative methods and other results for quasi variational inequalities are a beautiful blend of ideas of pure and applied mathematical sciences. In this paper, we have presented the main results regarding the development of various algorithms, Wiener-Hopf equations, dynamical systems, merit functions, nonexpansive mappings and the sensitivity analysis of the quasi variational inequalities. Although this paper is expository in nature, our choice has been rather to consider a number of familiar and to us some fascinating aspects of quasi variational inequalities. We also include some new results which we and our coworkers have recently obtained. The language used is necessarily that of functional analysis and some knowledge of elementary Hilbert space theory is assumed. The framework chosen should be seen as a model setting for more general results for other classes of quasi variational inequalities and related optimization problems. It is true that each of these areas of applications require special consideration of peculiarities of the physical problem at hand and the inequalities that model it. However, many of the concepts and techniques, we have discussed are fundamental to all of these applications. One of the main purposes of this expository paper is to demonstrate the close connection among various classes of algorithms for the solution of the quasi variational inequalities and to point out that researchers in different field of variational inequalities and optimization have been considering parallel paths. We would like to emphasize that the results obtained and discussed in this report may motivate and bring a large number of novel, innovate, potential applications, extensions and interesting topics in these areas. We have given only a brief introduction of this fast growing field. The interested reader is advised to explore this field further and discover novel and fascinating applications this theory in other areas of sciences.

2 Formulations and Basic Facts

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. First of all, we recall some concepts from convex analysis, see Cristescu et al. [12] and Niculescu et al. [33], which are needed in the derivation of the main results.

Definition 2.1. The set K in H is said to be a *convex set*, if

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

Definition 2.2. A function F is said to be a *convex function*, if

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

We remark that the minimum $u \in K$ of the differentiable convex function on the convex set K is equivalent to finding $u \in K$, such that

$$\langle F'(u), v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.1}$$

where $F'(u)$ is the differential of the convex function F at $u \in K$. The inequality of the type (2.1) is called the variational inequality, which was introduced by Stampacchia [64] in potential theory. It is amazing that this simple fact played in the developments of several fields of pure and applied sciences such as transportation, water resources, management, operation research and optimization.

In several problems as observed by Bensoussan and Lions [6], the underlying convex set may depend implicity or explicitly on the solution itself. In these situations, variational inequality is called the quasi variational inequality. To be more precise, for a given operator T, find $u \in K(u)$, a point-to-set mapping $K : u \longrightarrow K(u)$, which associates a closed convex-valued set $K(u)$ with any element u of H, such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K(u), \tag{2.2}$$

which is known as the quasi variational inequality. Quasi variational inequalities were introduced by Bensoussan and Lions [6] in the impulse control theory. For the numerical analysis, sensitivity analysis, dynamical systems, other aspects of quasi variational inequalities and related optimization programming problems. see [2, 6, 8, 25, 36, 37, 39, 41, 51, 54–57] and the references therein.

We now discuss some important special cases of the quasi variational inequalities (2.2).

(I). If $K^*(u) = \{u \in H : \langle u, v\rangle \geq 0, \quad \forall v \in K(u)\}$ is a polar (dual) cone of a convex-valued cone $K(u)$ in H, then problem (2.2) is equivalent to finding $u \in K(u)$ such that

$$Tu \in K^*(u) \quad \text{and} \quad \langle Tu, u\rangle = 0, \tag{2.3}$$

which is known as the quasi complementarity problems, which was studied and investigated by Noor [36]. Obviously quasi complementarity problems include the complementarity problems, which were introduced by Lemake [26] and Cottle [10] and in game theory, management sciences and quadratic programming as special cases. For details, see [11, 23, 30, 41, 51].

(II). If $K(u) = K$, a convex set in H, then problem (2.2) reduces to: For given nonlinear operator $T : H \to H$, find $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.4}$$

which is called the variational inequality, introduced and studied by Stampacchia [64] in the potential theory. For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational inequalities, see [13–15, 20, 24, 29, 31, 35, 37, 40, 41, 48, 55–57, 60–62, 64] and the references therein.

Remark 2.1. It is worth mentioning that for appropriate and suitable choices of the operator T, set-valued convex set $K(u)$ and the spaces, one can obtain several classes of variational inequalities (2.4), complementarity problems (2.3) and optimization problems as special cases of the quasi-variational inequalities. This shows that the problem (2.2) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the nonlinear quasi-variational inequalities.

Example 2.1. To convey an idea of the applications of the quasi variational inequalities (2.2), we consider the second-order implicit obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -u'' \geq f(x) & \text{on } \Omega = [a, b] \\ u \geq M(u) & \text{on } \Omega = [a, b] \\ [-u'' - f(x)][u - M(u)] = 0 & \text{on } \Omega = [a, b] \\ u(a) = 0, \quad u(b) = 0. & \end{array}\right\} \tag{2.5}$$

where $f(x)$ is a linear continuous function and $M(u)$ is the cost (obstacle) function. The prototype encountered is

$$M(u) = k + \inf_i \{u^i\}. \tag{2.6}$$

In (2.6), k represents the switching cost. It is positive, when the unit is turned on and equal to zero when the unit is turned off. Note that the operator M provides the coupling between the unknowns $u = (u^1, u^2, \ldots, u^i)$, see [23]. We study the problem (2.5) in the framework of quasi variational inequality approach. To do so, we first define the set K as

$$K(u) = \{v : v \in H_0^1(\Omega) : v \geq M(u), \quad \text{on } \Omega\},$$

which is a closed convex-valued set in $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is a Sobolev (Hilbert) space, see [65]. One can easily show that the energy functional associated with the problem (2.5) is

$$\begin{aligned} I[v] &= -\int_a^b \left(\frac{d^2v}{dx^2}\right) v dx - 2\int_a^b f(x) v dx, \quad \forall v \in K(u) \\ &= \int_a^b \left(\frac{dv}{dx}\right)^2 dx - 2\int_a^b f(x) v dx \\ &= \langle Tv, v\rangle - 2\langle f, v\rangle, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \langle Tu, v\rangle &= -\int_a^b \left(\frac{d^2u}{dx^2}\right) v dx = \int_a^b \frac{du}{dx}\frac{dv}{dx} dx \\ \langle f, v\rangle &= \int_a^b f(x) v dx. \end{aligned} \tag{2.8}$$

It is clear that the operator T defined by (2.8) is linear, symmetric and positive. Using the technique of Tonti [65], one can show that the minimum of the functional $I[v]$ defined by (2.7) associated with the problem (2.5) on the closed convex-valued set $K(u)$ can be characterized by the inequality of type

$$\langle Tu, v - u\rangle \geq \langle f, v - u\rangle, \quad \forall v \in K(u), \tag{2.9}$$

which is exactly the quasi variational inequality (2.2).

We also need the following result, known as the projection Lemma (best approximation) Lemma, which plays a crucial part in establishing the equivalence between the quasi variational inequalities and the fixed point problem. This result is used in the analysing the convergence analysis of the projective implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.1. *[23] Let $K(u)$ be a closed and convex-valued set in H. Then, for a given $z \in H$, $u \in K(u)$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K(u), \tag{2.10}$$

if and only if,

$$u = P_{K(u)}(z),$$

where $P_{K(u)}$ is implicit projection of H onto the closed convex-valued set $K(u)$.

It is well known that the projection operator $P_{K(u)}$ is not a nonexpansive mapping. However, the implicit operator $P_{K(u)}$ is required to satisfy the following assumption.

Assumption 2.1.

$$\|P_{K(u)}w - P_{K(u)}w\| \leq \nu\|u - v\|, \forall u, v, w \in H, \quad \nu > 0, \quad \textit{a constant.} \tag{2.11}$$

This implies that the operator $P_{K(u)}$ is Lipschitz continuous. We remark that Assumption 2.1 is true for the special case. In many important applications, the convex-valued set $K(u)$ has the form

$$K(u) = m(u) + K, \tag{2.12}$$

where K is a convex set in H and m is a point-to-point mapping. It is well known that

$$P_{K(u)}w = P_{m(u)+K}w = m(u) + P_K[w - m(u)], \quad \forall w, u \in H. \tag{2.13}$$

We remark that, if the mapping $m(u)$ is a Lipschitz continuous with constant $\nu_1 > 0$, then, from (2.12) and (2.13), we have

$$\begin{aligned}\|P_{m(u)K}w - P_{m(v)+K}w\| &= \|m(u) - m(v) + P_K[w - m(u)] - P_K[w - m(v)]\| \\ &\leq 2\|m(u) - m(v)\| \leq 2\nu_1\|u - v\|.\end{aligned}$$

This shows that the projection operator $P_{m(u)+K}$ is Lipschitz continuous with constant $2\nu_1 > 0$. and satisfies the Assumption 2.1 with $\nu = 2\nu_1$.

This property of the projection operator plays an important part in the derivation of the results.

Definition 2.3. An operator $T : H \to H$ is said to be:

(i) *Strongly monotone*, if there exist a constant $\alpha > 0$, such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|u - v\|^2, \quad \forall u, v \in H.$$

(ii) *Lipschitz continuous*, if there exist a constant $\beta > 0$, such that

$$\|Tu - Tv\| \leq \beta\|u - v\|, \quad \forall u, v \in H.$$

(iii) *Monotone*, if

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

(iv) *Pseudo monotone*, if

$$\langle Tu, v - u\rangle \geq 0 \quad \Rightarrow \quad \langle Tv, v - u\rangle \geq 0, \quad \forall u, v \in H.$$

(v) *hemicontinous*, if the mapping $\langle T(u + t(v - u)), v - u\rangle \quad \Rightarrow \quad \langle Tu, v - u\rangle$ as $t \to 0$.

Remark 2.2. Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true. Also, every continuous operator is hemicontinuous, but the converse is not true.

Lemma 2.2. *Let the operator T be pseudomonotone and hemicontinuous. Then $u \in K(u)$ is a solution of the quasi variational inequality (2.2), if and only if, $u \in K(u)$ satisfies the inequality*

$$\langle Tv, v-u\rangle \geq 0, \quad \forall v \in K(u). \tag{2.14}$$

Proof. Let $u \in K(u)$ be a solution of the quasi variational inequality (2.2). Then $\in K(u)$ satisfies (2.14), where we have used the pseudomonotonicity of T.

Conversely, let $\in K(u)$ satisfies (2.14). Since $K(u)$ is a convex-valued convex set, so $\forall u, v \in K(u), \quad t \in [0,1]$,

$$v_t = u + t(v-u) \in K(u).$$

Replacing v by v_t in (2.14), we have

$$0 \leq \langle T(v_t), v_t - u\rangle = t\langle T(v_t), v-u\rangle,$$

which implies that

$$0 \leq \langle T(v_t), v-u\rangle. \tag{2.15}$$

Using the hemicontinuity of the operator T and taking the limit as $t \to 0$ in (2.15), we have

$$\langle Tu, v-u\rangle \geq 0, \tag{2.16}$$

which is the required result (2.2). □

Remark 2.3. We point out that Lemma 2.2 is known as the Mintly lemma and the quasi variational inequality is called the dual quasi variational inequality. This results is useful to study the convexity of the solution set of the dual quasi variational inequality (2.2). This can be viewed as the linearization of the variational inequalities. For $K(u) = K$, this result is well known, see [23].

3 Projection Method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the strongly nonlinear quasi variational inequalities.

Using Lemma 2.1, one can show that the quasi variational inequality (2.2) is equivalent to the fixed point problem.

Lemma 3.1. [2] *The function $u \in K(u)$ is a solution of the quasi variational inequalities (2.2), if and only if, $u \in K(u)$ satisfies the relation*

$$u = P_{K(u)}[u - \rho T u], \tag{3.1}$$

where $P_{K(u)}$ is the projection operator and $\rho > 0$ is a constant.

Lemma 3.1 implies that the quasi variational inequality (2.2) is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation is used to suggest some implicit iterative methods for solving the quasi variational inequalities.

