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# A Certain Subclass of Multivalent Analytic Functions with Negative Coefficients for Operator on Hilbert Space

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## Abstract

In the present work, we introduce and study a certain subclass for multivalent analytic functions with negative coefficients defined on complex Hilbert space. We establish a number of geometric properties, like, coefficient estimates, convex set, extreme points and radii of starlikeness and convexity.

## 1. Introduction

Let $\mathcal{A}_p$ denote the family of functions $f$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and multivalent in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Also, let $S_p$ denote the subclass of $\mathcal{A}_p$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (a_{n+p} \geq 0,\ p \in \mathbb{N} = \{1, 2, \ldots\}). \tag{1.2}$$

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Denote by $H$ the Hilbert space on the complex field and $T$ is a linear operator on $H$. For a complex analytic function $f$ on the unit disk $U$, we denoted $f(T)$ the operator on $H$ defined by the usual Riesz-Dunford integral [2]

$$f(T) = \frac{1}{2\pi i}\int_c f(z)(zI - T)^{-1}dz,$$

where $I$ is the identity operator on $H$, $c$ is a positively oriented simple closed rectifiable contour lying in $U$ and containing the spectrum $\sigma(T)$ of $T$ in its interior domain [3]. Also $f(T)$ can be defined by the series

$$f(T) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}T^n,$$

which converges in the norm topology [4].

**Definition 1.1.** A function $f \in S_p$ is said to be in the class $\mathcal{A}S_p(\eta, \gamma, \delta, T)$ if satisfies the inequality:

$$\left\| \frac{Tf'(T)}{f(T)} - p \right\| < \eta \left\| \gamma \frac{Tf'(T)}{f(T)} + p - \delta(\gamma + 1) \right\|, \tag{1.3}$$

where $p \in \mathbb{N}$, $0 \leq \gamma < 1$, $0 \leq \delta < p$, $0 < \eta \leq 1$ and for all operator $T$ with $\| T \| < 1$, $T \neq \varnothing$ ($\varnothing$ denote the zero operator on $H$).

The operator on Hilbert space were consider recently be Pu [8], Joshi [6], Kim et al. [1], Ghanim and Darus [5], Selvaraj et al. [7], Wanas and Jebur [9] and Wanas and Frasin [10].

## 2. Main Results

**Theorem 2.1.** *Let* $f \in S_p$ *be given by* (1.2). *Then* $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T)$ *for all* $T \neq \varnothing$ *if and only if*

$$\sum_{n=1}^{\infty} [n(1 + \eta\gamma) + \eta(p - \delta)(\gamma + 1)]a_{n+p} \leq \eta(p - \delta)(\gamma + 1), \tag{2.1}$$

*where* $0 \leq \gamma < 1$, $0 \leq \delta < p$, $0 < \eta \leq 1$.

*The result is sharp for the function f given by*

$$f(z) = z^p - \frac{\eta(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} z^{n+p}, \quad n \geq 1. \tag{2.2}$$

**Proof.** Assume that the inequality (2.1) holds. Then, we obtain

$$\| Tf'(T) - pf(T) \| - \eta \| \gamma T f'(T) + (p - \delta(\gamma+1)) f(T) \|$$

$$= \left\| -\sum_{n=1}^{\infty} n a_{n+p} T^{n+p} \right\|$$

$$- \left\| \eta(p-\delta)(\gamma+1)T^p - \sum_{n=1}^{\infty} (\eta\gamma n + \eta(p-\delta)(\gamma+1)) a_{n+p} T^{n+p} \right\|$$

$$\leq \sum_{n=1}^{\infty} [n(1+\eta\gamma) + \eta(p-\delta)(\gamma+1)] a_{n+p} - \eta(p-\delta)(\gamma+1) \leq 0.$$

Thus, $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T)$.

To show the converse, let $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T)$. Then

$$\| Tf'(T) - pf(T) \| - \eta \| \gamma Tf'(T) + (p - \delta(\gamma+1)) f(T) \|,$$

gives

$$\left\| -\sum_{n=1}^{\infty} n a_{n+p} T^{n+p} \right\|$$

$$< \eta \left\| (p-\delta)(\gamma+1)T^p - \sum_{n=1}^{\infty} (\gamma n + (p-\delta)(\gamma+1)) a_{n+p} T^{n+p} \right\|.$$