Using (3.1), we define the mapping Φ associated with (3.1) as:

$$\Phi(u) = P_{K(u)}[u - \rho \mathcal{T} u]. \tag{3.2}$$

To prove the existence of the solution of problem (2.1), it is enough that the mapping Φ defined by (3.2) is a contraction mapping.

Theorem 3.1. *Let the operator $\mathcal{T}$ be strongly monotone with constants $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. If Assumption (2.1) holds and there exists a constant $\rho > 0$, such that*

$$\|\rho - \frac{\alpha}{\beta^2}\| < \frac{\sqrt{\alpha^2 - \beta^2 \eta(2 - k\eta)}}{\beta^2}, \quad \alpha > \beta\sqrt{\eta(2-\eta)}, \quad \eta < 1, \tag{3.3}$$

then there exists a solution $u \in K(u)$ satisfying problem (2.2).

Proof. Let $u \neq v \in K(u)$ be two solutions of problem (2.1). Then, from problem (3.2), we have

$$\begin{aligned} \|\Phi(v)-\Phi(u)\| &\leq \|P_{K(v)}[v-\rho Tv]-P_{K(v)}[u-\rho Tu]\| \\ &= \|P_{K(v)}[v-\rho Tv]-P_{K(u)}[u-\rho Tu]\| \\ &\quad +P_{K(u)}[v-\rho Tv]-P_{K(u)}[u-\rho Tu]\| \\ &\leq \|v-u-\rho(Tv-Tu)\|+\eta\|v-u\|. \end{aligned} \tag{3.4}$$

Since the operator $\mathcal{T}$ is strongly monotonicity with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, so

$$\begin{aligned} \|u-v-\rho(Tu-Tv)\|^2 &= u-v\|^2-\rho\langle Tu-Tv, u-v\rangle \\ &\quad +\rho^2\|Tu-Tv\|^2, \\ &\leq (1-2\alpha\rho+\beta^2\rho^2)\|u-v\|^2. \end{aligned} \tag{3.5}$$

Combining (3.5) and (3.4), we have

$$\begin{aligned} \|\Phi(v)-\Phi(u)\| &\leq \{\sqrt{(1-2\alpha\rho+\beta^2\rho^2)}+\eta\}\|v-u\| \\ &= \theta\|u-v\|, \end{aligned} \tag{3.6}$$

where

$$\theta = \{\sqrt{(1-2\alpha\rho+\beta^2\rho^2)}+\eta\}. \tag{3.7}$$

From (3.3), it follows that $\theta < 1$. Thus it follows that the mapping $\Phi(\mu)$ defined by (3.2) is a contraction mapping and consequently, the mapping $\Phi(\mu)$ has a fixed point $\Phi(\mu) = \mu \in K(u)$ satisfying (2.2), the required result. □

We now use the alternative fixed point (3.1) to suggest the following iterative methods for solving the quasi variational inequality (2.2).

Algorithm 3.1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_n)}[u_n-\rho Tu_n], \quad n = 0, 1, 2, ... \tag{3.8}$$

which is known as the projection method and has been studied extensively.

Algorithm 3.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_{n+1})}[u_n - \rho T u_{n+1}], \quad n = 0, 1, 2, ... \tag{3.9}$$

which is known as the implicit projection method.

To implement the Algorithm (3.2), one uses the predictor-corrector technique. Consequently, considering the Algorithm (3.1) as a predictor and Algorithm (3.2) as corrector, we have the following method.

Algorithm 3.3. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n] \\ u_{n+1} &= P_{K(y_n)}[u_n - \rho T y_n], \quad n = 0, 1, 2, ... \end{aligned}$$

In a similar way, using (3.1), we can suggest the following implement method

Algorithm 3.4. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_{n+1})}[u_{n+1} - \rho T u_{n+1}], \quad n = 0, 1, 2, ... \tag{3.10}$$

which is known as the modified implicit projection method.

Using the predictor-corrector technique, Algorithm (3.4) is equivalent to the following iterative method.

Algorithm 3.5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n] \\ u_{n+1} &= P_{K(y_n)}[y_n - \rho T y_n], \quad n = 0, 1, 2, ... \end{aligned}$$

which is two-step predictor-corrector method for solving the quasi variational inequality (2.2).

Such type methods have been studied by Noor [40, 41] for solving variational inequalities.

We can rewrite the equation (3.1) as:

$$u = P_{K(u)}[\frac{u+u}{2} - \rho Tu]. \tag{3.11}$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 3.6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_{n+1})}[\frac{u_n + u_{n+1}}{2} - \rho Tu_{n+1}], \quad n = 0, 1, 2, ... \tag{3.12}$$

For the implementation and numerical performance of Algorithm 3.6, we can suggest the following two-step iterative method for solving quasi variational inequalities.

Algorithm 3.7. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho Tu_n] \\ u_{n+1} &= P_{K(y_n)}[\frac{y_n + u_n}{2} - \rho Ty_n], \quad \lambda \in [0, 1], \quad n = 0, 1, 2, ... \end{aligned}$$

From equation (3.1), we have

$$u = P_{K(u)}[u - \rho T(\frac{u+u}{2})]. \tag{3.13}$$

This fixed point formulation (3.13) is used to suggest the implicit method for solving the quasi variational inequalities as

Algorithm 3.8. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_{n+1})}[u_n - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.14}$$

which is another implicit method.

To implement this implicit method, one can use the predictor-corrector technique to rewrite Algorithm 3.8 as equivalent two-step iterative method:

Algorithm 3.9. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n], \\ u_{n+1} &= P_{K(y_n)}[u_n - \rho T(\frac{u_n + y_n}{2})], \quad n = 0, 1, 2, ... \end{aligned}$$

is known as the mid-point implicit method for solving quasi variational inequalities.

We again use the above fixed formulation to suggest the following iterative method.

Algorithm 3.10. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_{n+1})}[u_{n+1} - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.15}$$

which is another implicit method.

To implement this implicit method, one can use the predictor-corrector technique to rewrite Algorithm 3.9 as equivalent two-step iterative method:

Algorithm 3.11. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n], \\ u_{n+1} &= P_{K(y_n)}[y_n - \rho T(\frac{u_n + y_n}{2})], \quad n = 0, 1, 2, ... \end{aligned}$$

which appears to be new one.

It is obvious that Algorithm 3.3 and Algorithm 3.4 have been suggested using different variant of the fixed point formulations of the equation (3.1). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the quasi variational inequalities and related optimization problems, which is the main motivation of this paper.

One can rewrite the (3.1) as

$$u = P_{K(u)}[\frac{u + u}{2} - \rho T(\frac{u + u}{2})]. \tag{3.16}$$

This equivalent fixed point formulation enables to suggest the following method for solving the quasi variational inequalities.

Algorithm 3.12. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_{n+1})}[\frac{u_n + u_{n+1}}{2} - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.17}$$

which is an implicit method.

We would like to emphasize that Algorithm 3.12 is an implicit method. To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 3.4 as the predictor and Algorithm 3.5 as corrector. Thus, we obtain a new two-step method for solving the quasi variational inequalities.

Algorithm 3.13. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n] \\ u_{n+1} &= P_{K(y_n)}\left[\left(\frac{y_n + u_n}{2}\right) - \rho T\left(\frac{y_n + u_n}{2}\right)\right], \quad n = 0, 1, 2, ... \end{aligned}$$

which is two-step method and appears to be a new one.

For a parameter α, one can rewrite the (3.1) as

$$u = P_{K(u)}(1 - \alpha)u + \alpha u - \rho T u]. \tag{3.18}$$

This equivalent fixed point formulation enables to suggest the following method for solving the quasi variational inequalities.

Algorithm 3.14. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_n)}[(1 - \alpha)u_n + \alpha u_{n-1} - \rho T u_n], \quad n = 0, 1, 2, ...$$

which is an inertial implicit method.

It is noted that Algorithm 3.14 is equivalent to the following two-step method.

Algorithm 3.15. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\alpha)u_n + \alpha u_{n-1} \\ u_{n+1} &= P_{K(u_n)}[y_n - \rho T u_n], \quad n = 0, 1, 2, ... \end{aligned}$$

Algorithm 3.15 is known as the inertial projection method. Such type of methods are mainly due to Noor [41] and Noor et al. [54, 56].

Using this idea, one can suggest the following iterative methods for solving quasi variational inequalities.

Algorithm 3.16. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\alpha)u_n + \alpha u_{n-1} \\ u_{n+1} &= P_{K(u_n)}[y_n - \rho T y_n], \quad n = 0, 1, 2, ... \end{aligned}$$

Algorithm 3.17. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\alpha)u_n + \alpha u_{n-1} \\ u_{n+1} &= P_{K(y_n)}[y_n - \rho T y_n], \quad n = 0, 1, 2, ... \end{aligned}$$

In a similar way, we can suggest the following four-step inertial method for solving the quasi variational inequalities (2.2).

Algorithm 3.18. *For given $u_0, u_1 \in H$, compute u_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= u_n - \theta_n (u_n - u_{n-1}), \\ x_n &= (1-\gamma_n)u_n + \gamma_n P_{K(\omega_n)} [\omega_n - \rho T \omega_n] \}, \\ y_n &= (1-\beta_n)u_n + \beta_n P_{K(x_n)} [x_n - \rho T x_n] \}, \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n P_{K(y_n)} [y_n - \rho T y_n] \}, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0, 1], \quad \forall n \geq 1$.

Using the essentially the technique of Shehu et al [63], Jabeen et al [20] and Noor et al. [54], one can investigate the convergence analysis of these inertial projection method. For appropriate suitable choice of the parameters θ, γ and α, one can obtain one-step, two-step and three-step inertial methods for solving the quasi variational inequalities.

4 Wiener-Hopf Equations Technique

In this section, we discuss the technique of Wiener-Hopf equations associated with nonlinear quasi variational inequalities. It is worth mentioning that the Wiener-Hopf equations associated with variational inequalities were introduced and studied by Shi [62] and Robinson [61] independently using different techniques. For the applications of the Wiener-Hopf equations in developing numerical methods, sensitivity analysis, dynamical systems and other aspects, see [35, 39, 41, 42, 56] and the references therein. Noor [39] proved that the quasi variational inequalities are equivalent to the implicit Wiener-Hopf equations to study the sensitivity analysis.

We now consider the problem of solving the Wiener-Hopf equations related to the quasi variational inequalities.

Let T be an operator and $R_{K(u)} = I - P_{K(u)}$, where I is the identity operator and $P_{K(u)}$ is the projection operator. We consider the problem of finding $z \in H$ such that

$$TP_{K(u)}z + \rho^{-1}R_{K(u)}z = 0. \tag{4.1}$$

The equations of the type (4.1) are called the implicit Wiener-Hopf equations. It have been shown that the Wiener-Hopf equations play an important part in the developments of iterative methods, sensitivity analysis and other aspects of the variational inequalities.

Lemma 4.1. *[39] The element $u \in K(u)$ is a solution of the quasi variational* *inequality* (2.2)*, if and only if,* $z \in H$ *satisfies the resolvent equation* (4.1)*, where*

$$u = P_{K(u)}z, \tag{4.2}$$

$$z = u - \rho Tu, \tag{4.3}$$

where $\rho > 0$ *is a constant.*

From Lemma 4.1, it follows that the quasi variational inequalities (2.2) and the Wiener-Hopf equations (4.1) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the quasi variational inequalities and related optimization problems.