Setting $T = rI$ $(0 < r < 1)$ in the above inequality, we get

$$\frac{\sum_{n=1}^{\infty} n a_{n+p} r^{n+p}}{(p-\delta)(\gamma+1)r^p - \sum_{n=1}^{\infty} (\gamma n + (p-\delta)(\gamma+1)) a_{n+p} r^{n+p}} < \eta. \tag{2.3}$$

Upon clearing denominator in (2.3) and letting $r \to 1,$ we obtain

$$\sum_{n=1}^{\infty} na_{n+p} < \eta(p-\delta)(\gamma+1) - \sum_{n=1}^{\infty} \eta(\gamma n + (p-\delta)(\gamma+1))a_{n+p}.$$

Thus

$$\sum_{n=1}^{\infty} [n(1+\eta\gamma) + \eta(p-\delta)(\gamma+1)]a_{n+p} \leq \eta(p-\delta)(\gamma+1),$$

which completes the proof.

**Corollary 2.1.** *If* $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T),$ *then*

$$a_{n+p} \leq \frac{n(p-\delta)(\gamma+1)}{n(1+\eta\gamma) + \eta(p-\delta)(\gamma+1)}, \quad n \geq 1.$$

**Theorem 2.2.** *The class* $\mathcal{A}S_p(\eta, \gamma, \delta, T)$ *is a convex set.*

**Proof.** Let $f_1$ and $f_2$ be the arbitrary elements of $\mathcal{A}S_p(\eta, \gamma, \delta, T).$ Then for every $t\ (0 \leq t \leq 1),$ we show that $(1-t)f_1 + tf_2 \in \mathcal{A}k_p(\alpha, \beta, \delta, T).$ Thus, we have

$$(1-t)f_1 + tf_2 = z^p - \sum_{n=1}^{\infty} ((1-t)a_{n+p} + tb_{n+p})z^{n+p}.$$

Hence

$$\sum_{n=1}^{\infty} [n(1+\eta\gamma) + \eta(p-\delta)(\gamma+1)]((1-t)a_{n+p} + tb_{n+p})$$

$$= (1-t)\sum_{n=1}^{\infty} [n(1+\eta\gamma) + \eta(p-\delta)(\gamma+1)]a_{n+p}$$

$$+ t\sum_{n=1}^{\infty} [n(1+\eta\gamma) + \eta(p-\delta)(\gamma+1)]b_{n+p}$$

$$\leq (1-t)\eta(p-\delta)(\gamma+1) + t\eta(p-\delta)(\gamma+1).$$

This completes the proof.

**Theorem 2.3.** *Let* $f_0(z) = z^p$ *and*

$$f_n(z) = z^p - \frac{\eta(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} z^{n+p}, \quad n \geq 1.$$

*Then* $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z), \tag{2.4}$$

*where* $\lambda_n \geq 0$ *and* $\sum_{n=0}^{\infty} \lambda_n = 1$.

**Proof.** Suppose that *f* can be expressed by (2.4). Then, we find that

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$$

$$= z^p - \sum_{n=0}^{\infty} \frac{\eta(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \lambda_n z^{n+p}.$$

Thus

$$\sum_{n=0}^{\infty} \frac{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)}{\eta(p-\delta)(\gamma+1)} \frac{\eta(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \lambda_n = \sum_{n=0}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1,$$

and so $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T)$.

Conversely, assume that *f* given by (1.2) is in the class $\mathcal{A}S_p(\eta, \gamma, \delta, T)$. Then by Corollary 2.1, we have

$$a_{n+p} \leq \frac{\eta(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)}.$$

Putting

$$\lambda_n = \frac{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)}{\eta(p-\delta)(\gamma+1)} a_n, \quad n \geq 1,$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$. Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z),$$

which completes the proof.

**Theorem 2.4.** *If* $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T)$ *and* $\| T \| < 1$, $T \neq \emptyset$, *then*

$$\| T \|^p - \frac{\eta(p-\delta)(\gamma+1)}{(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \| T \|^{p+1}$$

$$\leq \| f(T) \|$$

$$\leq \| T \|^p + \frac{\eta(p-\delta)(\gamma+1)}{(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \| T \|^{p+1}$$

*and*

$$p\| T \|^{p-1} - \frac{\eta(n+p)(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \| T \|^p$$

$$\leq \| f'(T) \|$$

$$\leq p\| T \|^{p-1} + \frac{\eta(n+p)(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \| T \|^p.$$

**Proof.** According to Theorem 2.1, we have

$$\sum_{n=1}^{\infty} a_{n+p} \leq \frac{\eta(p-\delta)(\gamma+1)}{(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)}.$$