We use the Wiener-Hopf equations (4.1) to suggest some new iterative methods for solving the quasi variational inequalities. From (4.2) and (4.3),

$$\begin{aligned} z &= J_{K(u)}z - \rho TP_{K(u)}z \\ &= P_{K(u)}[u - \rho Tu] - \rho TP_{K(u)}[u - \rho Tu]. \end{aligned}$$

Thus, we have

$$u = \rho Tu - +\big[P_{K(u)}[u - \rho Tu] - \rho TP_{K(u)}[u - \rho Tu].$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} u &= (1 - \alpha_n)u + \alpha_n P_{K(u)}\{P_{K(u)}[u - \rho Tu] + \rho Tu \\ &\quad -\rho TP_{K(u)}[u - \rho Tu]\} \\ &= (1 - \alpha_n)u + \alpha_n P_{K(u)}\{y - \rho Ty + \rho Tu\}, \end{aligned} \tag{4.4}$$

where

$$y = P_{K(u)}[u - \rho Tu]. \tag{4.5}$$

Using (4.4) and (4.5), we can suggest the following new predictor-corrector method for solving the strongly nonlinear quasi variational inequalities.

Algorithm 4.1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n] \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n P_{K(y_n)}\Big\{y_n - \rho T y_n + \rho T u_n\Big\}. \end{aligned}$$

If $\alpha_n = 1$, then Algorithm 4.1 reduces to

Algorithm 4.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n] \\ u_{n+1} &= P_{K(y_n)}[y_n - \rho T y_n + \rho T u_n], \end{aligned}$$

which appears to be a new one.

In a similar way. we can suggest and analyse the predictor-corrector method for solving the nonlinear quasi variational inequalities (2.2), which only involve only one projection.

Algorithm 4.3. *For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= u_n - \xi(u_n - u_{n-1}) \\ u_{n+1} &= P_{K(y_n)}[y_n - \rho T y_n + \rho T u_n]. \end{aligned}$$

One can study the convergence of the Algorithm 4.3 using the technique of Jabeen et al [14].

Remark 4.1. We have only given some glimpse of the technique of the Wiener-Hopf equations for solving the quasi variational inequalities. One can explore the applications of the Wiener-Hopf equations in developing efficient numerical methods for variational inequalities and related nonlinear optimization problems.

5 Dynamical Systems Technique

Dupuis and Nagurney [14] used the alternative fixed point formulation to consider the projected dynamical systems associated with variational inequalities. This dynamical system is in fact a first order initial value problem. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problem. Hence, equilibrium and nonlinear problems arising in various branches in pure and applied sciences, which can be formulated in the setting of the variational inequalities, can now be studied in the more general setting of dynamical systems. It has been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. In recent years, much attention has been given to study the globally asymptotic stability of these projected dynamical systems. In this section, we consider the projected dynamical systems associated with the quasi variational inequalities to propose some iterative methods. We investigate the convergence analysis of these new methods involving only the monotonicity of the operator.

We now define the residue vector $R(u)$ by the relation

$$R(u) = u - P_{K(u)}[u - \rho Tu]. \tag{5.1}$$

Invoking Lemma 3.1, one can easily conclude that $u \in H$ is a solution of (2.2), if and only if, $u \in H$ is a zero of the equation

$$R(u) = 0. \tag{5.2}$$

We now consider a resolvent dynamical system associated with the nonlinear quasi variational inequalities. Using the equivalent formulation (3.1), we suggest a class of project dynamical systems as

$$\frac{du}{dt} = \lambda\{P_{K(u)}[u - \rho Tu] - u\}, \quad u(t_0) = \alpha, \tag{5.3}$$

where λ is a parameter. The system of type (5.3) is called the project dynamical system associated with the quasi variational inequalities (2.2). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in H. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (5.3) can be studied. These project dynamical systems are associated with the quasi variational inequalities (2.2).

We use the project dynamical system (5.3) to suggest some iterative for solving the quasi variational inequalities (2.2). These methods can be viewed in the sense of Koperlevich [25] and Noor [40, 41] involving the double resolvent operator.

For simplicity, we take $\lambda = 1$. Thus the dynamical system(5.3) becomes

$$\frac{du}{dt} + u = P_{K(u)}[u - \rho Tu], \quad u(t_0) = \alpha. \tag{5.4}$$

We construct the implicit iterative method using the forward difference scheme. Discretizing (5.4), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = P_{K(u_n)}[u_n - \rho T u_{n+1}], \tag{5.5}$$

where h is the step size. Now, we can suggest the following implicit iterative method for solving the quasi variational inequality (2.2).

Algorithm 5.1. *For a given u_0,, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_{n+1})}[u_n - \rho T u_{n+1} - \frac{u_{n+1} - u_n}{h}],$$

which is an implicit method.

Algorithm 5.1 is equivalent to the following two-step method.

Algorithm 5.2. *For a given u_0,, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n] \\ u_{n+1} &= P_{K(y_n)}[u_n - \rho T y_n - \frac{y_n - u_n}{h}], \end{aligned}$$

We now suggest an other implicit iterative method for solving (2.2).

Discretizing (5), we have

$$\frac{u_{n+1}-u_n}{h}+u_{n+1}=P_{K(u_{n+1})}[u_{n+1}-\rho Tu_{n+1}], \tag{5.6}$$

where h is the step size.

This formulation enable us to suggest the following iterative method.

Algorithm 5.3. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n-\rho Tu_n] \\ u_{n+1} &= P_{K(y_n)}\left[y_n-\rho Ty_n-\frac{y_n-u_n}{h}\right], \quad n=0,1,2,\ldots. \end{aligned}$$

Again using the project dynamical systems, we can suggested some iterative methods for solving the quasi variational inequalities and related optimization problems.

Algorithm 5.4. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1}=P_{K(u_{n+1})}\left[\frac{(h+1)u_n-u_{n+1}}{h}-\rho Tu_n\right], \quad n=0,1,2,\ldots.$$

which can be written in the equivalent form as

Algorithm 5.5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n-\rho Tu_n] \\ u_{n+1} &= P_{K(y_n)}\left[\frac{(h+1)u_n-y_n}{h}-\rho Tu_n\right], \quad n=0,1,2,\ldots. \end{aligned}$$

Now, we can suggest the following implicit iterative method for solving the quasi variational inequality (2.2).

Algorithm 5.6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_{K(u_n)}[u_n - \rho T u_n] \\ u_{n+1} &= P_{K(y_n)}\left[\frac{(h+1)u_n - y_n}{h} - \rho T u_n\right], \quad n = 0, 1, 2, \ldots. \end{aligned}$$

We construct the another implicit iterative method using the forward difference scheme.

Discretizing (5.4), we have

$$\frac{u_{n+1} - u_n}{h} + u_n = P_{K(u_{n+1})}[u_n - \rho T u_{n+1}],$$

where h is the step size.

In particular, for $h = 1$, we can suggest the following implicit iterative method for solving the nonlinear quasi variational inequality (2.2).

Algorithm 5.7. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K(u_{n+1})}[u_n - \rho T u_{n+1}], \quad n = 0, 1, 2, 3, \ldots.$$

Algorithm 5.7 is an implicit iterative method in the sense of Koperlevich [25], which can be written the equivalent form:

Algorithm 5.8. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1} + u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0. \tag{5.7}$$

We study the convergence analysis of Algorithm 5.8, which is the main motivation of our next result.

Theorem 5.1. *Let $u \in K(u)$ be a solution of (2.2) and u_n be the approximate solution obtained from (5.7). If the operator T is monotone, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{5.8}$$

Proof. Let $u \in K(u)$ be a solution of (2.2). Then

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K(u),$$

implies that

$$\langle Tv, v - u\rangle \geq 0, \quad \forall v \in K(u), \tag{5.9}$$

since T is a monotone operator.

Taking $v = u_{n+1}$ in (5.9), we have

$$\langle Tu_{n+1}, u_{n+1} - u\rangle \geq 0. \tag{5.10}$$

Now taking $v = \bar{u}$ in (5.7), we obtain

$$\langle \rho Tu_{n+1} + u_{n+1} - u_n, u - u_{n+1}\rangle \geq 0. \tag{5.11}$$

From (5.10) and (5.11), we have

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq 0, \tag{5.12}$$

from which, using the inequality $2\langle a, b\rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in H$, we obtain

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2,$$

which is the required result (5.8). □

Theorem 5.2. *Let u_{n+1} be the approximate solution obtained from Algorithm 5.8 and $u \in K(u)$ be a solution of (2.2). Assume that*
(i). for any sequence $\{u_n\}$ with $u_n \to u$, and for any $v \in K(u)$, there exists a sequence $\{v_n\}$ such that $v_n \in K(u_n)$and $v_n \to v$, as $n \to \infty$
(ii). for all sequences $\{u_n\}$ and $\{v_n\}$ with $v_n \in K(u_n)$, if $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$, then $v \in K(u)$.

If the operator T is a monotone operator, then

$$\lim_{n \longrightarrow \infty} u_n = u.$$

Proof. Let $u \in K(u)$ be a solution of (2.2). Then, from (5.8), it follows that that the sequence $\{\|u-u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (5.8), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n \longrightarrow \infty} u_n = u. \tag{5.13}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (5.7) and taking the limit as $n_j \longrightarrow \infty$ and using (5.13), we have

$$\langle T\hat{u}, v - \hat{u} \rangle \geq 0, \quad \forall v \in K(u),$$

which implies that $\hat{u}$ solves the quasi inequality problem (2.2) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n \longrightarrow \infty} u_n = \hat{u}$. □

We now suggest some inertial type iterative methods for solving quasi variational inequalities (2.2).

From (5.4), we have

$$\frac{du}{dt} + u = P_{K((1-\alpha)u+\alpha u)}[(1-\alpha)u + \alpha u - \rho T((1-\alpha)u + \alpha u)], \tag{5.14}$$

where $\alpha \in [0, 1]$ is a constant.

Discretization (5.14 and taking $h = 1$, we have

$$u_{n+1} = P_{K((1-\alpha)u_n+\alpha u_{n-1})}\big[(1-\alpha)u_n + \alpha u_{n-1} - \rho T((1-\alpha)u_n + \alpha u_{n-1})\big], \tag{5.15}$$

which is an inertial type iterative method for solving the quasi variational inequality (2.2). Using the predictor-corrector techniques, we can suggest the following iterative method.

Algorithm 5.9. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= (1-\alpha)u_n + \alpha u_{n-1} \\ u_{n+1} &= P_{K(y_n}\big[y_n - \rho T(y_n)\big], \end{aligned}$$

which is known as the inertial two-step iterative method.

Remark 5.1. For appropriate and suitable choice of the operator T, convex-valued set, parameter α and the spaces, one can propose a wide class of implicit, explicit and inertial type methods for solving quasi variational inequalities and related optimization problems. Using the techniques and ideas of Noor et al. [54, 56, 57], one can discuss the convergence analysis of the proposed method.

6 Sensitivity Analysis

Quasi variational inequalities are being used as a mathematical programming tool in modeling various equilibria in economics, operations research, optimization, regional and transportation science. The behavior of such equilibrium solutions as a result of changes in the problem data is always of concern. In recent years, much attention has been devoted to developing general methods for the sensitivity analysis of variational inequalities. From the mathematical and engineering points of view, sensitivity properties of a variational inequality problem can provide new insight concerning the problem being studied and can stimulate ideas for problem solving. The techniques suggested so far vary with the problem setting being studied, see Dafermas [13] and Noor [39, 41]. To study the sensitivity analysis of quasi variational inequalities, first of all we show that parametric quasi variational inequalities are equivalent to the parametric Wiener-Hopf equations, essentially we use the ideas of Shi [62] and Noor [39]. This equivalence is used to study the sensitivity analysis of quasi-variational inequalities.

Related to the quasi variational inequality (2.2), we consider the problem of finding $z, u \in H$, such that

$$TP_{K(u)}z + \rho^{-1}R_{K(u)}z = 0, \tag{6.1}$$

where $\rho > 0$ is a constant and $R_{K(u)} = I - J_{K(u)}$. Here I is the identity operator. The equation of the type (6.1) is called the implicit Wiener-Hopf equations, which was introduced and studied by Noor [39].

If $K(u) = K$ the convex set, then the implicit Wiener-Hopf equation(6.1) becomes

$$TP_Kz + \rho^{-1}R_Kz = 0, \tag{6.2}$$

which are known as the Wiener-Hopf (normal maps) equations, introduced by Shi [62] and Robinson [61] independently. Using essentially the projection technique, Shi [62] and Robinson [61] have established the equivalence between the variational inequalities (2.2) and the Wiener-Hopf equations (6.2). For the generalization and the extensions of the Wiener-Hopf equations and their applications, see Noor [41–43], Noor et al. [55–57] and references therein.

We now consider the parametric versions of the problem (2.2) and (6.1). To formulate this problem, let M be an open subset of H in which the parameter λ takes values, and assume that $\{K_\lambda : \lambda \in M\}$ is a family of closed convex subsets of H. Let $T(u, \lambda)$ be given operator defined on $H \times M$ and taking value in H. From now onward, we denote $T_\lambda(.) \equiv T(., \lambda)$ unless otherwise specified. The parametric quasi variational problem is to find $u \in K_\lambda(u)$ such that

$$\langle T_\lambda u, v - u\rangle \geq 0, \quad \forall v \in K_\lambda(u) \tag{6.3}$$

We also assume that for some $\overline{\lambda} \in M$, the problem (6.3) has a unique solution $\overline{u}$.

Related to the parametric quasi variational inequality (6.3), we consider the parametric Wiener-Hopf equations. We consider the problem of finding $z, u \in H$, such that

$$T_\lambda P_{K_\lambda(u)}z + \rho^{-1}R_{K_\lambda(u)}z = 0, \tag{6.4}$$

where $\rho > 0$ is a constant and $P_{K_\lambda} z$ and $R_{K_\lambda} z$ are defined on the set of (z, λ) with $\lambda \in M$ and takes values in H. The equations of the type (6.4) are called the parametric implicit Wiener-Hopf equations.

We now establish the equivalence between the problem (6.3) and (6.4), which is the main motivation of our next result.

Lemma 6.1. *The parametric quasi variational inequality (6.3) has a solution $u \in K_\lambda(u)$, if and only if, the parametric Wiener-Hopf equations (6.4) have a solution $z, u \in H$, where*

$$u = P_{K_{\lambda(u)}} z \tag{6.5}$$

and

$$z = u - \rho T_\lambda(u). \tag{6.6}$$

From Lemma 6.1, we see that the parametric quasi variational inclusion (6.3) and the parametric Wiener-Hopf equations (6.4) are equivalent. We use this equivalence to study the sensitivity analysis of the quasi variational inequalities. We assume that for some $\overline{\lambda} \in M$, problem (6.4) has a solution $\overline{z}$ and X is a closure of a ball in H centered at $\overline{z}$. We want to investigate those conditions under which, for each λ in a neighbourhood of $\overline{\lambda}$, problem (6.4) has a unique solution $z(\lambda)$ near $\overline{z}$ and the function $z(\lambda)$ is continuous (Lipschitz continuous) and differentiable.

Definition 6.1. Let T_λ be an operator on $X \times M$. Then for all $\lambda \in M$, $u, v \in X$, the operator T_λ is said to be :

(a) *Locally strongly monotone,* if there exists a constant $\alpha > 0$ such that

$$\langle T_\lambda(u) - T_\lambda(v), u - v \rangle \geq \alpha \|u - v\|^2.$$

(b) *Locally Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|T_\lambda(u) - T_\lambda(v)\| \leq \beta \|u - v\|.$$

We also need the following condition for the operator $P_{K_\lambda(u)}$.

We consider the case, when the solutions of the parametric Wiener-Hofp equations (6.4) lie in the interior of X. Following the ideas of Dafermos [13] and Noor [39, 41], we consider the map

$$F_\lambda(z) = P_{K_\lambda(u)}z - \rho T_\lambda P_{K_\lambda(u)}z, \ \forall (z,\lambda) \in X \times M$$

$$= u - \rho T_\lambda(u), \tag{6.7}$$

where

$$u = P_{K_{\lambda(u)}}z. \tag{6.8}$$

We have to show that the map $F_\lambda(z)$ has a fixed point, which is a solution of the Wiener-Hopf equations (6.4). First of all, we prove that the map $F_\lambda(z)$, defined by (6.7), is a contraction map with respect to z uniformly in $\lambda \in M$.

Lemma 6.2. *Let T be a locally strongly monotone with constant $\alpha \geq 0$ and locally Lipschitz continuous with constant $\beta \geq 0$. If Assumption 2.1 holds ,then $\forall \quad z_1, z_2 \in X$ and $\lambda \in M$, we have*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \theta \|z_1 - z_2\|,$$

where

$$\theta = (\sqrt{1 - 2\rho\alpha + \rho^2\beta^2})/(1-\gamma) < 1$$

for

$$|\rho - \alpha/\beta^2| < \sqrt{\alpha^2 - \beta^2\gamma(2-\gamma)}/\beta^2 \tag{6.9}$$

$$\alpha > \beta\sqrt{\gamma(2-\gamma)}, \gamma < 1. \tag{6.10}$$

Proof. $\forall z_1, z_2 \in X$, $\lambda \in M$, we have, from (6.7),

$$\begin{aligned}\|F_\lambda(z_1) - F_\lambda(z_2)\|^2 &= \|u_1 - u_2 - \rho(T_\lambda(u_1) - T_\lambda(u_2)\|^2 \\ &= \|u_1 - u_2\|^2 - 2\rho\langle T_\lambda(u_1) - T_\lambda(u_2), u_1 - u_2\rangle \\ &\quad +\rho^2\|T_\lambda(u_1) - T_\lambda(u_2)\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2,\end{aligned}$$

which implies that

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq (\sqrt{1 - 2\rho\alpha + \rho^2\beta^2})\|u_1 - u_2\|. \tag{6.11}$$

Now from (6.5) and Assumption 2.1, we have

$$\begin{aligned}\|u_1 - u_2\| &= \|P_{K_\lambda(u_1)}z_1 - P_{K_\lambda(u_2)}z_2\| \\ &\leq \|P_{K_\lambda(u_1)}z_1 - P_{K_\lambda(u_2)}z_1\| + \|P_{K_\lambda(u_2)}z_1 - P_{K_\lambda(u_2)}z_2\| \\ &\leq \gamma\|u_1 - u_2\| + \|z_1 - z_2\|,\end{aligned}$$

which implies that

$$\|u_1 - u_2\| \leq (1/1 - \gamma)\|z_1 - z_2\|. \tag{6.12}$$

Combining (6.11) and (6.12), we have

$$\begin{aligned}\|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq [(\sqrt{1 - 2\rho\alpha + \rho^2\beta^2})/(1 - \gamma)]\|z_1 - z_2\| \\ &= \theta\|z_1 - z_2\|,\end{aligned}$$

where $\theta = (\sqrt{1 - 2\rho\alpha + \rho^2\beta^2})/(1 - \gamma) < 1$.

From (6.9) and (6.10), it follows that $\theta < 1$ and consequently the map $F_\lambda(z)$ defined by (6.7) is a contraction map and has a fixed point $z(\lambda)$, which is the solution of the Wiener-Hopf equation (6.4). □

Remark 6.1. From Lemma 6.2, we see that the map $F_\lambda(z)$ defined by (6.7) has a unique fixed point $z(\lambda)$, that is,

$$z(\lambda) = F_\lambda(z).$$

Also, by assumption, the function $\overline{z}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric Wiener-Hopf equations (6.4). Again using Lemma 6.2, we see that $\overline{z}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(z)$ and it is also a fixed point of $F_{\overline{\lambda}}(z)$. consequently, we conclude that

$$z(\overline{\lambda}) = \overline{z} = F_{\overline{\lambda}}(z(\overline{\lambda})). \tag{6.13}$$

Using Lemma 6.2, we prove the continuity of the solution $z(\lambda)$ of the parametric Wiener-Hopf equations (6.4), which is the main motivation of our next result.

Lemma 6.3. *If the operator T_λ is locally strongly monotone Lipischitz continuous and the map $\lambda \to J_{A_\lambda(u)}z$ is continuous (or Lipschitz continuous), then the function $z(\lambda)$ satisfying (6.13) is continuous (or Lipschitz continuous) at $\lambda = \overline{\lambda}$.*

Proof. $\forall \lambda \in M$, invoking Lemma 6.2 and the triangle inequality, we have

$$\begin{aligned} \|z(\lambda) - z(\overline{\lambda}\| &= \|F_\lambda(z(\lambda)) - F_{\overline{\lambda}}(z(\overline{\lambda}))\| \\ &\leq \|F_\lambda(z(\lambda)) - F_\lambda(z(\overline{\lambda}))\| + \|F_\lambda(z(\overline{\lambda})) - F_{\overline{\lambda}}(z(\overline{\lambda}))\| \\ &\leq \theta\|z(\lambda) - z(\overline{\lambda})\| + \|F_\lambda(z(\overline{\lambda})) - F_{\overline{\lambda}}(z(\overline{\lambda}))\|. \end{aligned} \tag{6.14}$$

From (6.7) and the fact that the operator T_λ is Lipschitz continuous, we have

$$\begin{aligned} \|F_\lambda(z(\overline{\lambda})) - F_{\overline{\lambda}}(z(\overline{\lambda}))\| &= \|u(\overline{\lambda}) - u(\overline{\lambda}) - \rho(T_\lambda(u(\overline{\lambda})) - T_{\overline{\lambda}}(u(\overline{\lambda}))\| \\ &\leq \rho\|T_\lambda(u(\overline{\lambda})) - T_{\overline{\lambda}}(u(\overline{\lambda}))\| \\ &\leq \rho\mu\|\lambda - \overline{\lambda}\|, \end{aligned} \tag{6.15}$$

where $\mu > 0$ is a Lipschitz continuity constant of T_λ.

Combining (6.14) and (6.15), we obtain

$$\|z(\lambda) - z(\overline{\lambda}\| \leq \frac{\rho\mu}{1-\theta}\|\lambda - \overline{\lambda}\|, \quad \text{for all} \quad \lambda, \overline{\lambda} \in X,$$

for which, the required result follows. □

Now using the technique of Dafermos [13], we can prove the following result.

Lemma 6.4. *If the assumptions of Lemma 6.3 hold, then there exists a neighbourhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, $z((\lambda)$ is the unique solution of the parametric Wiener-Hopf equations (6.4) in the interior of X.*

Theorem 6.1. *Let $\overline{u}$ be the solution of the parametric quasi variational inclusion (6.3) and $\overline{z}$ be the solution of the parametric Wiener-Hopf equations (6.4) for $\lambda = \overline{\lambda}$. Let $T_\lambda(u)$ be the locally strongly monotone Lipschitz continuous operator for all $u, v \in X$. If the the map $\lambda \rightarrow J_{A_\lambda(u)}(z)$ is continuous (or Lipschitz continuous) at $\lambda = \overline{\lambda}$, then there exists a neighbourhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric Wiener-Hopf equations (6.4) have a unique solution $z(\lambda)$ in the interior of $X, z(\overline{\lambda}) = \overline{z}$ and $z(\lambda)$ is continuous (or Lipschitz continuous) at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from Lemmas 6.2, 6.3, 6.4 and Remark 6.1. □

Remark 6.2. Since the quasi variational inclusions include the classical variational inequalities, and complementarity problems as special cases, the technique developed in this paper can be used to study the sensitivity analysis of these problems. The fixed formulation of the quasi variational inclusions allows us to study the Holder and Lipschitz continuity of the solution of the parametric problems essentially. In fact, our results represents a refinement and significant improvement of previous known results of Dafermos [13] and Noor [39] and others in this field. It is worth mentioning that the Wiener-Hopf equations technique does not require the differentiability of the given data.