Thus

$$\| f(T) \| \geq \| T \|^p - \sum_{n=1}^{\infty} a_{n+p} \| T \|^{n+p}$$

$$\geq \| T \|^p - \| T \|^{p+1} \sum_{n=1}^{\infty} a_{n+p}$$

$$\geq \| T \|^p - \frac{\eta(p-\delta)(\gamma+1)}{(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \| T \|^{p+1}.$$

Also,

$$\| f(T) \| \leq \| T \|^p + \sum_{n=1}^{\infty} a_{n+p} \| T \|^{n+p}$$

$$\leq \| T \|^p + \frac{\eta(p-\delta)(\gamma+1)}{(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \| T \|^{p+1}.$$

In view of Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty} (n+p) a_{n+p} \leq \frac{\eta(n+p)(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)}.$$

Hence

$$\| f'(T) \| \geq p \| T \|^{p-1} - \sum_{n=1}^{\infty} (n+p) a_{n+p} \| T \|^{n+p-1}$$

$$\geq p \| T \|^{p-1} - \| T \|^p \sum_{n=1}^{\infty} (n+p) a_{n+p}$$

$$\geq p \| T \|^{p-1} - \frac{\eta(n+p)(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \| T \|^p$$

and

$$\| f'(T) \| \leq p \| T \|^{p-1} + \| T \|^p \sum_{n=1}^{\infty} (n+p) a_{n+p}$$

$$\leq p \| T \|^{p-1} + \frac{\eta(n+p)(p-\delta)(\gamma+1)}{n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)} \| T \|^p.$$

Therefore the proof is complete.

**Theorem 2.5.** *If* $f \in \mathcal{AS}_p(\eta, \gamma, \delta, T)$, *then* $f$ *is multivalent starlike of order* $\theta$ $(0 \leq \theta < p)$ *in the disk* $|z| < r_1$, *where*

$$r_1 = \inf_n \left\{ \frac{(p-\theta)[n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)]}{\eta(p-\delta)(\gamma+1)(n+p-\theta)} \right\}^{\frac{1}{n}}, \quad (n \geq 1).$$

*The result is sharp for the function given f by* (2.2).

**Proof.** It is sufficient to show that

$$\left\| \frac{Tf'(T)}{f(T)} - p \right\| \leq p - \theta. \tag{2.5}$$

We get

$$\left\| \frac{Tf'(T)}{f(T)} - p \right\| \leq \frac{\sum_{n=1}^{\infty} n a_{n+p} \| T \|^n}{1 - \sum_{n=1}^{\infty} a_{n+p} \| T \|^n}.$$

Hence (2.5) will be satisfied if

$$\sum_{n=1}^{\infty} \left( \frac{n+p-\theta}{p-\theta} \right) a_{n+p} \| T \|^n \leq 1. \tag{2.6}$$

In view of Theorem 2.1, if $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T)$, then

$$\sum_{n=1}^{\infty} \frac{[n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)]}{\eta(p-\delta)(\gamma+1)} a_{n+p} \leq 1. \tag{2.7}$$

By using (2.7), we note that (2.6) holds true if

$$\frac{n+p-\theta}{p-\theta} \| T \|^n \leq \frac{[n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)]}{\eta(p-\delta)(\gamma+1)},$$

or equivalently

$$\| T \| \leq \left\{ \frac{(p-\theta)[n(1+\eta\gamma)+\eta(p-\delta)(\gamma+1)]}{\eta(p-\delta)(\gamma+1)(n+p-\theta)} \right\}^{\frac{1}{n}},$$

This gives the desired result.

**Theorem 2.6.** *If* $f \in \mathcal{A}S_p(\eta, \gamma, \delta, T)$, *then* $f$ *is multivalent convex of order* $\theta$ $(0 \le \theta < p)$ *in the disk* $|z| < r_2$, *where*

$$r_2 = \inf_n \left\{ \frac{p(p-\theta)[n(1+\eta\gamma) + \eta(p-\delta)(\gamma+1)]}{\eta(n+p)(p-\delta)(\gamma+1)(n+p-\theta)} \right\}^{\frac{1}{n}}, \quad (n \ge 1).$$

*The result is sharp for the function given by* (2.2).

**Proof.** It is sufficient to show that

$$\left\| \frac{Tf''(T)}{f'(T)} + 1 - p \right\| \le p - \theta.$$

The result follows by application of arguments similar to the proof of Theorem 2.6.

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