7 Merit Functions

In recent years, much attention has been given to reformulate the variational inequality as an optimization problem. A function which can constitute an equivalent optimization problem is called a merit (gap) function. Merit functions turn out to be very useful in designing new globally convergent algorithms and in analyzing the rate of convergence of some iterative methods. Various merit (gap) functions for variational inequalities and complementarity problems have been suggested and proposed by many authors. See Noor [41, 44, 56, 60] and the references therein. Error bounds are functions which provide a measure of the distance between a solution set and an arbitrary point. Therefore, error bounds play an important role in the analysis of global or local convergence analysis of algorithms for solving variational inequalities. To the best of our knowledge, very few merit functions have been considered for quasi variational inequalities.

In this section, we construct three merit functions for the quasi variational inequalities using the equivalence between the fixed-point and the quasi variational inequalities coupled with the auxiliary principle technique. These results are mainly due to Noor [44]. We also obtain error bounds for the solutions of the quasi variational inequalities under some weaker conditions. Proofs of our results are simple and straightforward as compared with other methods. Since the quasi variational inequalities include variational inequalities and the implicit (quasi) complementarity problems as special cases, our results continue to hold for these problems. In this respect, our results can be considered as refinement of the previously known results for variational inequalities and related optimization problems.

From Lemma 3.1, we conclude that the quasi variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation plays an important part in suggesting and analyzing several iterative methods for solving variational inequalities. This fixed-point formulation has been used to suggest and analyze several iterative methods for solving the quasi variational inequalities (2.2).

We now consider the residue vector

$$R_\rho(u) \equiv R(u) := u - P_{K(u)}[u - \rho Tu]. \tag{7.1}$$

It is clear from Lemma 3.1 that (2.2) has a solution $u \in K(u)$, if and only if,

$u \in K(u)$ is a root of the equation

$$R(u) = 0. \tag{7.2}$$

It is known that the normal residue vector $\|R(u)\|$ is a merit function for the quasi variational inequalities (2.2). We use the relation (7.2) to derive the error bound for the solution of (2.2).

Theorem 7.1. *Let $\overline{u} \in K(u)$ be a solution of (2.2) and let Assumption 2.1 hold. Let the operator T be both strongly monotone and Lipschitz continuous with constants $\alpha > 0$ and $\beta > 0$ respectively. If $\overline{u} \in K(u)$ satisfies*

$$\langle T\overline{u}, v - \overline{u}\rangle \geq 0, \forall v \in K(\overline{u}), \tag{7.3}$$

then

$$k_1\|R(u)\| \leq \|u - \overline{u}\| \leq k_2\|R(u)\|, \quad \forall u \in K(u), \tag{7.4}$$

where k_1, k_2 are generic constants.

Proof. Let $\overline{u} \in K(u)$ be solution of (2.2). Then, taking $v = P_{K(u)}[u - \rho Tu]$ in (2.2), we have

$$\langle T\overline{u}, P_{K(u)}[u - \rho Tu] - \overline{u}\rangle \geq 0. \tag{7.5}$$

Letting $u = P_{K(u)}[u - \rho Tu], \quad z = u - \rho Tu$ and $v = \overline{u}$ in (2.10), we have

$$\langle \rho Tu + P_{K(u)}[u - \rho Tu] - u, \overline{u} - P_{K(u)}[u - \rho Tu]\rangle \geq 0. \tag{7.6}$$

Adding (7.5) and (7.6), we obtain

$$\langle T\overline{u} - Tu + (1/\rho)(u - P_{K(u)}[u - \rho Tu]), P_{K(u)}[u - \rho Tu] - \overline{u}\rangle \geq 0. \tag{7.7}$$

Since T is a strongly monotone, there exists a constant $\alpha > 0$, such that

$$\begin{aligned}
\alpha\|\overline{u}-u\|^2 &\leq \langle T\overline{u}-Tu, \overline{u}-u\rangle \\
&= \langle T\overline{u}-Tu, \overline{u}-P_{K(u)}[u-\rho Tu]\rangle \\
&\quad +\langle T\overline{u}-Tu, P_{K(u)}[u-\rho Tu]-u\rangle \\
&\leq (1/\rho)\langle u-P_{K(u)}[u-\rho Tu], P_{K(u)}[u-\rho Tu]-u+u-\overline{u}\rangle \\
&\quad +\langle T\overline{u}-Tu, P_{K(u)}[u-\rho Tu]-u\rangle \\
&\leq -(1/\rho)\|R(u)\|^2+(1/\rho)\|R(u)\|\|u-\overline{u}\| \\
&\quad +\|T\overline{u}-Tu\|\|R(u)\| \\
&\leq (1/\rho)(1+\beta\rho)\|R(u)\|\|\overline{u}-u\|,
\end{aligned}$$

which implies that

$$\|\overline{u}-u\| \leq k_2\|R(u)\|, \tag{7.8}$$

the right-hand inequality in (13) with $k_2 = (1/\alpha\rho)(1+\rho\beta)$.

Now from Assumption 2.1 and Lipschitz continuity of T, we have

$$\begin{aligned}
\|R(u)\| &= \|u-P_{K(u)}[u-\rho Tu]\| \\
&= \|u-\overline{u}+P_{K(\overline{u})}[\overline{u}-\rho T\overline{u}]-P_{K(u)}[u-\rho Tu]\| \\
&\leq \|u-\overline{u}\|+\|P_{K(\overline{u})}[\overline{u}-\rho T\overline{u}]-P_{K(\overline{u})}[u-\rho Tu]\| \\
&\quad +\|P_{K(\overline{u})}[u-\rho Tu]-P_{K(u)}[u-\rho Tu]\| \\
&\leq \|u-\overline{u}\|+\nu\|u-\overline{u}\|+\|u-\overline{u}+\rho(Tu-T\overline{u})\| \\
&\leq \{2+\nu+\rho\beta\}\|u-\overline{u}\| = k_1\|u-\overline{u}\|,
\end{aligned}$$

from which we have

$$(1/k_1)\|R(u)\| \leq \|u-\overline{u}\|, \tag{7.9}$$

the left-most inequality in (7.4) with $k_1 = (2+\nu+\rho\beta)$.

Combining (7.8) and (7.9), we obtain the required (7.4). □

Letting $u = 0$ in (7.4), we have

$$(1/k_1)\|R(0)\| \leq \|\overline{u}\| \leq k_2\|R(0)\|. \tag{7.10}$$

Combining (7.4) and (7.10), we obtain a relative error bound for any point $u \in K(u)$.

Theorem 7.2. *Assume that all the assumptions of Theorem 7.1 hold. If* $0 \neq \overline{u} \in K(u)$ *is a solution of (2.2), then*

$$c_1\|R(u)\|/\|R(0)\| \leq \|u - \overline{u}\|/\|\overline{u}\| \leq c_2\|R(u)\|/\|R(0)\|.$$

Note that the normal residue vector (merit function) $R(u)$ defined by (7.1) is nondifferentiable. To overcome the nondifferentiability, which is a serious drawback of the residue merit function, we consider another merit function associated with problem (2.2). This merit function can be viewed as a regularized merit function. We consider the function

$$\begin{aligned} M_\rho(u) &= \langle Tu, u - P_{K(u)}[u - \rho Tu]\rangle \\ &\quad -(1/2\rho)\|u - P_{K(u)}[u - \rho Tu]\|^2, \quad \forall u \in K(u). \end{aligned} \tag{7.11}$$

from which it follows that $M_\rho(u) \geq 0, \quad \forall u \in K(u)$.

We now show that the function $M_\rho(u)$ defined by (7.11) is a merit function and this is the main motivation of our next result.

Theorem 7.3. $\forall u \in K(u)$, *we have*

$$M_\rho(u) \geq (1/2\rho)\|R(u)\|^2. \tag{7.12}$$

In particular, we have $M_\rho(u) = 0$, if and only if, $u \in K(u)$ is a solution of (2.2).

Proof. Setting $v = u$, $u = P_{K(u)}[u - \rho Tu]$ and $z = u - \rho Tu$ in (2.10), we have

$$\langle Tu - (1/\rho)(u - P_{K(u)}[u - \rho Tu]), u - P_{K(u)}[u - \rho Tu]\rangle \geq 0.$$

which implies that

$$\langle Tu, R(u)\rangle \geq (1/\rho)\|R(u)\|^2. \tag{7.13}$$

Combining (7.11) and (7.13), we have

$$\begin{aligned} M_\rho(u) &= \langle Tu, R(u)\rangle - (1/2\rho)\|R(u)\|^2 \\ &\geq (1/\rho)\|R(u)\|^2 - (1/2\rho)\|R(u)\|^2 \\ &= (1/2\rho)\|R(u)\|^2, \end{aligned}$$

the required result (7.12). Clearly we have $M_\rho(u) \geq 0, \quad \forall u \in K(u)$.

Now if $M_\rho(u) = 0$, then clearly $R(u) = 0$. Hence by Lemma 3.1, we see that $u \in K(u)$ is a solution of (2.2). Conversely, if $u \in K(u)$ is a solution of (2.2), then $u = P_{K(u)}[u - \rho Tu]$ by Lemma 3.1. Consequently, from (7.12), we see that $M_\rho(u) = 0$, the required result. □

From Theorem 7.3, we see that the function $M_\rho(u)$ defined by (7.11) is a merit function for the quasi variational inequalities (2.2). We now derive the error bounds without using the Lipschitz continuity of the operator T.

Theorem 7.4. *Let T be a strongly monotone with a constant $\alpha > 0$. If $\overline{u} \in K(u)$ is a solution of (2.2), then*

$$\|u - \overline{u}\|^2 \leq (2\rho)/(2\alpha\rho - 1)M_\rho(u), \quad \forall u \in H. \tag{7.14}$$

Proof. From (7.11), we have

$$\begin{aligned} M_\rho(u) &\geq \langle Tu, u - \overline{u}\rangle - (1/2\rho)\|u - \overline{u}\|^2 \\ &= \langle Tu - T\overline{u} + \overline{u}, u - \overline{u}\rangle - -(1/2\rho)\|u - \overline{u}\|^2 \\ &\geq \langle T\overline{u}, u - \overline{u}\rangle + \alpha\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2, \end{aligned} \tag{7.15}$$

where we have used the fact that the operator T is strongly monotone with a constant $\alpha > 0$.

Let $\overline{u} \in K(u)$ be solution of (2.2). Then

$$\langle T\overline{u}, u - \overline{u}\rangle \geq 0. \tag{7.16}$$

Taking $v = u$ in (7.16), we have

$$\langle T\overline{u}, u - \overline{u}\rangle \geq 0. \tag{7.17}$$

From (7.11) and (7.17), we have

$$\begin{aligned} M_\rho(u) &\geq \alpha\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &= (\alpha - 1/2\rho)\|u - \overline{u}\|^2, \end{aligned}$$

from which the result (7.14) follows. □

We consider another merit function associated with quasi variational inequalities (2.2), which can be viewed as a difference of two regularized merit functions. Such type of the merit functions functions were introduced and studied by many authors for solving variational inequalities and complementarity problems. Here we define the D-merit function by a formal difference of the regularized merit function defined by (7.11). To this end, we consider the following function

$$\begin{aligned} D_{\rho,\mu}(u) &= \langle Tu, P_{K(u)}[u - \mu Tu] - P_{K(u)}[u - \rho Tu]\rangle \\ &\quad +(1/2\mu)\|u - P_{K(u)}[u - \mu Tu]\|^2 - (1/2\rho)\|u - P_{K(u)}[u - \rho Tu]\|^2 \\ &= \langle Tu, R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2, \quad u \in K(u), \quad \rho > \mu > 0. \end{aligned} \tag{7.18}$$

It is clear that the $D_{\rho,\mu}(u)$ is everywhere finite. We now show that the function $D_{\rho,\mu}(u)$ defined by (26) is indeed a merit function for the mixed quasi variational inequalities (2.2) and this is the motivation of our next result.

Theorem 7.5. *$\forall u \in K(u), \rho > \mu > 0$, we have*

$$(\rho - \mu)\|R_\rho(u)\|^2 \geq 2\rho\mu D_{\rho,\mu}(u) \geq (\rho - \mu)\|R_\mu(u)\|^2. \quad (7.19)$$

In particular, $D_{\rho,\mu}(u) = 0$, iff $u \in K(u)$ solves problem (2.2).

Proof. Taking $v = P_{K(u)}[u - \mu Tu], u = P_{K(u)}[u - \rho Tu]$ and $z = u - \rho Tu$ in (2.10), we have

$$\langle P_{K(u)}[u - \rho Tu] - u + \rho Tu, P_{K(u)}[u - \mu Tu] - P_{K(u)}[u - \rho Tu]\rangle \geq 0$$

which implies that

$$\langle Tu, R_\rho(u) - R_\mu(u)\rangle \geq (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle. \quad (7.20)$$

From (7.18) and (7.20), we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2 \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad -(1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 - (1/\rho)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\geq 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2, \end{aligned} \quad (7.21)$$

which implies the right-most inequality in (7.19).

In a similar way, by taking $u = P_{K(u)}[u - \mu Tu], z = u - \mu Tu$ and $v = P_{K(u)}[u - \mu Tu]$ in (2.10), we have

$$\langle P_{K(u)}[u - \mu Tu] - u + \mu Tu, P_{K(u)}[u - \mu Tu] - P_{K(u)}[u - \mu Tu]\rangle \geq 0,$$

which implies that

$$\langle Tu, R_\rho(u) - R_\mu(u)\rangle \leq (1/\mu)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle. \quad (7.22)$$

Consequently, from (7.22) and (7.21), we obtain

$$\begin{aligned} D_{\rho,\mu}(u) &\leq (1/\mu)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2 \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad -(1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 - (1/\rho)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= 1/2(1/\mu - 1/\rho)\|R_\rho(u)\|^2 - (1/2\mu)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\leq 1/2(1/\mu - 1/\rho)\|R_\rho(u)\|^2, \end{aligned} \tag{7.23}$$

which implies the left-most inequality in (7.19).

Combining (7.21) and (7.23), we obtain (7.19), the required result. □

Using essentially the technique of Theorem 7.5, we can obtain the following result.

Theorem 7.6. *Let $\overline{u} \in K(u)$ be a solution of (2.2). If the operator T is strongly monotone with constant $\alpha > 0$, then*

$$\|u - \overline{u}\|^2 \leq (2\rho\mu)/(\rho(2\mu\alpha + 1) - \mu)D_{\rho,\mu}, \quad \forall u \in K(u). \tag{7.24}$$

Proof. Let $\overline{u} \in K(u)$ be a solution of (7.15). Then, taking $v = u$ in (7.15), we have

$$\langle T\overline{u}, u - \overline{u}\rangle \geq 0. \tag{7.25}$$

Also from (7.23), (7.25) and strongly monotonicity of T, we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq \langle Tu, u - \overline{u}\rangle + (1/2\mu)\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &\geq \langle T\overline{u}, u - \overline{u}\rangle \\ &\quad +\alpha\|u - \overline{u}\|^2 + (1/2\mu)\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &\geq (\alpha + (1/2\mu) - (1/2\rho))\|u - \overline{u}\|^2, \end{aligned}$$

from which the required result (7.24) follows. □

8 Nonexpansive Mappings

It is well known that the solution of the quasi variational inequalities can be computed using the iterative projection method, the convergence of which requires the strongly monotonicity and Lipschitz continuity of the involved operator. These strict conditions rule out its applications in important problem. To overcome these drawback, we use the concept of the relaxed co-coercive concept, which is weaker than the strongly monotonicity. In this respect our results represent a refinement of the previously known results. Noor [40] suggested and analyzed several three-step iterative methods for solving different classes of variational inequalities. It has been shown that three-step schemes are numerically better than two-step and one-step methods. Related to the quasi variational inequalities is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. Motivated by the research going on these fields, we suggest and analyze several new three-step iterative methods for finding the common solution of these problems. We also prove the convergence criteria of these new iterative schemes under some mild conditions. These iterative methods contain the Mann Iterations [28], Ishikawa iterations [19] and Noor iterations [40,41]as special cases for solving quasi variational inequalities and quasi complementarity problems. Since the quasi variational inequalities include the variational inequalities and implicit complementarity problems as special cases, results obtained in this section continue to hold for these problems. Results proved in this section may be viewed as a significant and improvement of the previously known results.

Remark 8.1. Lemma 3.1 implies that quasi variational inequalities (2.2) and the fixed point problems (3.1) are equivalent. This alternative equivalent formulation has played a significant role in the studies of the quasi variational inequalities and related optimization problems.

Let S be a nonexpansive mapping. We denote the set of the fixed points of S by $F(S)$ and the set of the solutions of the quasi variational inequalities (2.2) by $QVI(K(u), T)$. We can characterize the problem. If $x^* \in F(S) \cap QVI(K(u), T)$, then $x^* \in F(S)$ and $x^* \in QVI(K(u), T)$. Thus from Lemma 8.1, it follows that

$$x^* = Sx^* = P_{K(u)}[x^* - \rho Tx^*] = SP_{K(u)}[x^* - \rho Tx^*], \tag{8.1}$$

where $\rho > 0$ is a constant.

This fixed point formulation (8.1) is used to suggest the following three-step iterative methods for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings S and the quasi variational inequalities (2.2).

Algorithm 8.1. *For a given $x_0 \in K(x_0)$, compute the approximate solution x_n by the iterative schemes*

$$z_n = (1 - c_n)x_n + c_n SP_{K(x_n)}[x_n - \rho Tx_n], \tag{8.2}$$

$$y_n = (1 - b_n)x_n + b_n SP_{K(z_n)}[z_n - \rho Tz_n], \tag{8.3}$$

$$x_{n+1} = (1 - a_n)x_n + a_n SP_{K(y_n)}[y_n - \rho Ty_n], \tag{8.4}$$

where $a_n, b_n, c_n \in [0, 1]$ for all $n \geq 0$ and S is the nonexpansive operator.

For $S = I$, the identity operator, we obtain a new three-step Algorithm 8.1 for solving the quasi variational inequalities (2.2).

Algorithm 8.2. *For a given $x_0 \in K(x_0)$, compute the approximate solution x_n by the iterative schemes*

$$z_n = (1 - c_n)x_n + c_n P_{K(x_n)}[x_n - \rho Tx_n], \tag{8.5}$$

$$y_n = (1 - b_n)x_n + b_n P_{K(z_n)}[z_n - \rho Tz_n], \tag{8.6}$$

$$x_{n+1} = (1 - a_n)x_n + a_n P_{K(y_n)}[y_n - \rho Ty_n], \tag{8.7}$$

where $a_n, b_n, c_n \in [0, 1]$ for all $n \geq 0$.

For $c_n \equiv 0$, Algorithm 8.1 reduces to:

Algorithm 8.3. *For an arbitrarily chosen initial point $x_0 \in K(x_0)$, compute the approximate solution$\{x_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-b_n)x_n + b_n SP_{K(x_n)}[x_n - \rho T x_n], \\ x_{n+1} &= (1-a_n)x_n + a_n SP_{K(y_n)}[y_n - \rho T y_n], \end{aligned}$$

where $a_n, b_n \in [0,1]$ for all $n \geq 0$ and S is the nonexpansive operator. Algorithm 8.6 is called the two-step Ishikawa iterations [28].

For $b_n \equiv 0,\ c_n \equiv 0$, Algorithm 8.1 collapses to the following iterative method.

Algorithm 8.4. *For a given $x_0 \in K(x_0)$, compute the approximate solution x_{n+1} by the iterative schemes:*

$$x_{n+1} = (1-a_n)x_n + a_n SP_{K(x_n)}[x_n - \rho T x_n], \tag{8.8}$$

which is known as the one-step Mann iteration [28] and appears to be a new one.

For $K(u) \equiv K$, Algorithm 8.5 reduces to the following three-step iterative methods for solving the problem $F(S) \cap VI(K,T)$, which is due to Noor and Huang [44].

Algorithm 8.5. *For a given $x_0 \in K$, compute the approximate solution x_n by the iterative schemes*

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n SP_K[x_n - \rho T x_n], \\ y_n &= (1-b_n)x_n + b_n SP_K[z_n - \rho T z_n], \\ x_{n+1} &= (1-a_n)x_n + a_n SP_K[y_n - \rho T y_n], \end{aligned}$$

where $a_n, b_n, c_n \in [0,1]$ for all $n \geq 0$ and S is the nonexpansive operator.

Algorithm 8.5 is a three-step predictor-corrector method. It is worth mentioning that three-step methods are also known as Noor iterations. Clearly Noor iterations include Mann iteration and Ishikawa iterations as special cases. In particular, three-step methods suggested in this paper are quite general and include several new and previously known algorithms for solving variational inequalities and nonexpansive mappings.

Definition 8.1. A mapping $T : K \to H$ is called *μ-Lipschitzian*, if $\forall x, y \in K$, there exists a constant $\mu > 0$, such that

$$\|Tx - Ty\| \leq \mu \|x - y\|.$$

Definition 8.2. A mapping $T : K \to H$ is called *α-inverse strongly monotonic* (or *co-coercive*), if $\forall x, y \in K$, there exists a constant $\alpha > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2.$$

Definition 8.3. A mapping $T : K \to H$ is called *r-strongly monotone*, if $\forall x, y \in K$, there exists a constant $r > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq r \|x - y\|^2.$$

Definition 8.4. A mapping $T : K \to H$ is called *relaxed* (γ, r)-*cocoercive* if for all $x, y \in K$, there exists constants $\gamma > 0, r > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma \|Tx - Ty\|^2 + r \|x - y\|^2.$$

Remark 8.2. Clearly a r-strongly monotonic mapping or a γ-inverse strongly monotonic mapping must be a relaxed (γ, r)-cocoercive mapping, but the converse is not true. Therefore the class of the relaxed (γ, r)-cocoercive mappings is the most general class, and hence Definition 8.4 includes both the Definition 8.2 and the Definition 8.3 as special cases.

Lemma 8.1. *Suppose $\{\delta_k\}_{k=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{k+1} \leq (1 - \lambda_k)\delta_k + \sigma_k, \ \ k \geq 0$$

with $\lambda_k \in [0, 1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$, and $\sigma_k = o(\lambda_k)$. Then $\lim_{k \to \infty} \delta_k = 0$.

In this section, we investigate the strong convergence of Algorithms 8.1, 8.3 and 8.4 in finding the common element of two sets of solutions of the quasi variational inequalities (2.2) and $F(S)$ and this is the main motivation of this section.

Theorem 8.1. *Let $K(u)$ be a closed convex-valued subset of a real Hilbert space H. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of $K(u)$ into H, and S be a nonexpansive mapping of $K(u)$ into $K(u)$ such that $F(S) \cap QVI(K(u), T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by Algorithm 8.1, for any initial point $x_0 \in K(x_0)$, with conditions*

$$\left|\rho - \frac{r - \gamma\mu^2}{\mu^2}\right| < \frac{\sqrt{(r-\gamma\mu^2)^2 - \mu^2(2\nu - \nu^2)}}{\mu^2} \tag{8.9}$$

$$r_1 > \gamma_1\mu^2 + \mu_1\sqrt{\nu(2-\nu)}, \quad \nu \in (0,1), \tag{8.10}$$

$a_n, b_n, c_n \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$. If Assumption 2.1 holds, then x_n obtained from Algorithm 8.5 converges strongly to $x^* \in F(S) \cap QVI(K(u), T)$.

Proof. Let $x^* \in K(u)$ be the solution of $F(S) \cap QVI(K(u), T)$. Then

$$\begin{aligned} x^* &= (1-c_n)x^* + c_n SP_{K(x^*)}[x^* - \rho Tx^*] \quad &(8.11)\\ &= (1-b_n)x^* + b_n SP_{K(x^*)}[x^* - \rho Tx^*] \quad &(8.12)\\ &= (1-a_n)x^* + a_n SP_{K(x^*)}[x^* - \rho Tx^*] \quad &(8.13)\end{aligned}$$

where $a_n, b_n, c_n \in [0,1]$ are some constants.

From (8.7), (8.11), Assumption 2.1, and the nonexpansive mapping S, we have

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ \leq\ & (1-a_n)\|x_n - x^*\| + a_n\|SP_{K(y_n)}[y_n - \rho Ty_n] - SP_{K(x^*)}[x^* - \rho Tx^*]\| \\ \leq\ & (1-a_n)\|x_n - x^*\| + a_n\|P_{K(y_n)}[y_n - \rho Ty_n] - P_{K(y_n)}[x^* - \rho Tx^*]\| \\ & + \|P_{K(y_n)}[x^* - \rho Tx^*]P_{K(x^*)}[x^* - \rho Tx^*] \\ \leq\ & (1-a_n)\|x_n - x^*\| + a_n\|y_n - x^* - \rho(Ty_n - Tx^*)\| + a_n\nu\|y_n - x^*\|. \quad (8.14)\end{aligned}$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T,

$$\begin{aligned}
& \|y_n - x^* - \rho(Ty_n - Tx^*)\|^2 \\
= \;& \|y_n - x^*\|^2 - 2\rho\langle Ty_n - Tx^*, y_n - x^*\rangle + \rho^2\|Ty_n - Tx^*\|^2 \\
\leq \;& \|y_n - x^*\|^2 - 2\rho[-\gamma\|Ty_n - Tx^*\|^2 + r\|y_n - x^*\|^2] \\
& +\rho^2\|Ty_n - Tx^*\|^2 \\
\leq \;& \|y_n - y^*\|^2 + 2\rho\gamma\mu^2\|y_n - x^*\|^2 - 2\rho r\|y_n - x^*\|^2 + \rho^2\mu^2\|y_n - x^*\|^2 \\
= \;& [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|y_n - x^*\|^2. \qquad (8.15)
\end{aligned}$$

Combining (8.14) and (8.11), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| \leq \;& (1 - a_n)\|x_n - x^*\| \\
& +a_n\left\{\sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2} + \nu\right\}\|y_n - x^*\| \\
= \;& (1 - a_n)\|x_n - x^*\| + a_n\|y_n - x^*\|, \qquad (8.16)
\end{aligned}$$

where

$$\theta = \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2} + \nu. \qquad (8.17)$$

It follows from (8.9) and (8.10) that $\theta < 1$.

From (8.6), (8.12), Assumption 2.1, and nonexpansivity of S, we have

$$\begin{aligned}
\|y_n - x^*\| \leq \;& (1 - b_n)\|x_n - x^*\| \\
& +b_n\|SP_{K(z_n)}[z_n - \rho Tz_n] - SP_{K(x^*)}[x^* - \rho Tx^*]\| \\
\leq \;& (1 - b_n)\|x_n - x^*\| + b_n\|P_{K(z_n)}[x^* - \rho Tx^*] - P_{K(x^*)}[x^* - \rho Tx^*]\| \\
& +b_n\|P_{K(z_n)}[z_n - \rho Tz_n] - P_{K(z_n)}[x^* - \rho Tx^*]\| \\
\leq \;& (1 - b_n)\|x_n - x^*\| + b_n\nu\|z_n - x^*\| \\
& +b_n\|z_n - x^* - \rho(Tz_n - Tx^*)\|. \qquad (8.18)
\end{aligned}$$

Now from the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T, we have

$$\begin{aligned} & \|z_n - x^* - \rho[Tz_n - Tx^*]\|^2 \\ = \;& \|z_n - x^*\|^2 - 2\rho\langle Tz_n - Tx^*, z_n - x^*\rangle + \rho^2\|Tz_n - Tx^*\|^2 \\ \leq \;& \|z_n - x^*\|^2 - 2\rho[-\gamma\|Tz_n - Tx^*\|^2 + r\|z_n - x^*\|^2] \\ & +\rho^2\|Tz_n - Tx^*\|^2 \\ \leq \;& \|z_n - x^*\|^2 + 2\rho\gamma\mu^2\|z_n - x^*\|^2 - 2\rho r\|z_n - x^*\|^2 \\ & +\rho^2\mu^2\|z_n - x^*\|^2 \\ = \;& [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|z_n - x^*\|^2. \end{aligned} \tag{8.19}$$

From (8.17), (8.18) and (8.19), we have

$$\|y_n - x^*\| \leq (1 - b_n)\|x_n - x^*\| + b_n\theta\|z_n - x^*\|. \tag{8.20}$$

In a similar way, from (8.5) and (8.11), it follows that

$$\begin{aligned} \|z_n - x^*\| &\leq (1 - c_n)\|x_n - x^*\| + c_n\theta\|x_n - x^*\|, \\ &= \{(1 - c_n(1 - \theta))\}\|x_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{8.21}$$

From (8.21), (8.20) and (8.17), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\| \\ &\leq (1 - a_n)\|x_n - x^*\| + a_n\theta\|z_n - x^*\| \\ &\leq (1 - a_n)\|x_n - x^*\| + a_n\theta\|x_n - x^*\| \\ &= [1 - a_n(1 - \theta)]\|x_n - x^*\|, \end{aligned} \tag{8.22}$$

and hence by Lemma 8.1, we have $\lim_{n\to\infty} ||x_n - x^*|| = 0$, the required result. □

If the convex-valued set $K(x^*)$ is independent of the solution x^*, that is, $K(x^*) \equiv K$, then Theorem 8.1 reduces to the following result, which is due to Noor and Huang [44].

Theorem 8.2. *Let K be a closed convex subset of a real Hilbert space H. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of K into H, and S be a nonexpansive mapping of K into K such that $F(S) \cap VI(K,T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by Algorithm 8.9, for any initial point $x_0 \in K$, with conditions*

$$0 < \rho < 2(r - \gamma\mu^2)/\mu^2, \quad \gamma\mu^2 < r,$$

$a_n, b_n, c_n \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, then x_n obtained from Algorithm 8.9 converges strongly to $x^* \in F(S) \cap VI(K,T)$.

Next we will provide and prove the strong convergence theorem of Algorithm 8.4 under the α-inverse strongly monotonicity.

Theorem 8.3. *Let K be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$. Let T be an α-inverse strongly monotone mapping of $K(u)$ into H, and S be a nonexpansive mapping of $K(u)$ into $K(u)$ such that $F(S) \cap QVI(K(u),T) \neq \emptyset$. If the Assumption 2.1 hold and*

$$|\rho - \alpha| \leq \alpha(1 - \nu), \tag{8.23}$$

then the approximate solution obtained from Algorithm 8.3 converges strongly to $x^* \in F(S) \cap QVI(K(x^*),T)$.

Proof. It is well known that, if T is α-inverse strongly monotonic with the constant $\alpha > 0$, then T is $\frac{1}{\alpha}$–Lipschitzian continuous.

Consider

$$\begin{aligned}
& \|x_n - x^* - \rho[Tx_n - Tx^*]\|^2 \\
= \ & \|x_n - x^*\|^2 + \rho^2\|Tx_n - Tx^*\|^2 - 2\rho\langle Tx_n - Tx^*, x_n - x^*\rangle \\
\leq \ & \|x_n - x^*\|^2 + \rho^2\|Tx_n - Tx^*\|^2 - 2\rho\alpha\|Tx_n - Tx^*\|^2 \\
= \ & \|x_n - x^*\|^2 + (\rho^2 - 2\rho\alpha)\|Tx_n - Tx^*\|^2 \\
\leq \ & \|x_n - x^*\|^2 + (\rho^2 - 2\rho\alpha) \cdot \frac{1}{\alpha^2}\|x_n - x^*\|^2 \\
= \ & (1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2})\|x_n - x^*\|^2.
\end{aligned} \tag{8.24}$$

From (8.7), (8.9) and Assumption 2.1, we have

$$
\begin{aligned}
& \|x_{n+1} - x^*\| \\
\leq\ & (1-a_n)\|x_n - x^*\| + a_n\|SP_{K(x_n)}[x_n - \rho T x_n] - SP_{K(x^*)}[x^* - \rho T x^*]\| \\
\leq\ & (1-a_n)\|x_n - x^*\| + a_n\|P_{K(x_n)}[x_n - \rho T x_n] - P_{K(x^*)}[x^* - \rho T x^*]\| \\
\leq\ & (1-a_n)\|x_n - x^*\| + a_n\|P_{K(x_n)}[x_n - \rho T x_n] - P_{K(x_n)}[x^* - \rho T x^*]\| \\
& + P_{K(x_n)}[x^* - \rho T x^*] - P_{K(x^*)}[x^* - \rho T x^*]\| \\
\leq\ & (1-a_n)\|x_n - x^*\| + a_n\|x_n - x^* - \rho(Tx_n - Tx^*)\| + a_n\nu\|x_n - x^*\| \\
\leq\ & (1-a_n)\|x_n - x^*\| + a_n\theta_1\|x_n - x^*\| \\
=\ & [1 - a_n(1-\theta_1)]\|x_n - x^*\|,
\end{aligned}
$$

where

$$\theta_1 = \sqrt{1 + \frac{\rho^2 - 2\rho\alpha}{\alpha^2}} + \nu. \tag{8.25}$$

From (8.23), it follows that $\theta_1 < 1$ and consequently using Lemma 8.1, we have

$$\lim_{n\to\infty} \|x_n - x^*\| = 0$$

the required result. □

Related to the variational inequalities, we have the problem of solving the Wiener-Hopf equations. To be more precise, let $Q_{K(u)} = I - SP_{K(u)}$, where $P_{K(u)}$ is the projection of H onto the closed convex set $K(u)$, I is the identity operator and S is the nonexpansive operator. We consider the problem of finding $z \in H$ such that

$$TSP_{K(u)}z + \rho^{-1}Q_{K(u)}z = 0, \tag{8.26}$$

which is called the implicit Wiener-Hopf equation involving the nonexpansive operator S. For $S = I$, the identity operator, we obtain the implicit Wiener-Hopf equation, introduced by Noor [14]. If $S = I$, and $K(u) = K$, then the implicit Wiener-Hopf equations (8.26) reduces to the original Wiener-Hopf equations considered and studied in relation with the classical variational inequalities.

In this section, we use the Wiener-Hopf equations to suggest and analyze an iterative method for finding the common element of the nonexpansive mappings and the quasi variational inequality (2.2). For this purpose, we need the following result, which can be proved by using Lemma 3.1.

Lemma 8.2. *The element $u \in K(u)$ is a solution of quasi variational inequality (2.2, if and only if, $z \in H$ satisfies the implicit Wiener-Hopf equation (8.26), where*

$$u = P_{K(u)}z, \tag{8.27}$$
$$z = u - \rho Tu, \tag{8.28}$$

where $\rho > 0$ is a constant.

Proof. Let $u \in K(u)$ be a solution of quasi variational inequality (2.2) Then, from Lemma 3.1, we have

$$u = SP_{K(u)}[u - \rho Tu]. \tag{8.29}$$

Let

$$z = u - \rho Tu. \tag{8.30}$$

From (8.29) and (8.29), we have

$$u = SP_{K(u)}z, \qquad z = u - \rho Tu,$$

from which, we have

$$z = SP_{K(u)}z - \rho TSP_{K(u)}z,$$

which is exactly the implicit Wiener-Hopf equation (8.26), the required result. □

From Lemma 8.2, it follows that the quasi variational inequality (2.2) and the implicit Wiener-Hopf equation (8.26) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving quasi variational inequalities and related optimization problems.

Using Lemma 8.1, we now suggest and analyze a new iterative algorithm for finding the common element of the solution sets of the quasi variational inequalities and nonexpansive mappings S.

Algorithm 8.6. *For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes*

$$u_n = SP_{K(u_n)}z_n \tag{8.31}$$
$$z_{n+1} = (1-a_n)z_n + a_n\{u_n - \rho Tu_n\}, \tag{8.32}$$

where $a_n \in [0,1]$ for all $n \geq 0$ and S is a nonexpansive operator.

For $S = I$, the identity operator, Algorithm 8.6 reduces to the following iterative method for solving quasi variational inequalities(2.2) and appears to be a new one.

Algorithm 8.7. *For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes*

$$u_n = P_{K(u_n)}z_n$$
$$z_{n+1} = (1-a_n)z_n + a_n\{u_n - \rho Tu_n\}.$$

For $a_n = 1$ and $S = I$, the identity operator, Algorithm 8.6 collapses to the iterative method for solving quasi variational inequalities (2.2).

Algorithm 8.8. *For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes*

$$u_n = P_{K(u_n)}z_n$$
$$z_{n+1} = u_n - \rho Tu_n.$$

If $K(u) = K$, the convex set in H, then Algorithm 8.6, Algorithm 8.11 and Algorithm 8.12 reduce to the following algorithms for solving variational inequalities and nonexpansive mapping, which are due to Noor and Huang [44].

Algorithm 8.9. *For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes*

$$u_n = SP_K z_n \tag{8.33}$$

$$z_{n+1} = (1-a_n)z_n + a_n\{u_n - \rho T u_n\}, \tag{8.34}$$

where $a_n \in [0,1]$ for all $n \geq 0$ and S is a nonexpansive operator.

For $S = I$, the identity operator, Algorithm 8.8 reduces to the following iterative method for solving variational inequalities and appears to be a new one.

Algorithm 8.10. *For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes*

$$u_n = P_K z_n$$

$$z_{n+1} = (1-a_n)z_n + a_n\{u_n - \rho T u_n\}.$$

For $a_n = 1$ and $S = I$, the identity operator, Algorithm 8.9 collapses to the following iterative method for solving quasi variational inequalities.

Algorithm 8.11. *For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes*

$$u_n = P_K z_n$$

$$z_{n+1} = u_n - \rho T u_n.$$

We now study the convergence criteria of Algorithm 8.6.

Theorem 8.4. *Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping and S be a nonexpansive mapping such that $F(S) \cap IWHE(H,T,S) \neq \emptyset$. Let $\{z_n\}$ be a sequence defined by Algorithm 8.6, for any initial point $z_0 \in H$. If Assumption 2.1 holds and*

$$|\rho - \frac{r-\gamma\mu^2}{\mu^2}| \leq \frac{\sqrt{(r-\gamma\mu^2)^2 - \mu^2\nu(2-\nu)}}{\mu^2}, \tag{8.35}$$

$$r > \gamma\mu^2 + \mu\sqrt{\nu(2-\nu)}, \quad \nu \in (0,1),$$

$a_n \in [0,1]$ *and* $\sum_{n=0}^{\infty} a_n = \infty$*, then* z_n *converges strongly to* $z^* \in F(S) \cap IWHE(H,T,S)$*.*

Proof. Let $z^* \in H$ be a solution of $F(S) \cap IWHE(H,T,S)$. Then, from Lemma 8.5, we have

$$u^* = SPK_{(u^*)} z^* \tag{8.36}$$

$$z^* = (1-a_n)z^* + a_n\{u^* - \rho T u^*\}, \tag{8.37}$$

where $a_n \in [0,1]$ and $u^* \in K$ is a solution of variational inequality (2.2). From (8.34) and (8.37), we have

$$\begin{aligned} \|z_{n+1} - z^*\| &= \|(1-a_n)z_n + a_n\{u_n - \rho T u_n\} \\ &\quad -(1-a_n)z^* - a_n\{u^* - \rho T u^*\}\| \\ &\leq (1-a_n)\|z_n - z^*\| + a_n\|u_n - u^* - \rho(Tu_n - Tu^*)\|. \end{aligned} \tag{8.38}$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T, we have

$$\begin{aligned} & \|u_n - u^* - \rho(Tu_n - Tu^*)\|^2 \\ = \ & \|u_n - u^*\|^2 - 2\rho\langle Tu_n - Tu^*, u_n - u^*\rangle + \rho^2\|Tu_n - Tu^*\|^2 \\ \leq \ & \|u_n - u^*\|^2 - 2\rho[-\gamma||Tu_n - Tu^*|||^2 + r\|u_n - u^*||^2] \\ & +\rho^2\|Tu_n - Tu^*\|^2 \\ \leq \ & \|u_n - u^*\|^2 + 2\rho\gamma\mu^2|u_n - u^*\|^2 - 2\rho r\|u_n - u^*\|^2 + \rho^2\mu^2\|u_n - u^*\|^2 \\ = \ & [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]\|u_n - u^*\|^2 \\ = \ & \theta_1^2\|u_n - u^*\|^2, \end{aligned} \tag{8.39}$$

where

$$\theta_1 = \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}. \tag{8.40}$$

Combining (8.38) and (8.394), we have

$$\|z_{n+1} - z^*\| \leq (1-a_n)\|z_n - z^*\| + a_n\theta_1\|u_n - u^*\|. \tag{8.41}$$

Using Assumption 2.1, we have

$$\begin{aligned}
\|u_n - u^*\| &\leq a_n\|SP_{K(u_n)}z_n - SP_{K(u^*)}z^*\| \\
&\leq \|P_{K(u_n)}z_n - P_{K(u_n)}z^*\| + \|P_{K(u_n)}z^* - P_{K(u^*)}z^*\| \\
&\leq \nu\|u_n - u^*\| + \|z_n - z^*\|,
\end{aligned}$$

which implies that

$$\|u_n - u^*\| \leq \frac{1}{1-\nu}\|z_n - z^*\|. \tag{8.42}$$

From (8.41) and (8.42), we obtain that

$$\begin{aligned}
\|z_{n+1} - z^*\| &\leq (1 - a_n)\|z_n - z^*\| + a_n\theta\|z_n - z^*\| \\
&= [1 - a_n(1-\theta)]\|z_n - z^*\|,
\end{aligned}$$

where

$$\theta = \frac{\sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}}{1-\nu} < 1, \quad \text{using (3.9)},$$

and hence by Lemma 8.1,

$$\lim_{n\to\infty}\|z_n - z^*\| = 0,$$

completing the proof. □

Remark 8.3. In this section, we have shown that the quasi variational inequalities are equivalent to a new class of Wiener-Hopf equations and fixed point problems involving the nonexpansive operator. This equivalence is used to suggest and analyze an iterative method for finding the common element of set of the solutions of the quasi variational inequalities and the set of the fixed-points of the nonexpansive operator. It is worth mentioning that Noor [43] used the Wiener-Hopf equations technique to develop some very efficient and numerically implementable iterative methods for solving variational inequalities and related optimization problems. The results are encouraging and perform better than the other methods. It is interesting to use the techniques and ideas of this section to develop other new iterative methods for solving the quasi variational inequalities involving the nonexpansive operators. This is another direction for future work.

9 Applications

In this section, we show that the quasi variational inequalities are equivalent to the general variational inequalities, which were introduced and investigated by Noor [40].

In many applications, the convex-valued set $K(u)$ is of the form:

$$K(u) = m(u) + K, \tag{9.1}$$

where K is a convex set and m is a point-to-point mapping. The convex-valued set $K(u)$ defined by (9.1) is known as the moving convex-valued set.

Let $u \in K(u)$ be a solution of problem (2.2). Then, from Lemma 3.1, it follows that $u \in K(u)$ such that

$$u = P_{K(u)}\big[u - \rho Tu\big]. \tag{9.2}$$

From (9.1) and (9.2), we obtain

$$\begin{aligned} u &= P_{K(m(u)+K)}\big[u - \rho Tu\big] \\ &= m(u) + P_K\big[u - m(u) - \rho Tu\big]. \end{aligned}$$

This implies that

$$g(u) = P_K\big[g(u) - \rho Tu\big]$$

which is equivalent to finding $u \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in K. \tag{9.3}$$

The inequality of the type (9.3) is called the general variational inequality, which was introduced and investigated by Noor [37]. Noor [40] have shown that odd-order and nonsymmetric obstacle boundary value problems can be studied in the general variational inequalities. For more details, see Noor [37, 38, 40, 41, 43] and Noor et al. [56,57]. Thus all the results proved for quasi variational inequalities continue to hold for general variational inequalities of the type (9.3) with suitable modifications and adjustment. Despite the research activates, very few results are available. The development of efficient numerical methods requires further efforts.

10 Generalizations and Future Research

We would like to emphasize that the results obtained in this paper can extended for general quasi variational inequalities. To be more precise, for given operators $T, g, h : H \to H$, consider the problem of finding $u \in K(u)$, such that

$$\langle Tu, h(v) - g(u) \rangle \geq 0, \quad \forall u, v \in K(u), \tag{10.1}$$

which is called the extended general quasi variational inequality, considered and analyzed by Noor et al. [52, 53].

We now discuss some important special cases of the problem (10.1).

(I). If $K(u) = K$, a closed convex set, then problem (10.1) reduces to finding $u \in K$ such that

$$\langle Tu, h(v) - g(u) \rangle \geq 0, \quad \forall u, v \in K, \tag{10.2}$$

which is called the extended general variational inequality, introduced and studied by Noor [45]. For more details, see [9, 27, 32, 46, 47, 58] and the references therein.

(II). If $g = h$, then the problem (10.1) reduces to finding $u \in K(u)$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall u, v \in K(u), \tag{10.3}$$

is called the general quasi variational inequality, which has been studied extensively, see [63].

(III). If $g = h = I$, the identity operator, then the problem (10.1) reduces to finding $u \in K(u)$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall u, v \in K(u), \tag{10.4}$$

which is exactly quasi variational inequality (2.2) studied in this paper.

(IV). If $Tu = u$, then the problem (10.1) reduces to finding $u \in K(u)$ such that

$$\langle u, h(v) - g(u) \rangle \geq 0, \quad \forall u, v \in K(u), \tag{10.5}$$

which is called the inverse extended general quasi variational inequalities. Several important special cases of the problem (10.5) have been considered by several authors including [7, 18, 22, 38]. It is an interesting problem to develop new numerical methods for solving inverse variational inequalities and explore their applications in various branches of pure and applied sciences. These are new problems and have not investigated recently. See also [50, 63, 67, 68] for the applications of quasi-hemivariational inequalities and the references therein.

Conclusion

In this paper, we have used the equivalence between the quasi variational inequalities and fixed point formulation to suggest some new iterative methods for solving the variational inequalities. These new methods include extragradient method, modified double projection methods and inertial type are suggested using the techniques of projection method, Wiener-Hopf equations and dynamical systems. Sensitivity analysis the is discussed using Wiener-Hofp technique. Merit functions are used to derive the error bounds for the solutions of the quasi variational inequalities under some weaker conditions. Proofs of our results are simple and straightforward as compared with other methods. We have shown that the quasi variational inequalities are equivalent to the problem of finding the fixed points of the nonexpansive mappings. We have suggested and analyzed several new three-step iterative method for finding the common solution of these problems of the quasi variational inequalities and nonexpansive mappings. Convergence analysis of the proposed methods is discussed. Ishikawa iterations, Mann iterations and Noor iterations are some important special cases of suggested three-step iterative methods. Jabeen et al. [21] have proposed and suggested inertial type methods for solving system of quasi variational inequalities. It is an open problem to compare these proposed methods with other methods. We have shown that the quasi variational inequalities are equivalent to the general variational inequalities under suitable conditions of the convex-valued set. Using the ideas and techniques of this paper, one can suggest and investigate several new implicit methods for solving various classes of variational inequalities and related problems.

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