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Analytic Property of a Generalized Poisson Distribution

O.E. Opaleye[1,*], S.O. Fawale[2], P.O. Awoleye[3] and S. Sunday[4]

Department of Pure and Applied Mathematics
Ladoke Akintola University of Technology
Ogbomoso P.M.B. 4000, Ogbomoso, Oyo State, Nigeria
e-mail: oeopaleye@lautech.edu.ng[1]
e-mail: sofawale13@student.lautech.edu.ng[2]
e-mail: poawoleye@student.lautech.edu.ng[3]
e-mail: ssunday@student.lautech.edu.ng[4]

Abstract

Basic properties of probability with Poisson distribution is used in obtaining the coefficient bound by subordination principle which is the fundamental purpose of this work. A class of analytic function $f : \xi \to \mathbb{C}$ with unit disc $\xi : \{z \in \mathbb{C} : |z| < 1\}$ is established. Likewise known results of Fekete-Szegö inequalities type and the second bound of Toeplitz determinant are obtained.

1 Introduction and Preliminaries

Let $\xi = \{z \in \mathbb{C} : |z| < 1\}$. The class $\mathcal{A}$ consists of all analytic functions $f : \xi \to \mathbb{C}$ normalized by $f(0) = f'(0) - 1 = 0$ is of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \xi. \tag{1.1}$$

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*Corresponding author

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. Suppose Ω is the class of analytic functions $w : \xi \to \mathbb{C}$ with $w(0) = 0$ and $|w(z)| < 1$. Let the functions f and g be analytic in ξ; we say that f is subordinate to g in ξ, written as $f(z) \prec g(z)$ or simply $f \prec g$, see details [6], if there is some function $w \in \Omega$ such that $f(z) = g(w(z))$ for all $z \in \xi$. If g is univalent in ξ, then the following relation follows $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\xi) \subset g(\xi)$.

Let $\varphi(z)$ be an analytic and univalent function with positive real part in ξ with $\varphi(0) = 1$; $\varphi'(0) > 0$ and $\varphi(z)$ maps the unit disk ξ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

The familiar subclasses of $\mathcal{S}$ consisting of starlike and convex function is represented as $\mathcal{S}^*$ and $\mathcal{K}$ respectively.

Consider the convergent series $\sum_{k=0}^{\infty} a_k$ of non-negative terms and let s denote its sum:

$$s = \sum_{k=0}^{\infty} a_k, \quad a_k \geq 0.$$

This defines a generalized discrete probability distribution whose probability mass functions is given by $p(k) = \frac{a_k}{S}, \quad k = 0, 1, 2, \cdots$; the function $p(k)$ is the probability mass function because $p(k) \geq 0$ and $\sum_k p(k) = 1$. Of interest is the function ϕ defined by $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$. Since $\sum_{k=0}^{\infty} a_k$ is convergent, it follows that ϕ is convergent for $|z| < 1$ and $z = 1$ see details [1,2,5] which investigated generalized discrete probability distribution in conical domain. The power series, whose coefficients are probabilities of the generalized distributions in relation to (1.1) is of the form;

$$\nu(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda] \frac{a_{k-1}}{s} z^k, \quad 0 \leq \lambda < 1 \quad z \in \xi \tag{1.2}$$

A Toeplitz determinants is an upside down Hankel determinants, that is Hankel determinants have constant entries along the reverse diagonal while Toeplitz determinants have constant entries the diagonal.

Thomas and Halim [3] introduced the symmetric Toeplitz determinant $T_q(n)$ for analytic function $f(z)$ of the form $f(z) = z + a_2z^2 + a_3z^3 + ...$ defined as follows;

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}$$

(where $n, q = 1, 2, 3, ...a_1 = 1$ for $f(z) \in \mathcal{S}$).

In particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix},$$

$T_2(2) = |a_3^2 - a_2^2|$

Definition 1. A function $\nu(z) \in A$ given by (2.1) is in the class $S(\beta, \lambda, z)$ if it satisfies the condition:

$$\nu'(z)\left(\frac{z}{\nu(z)}\right)^{\beta} \prec \varphi(z), \quad (z \in \xi; 0 < \beta \leq 1)$$

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots, B_1 > 0$$

$Re(\varphi) > 0$ and $\nu(z)$ as defined in (1.2).

In this work, the following Lemmas are employed

Lemma 1. *If $p(z) = 1 + p_1z + p_2z^2 + \cdots \in P$, then the following shape inequality holds:*

$$|p_k| \leq 2, \quad (n \in \mathbb{N}). \tag{1.3}$$

Lemma 2. [1, 2] *If $\omega(z) = \omega_1z + \omega_2z^2 + \cdots \in \Omega$, then*

$$|w_2 - tw_1^2| \leq \max\{1, |t|\} \tag{1.4}$$

for any complex number t. The result is sharp for the function $w(z) = z$ or $w(z) = z^2$.

Lemma 3. [3] *If* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in P$, *then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{1.5}$$

for some $x, |x| \leq 1$ *and*

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\rho \tag{1.6}$$

for some complex value $\rho, |\rho| \leq 1$.

Lemma 4. *Let the function* $P(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *be a member of a class* $S(\alpha, \lambda, z)$. *Then,* $|p_2 - vp_1^2| \leq 2max\{1, |2v - 1|\}$.

2 Coefficient Bounds

Theorem 2.1. *If the function* $\nu(z) \in S(\beta, \lambda, z)$, *then*

$$\left|\frac{a_1}{s}\right| \leq \frac{2|B_1|}{(\lambda + 1)(2 - \beta)}$$

$$\left|\frac{a_2}{s}\right| \leq \frac{|B_1|(2 + 4Y)}{(2\lambda + 1)(3 - \beta)}$$

$$\left|\frac{a_3}{s}\right| \leq \frac{1}{(1 + 3\lambda)(4 - \beta)}\left(2|B_1| + 4X + 8\left(|B_3| + \frac{\beta|B_1|^3(\beta + 1)(\beta - 4)}{6(2 - \beta)^3} - \frac{|B_1|^2}{(2 - \beta)(3 - \beta)}Y\right)\right)$$

where $X = \left(2B_2 - \frac{B_1^2(\beta^2 - 4\beta)}{(2-\beta)(3-\beta)}\right)$ *and* $Y = \left(\frac{B_2}{B_1} - \frac{B_1(\beta^2 - 3\beta)}{2(2-\beta)^2}\right)$.

Proof. For $\nu(z) \in S(\beta, \lambda, z)$ then there exist $\omega(z)$ called a Schwarz function with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\nu'(z)\left(\frac{z}{\nu(z)}\right)^{\beta} = \varphi(\omega(z)). \tag{2.1}$$

Now, insert the values of $\nu(z)$ and $\nu'(z)$ in (2.1) we have

$$\left(1+\sum_{k=2}^{\infty}k[1+(k-1)\lambda]\frac{a_{k-1}}{s}z^{k-1}\right)\left(1+\sum_{k=2}^{\infty}[1+(k-1)\lambda]\frac{a_{k-1}}{s}z^{k-1}\right)^{-\beta} \quad (2.2)$$

then (2.2) becomes

$$\left(1+2(\lambda+1)\frac{a_1}{s}z+3(2\lambda+1)\frac{a_2}{s}z^2+4(3\lambda+1)\frac{a_3}{s}z^3+\cdots\right)\left(1+(\lambda+1)\frac{a_1}{s}z+(2\lambda+1)\frac{a_2}{s}z^2 + (3\lambda+1)\frac{a_3}{s}z^3+\cdots\right)^{-\beta} \quad (2.3)$$

simplifying $\left(1+(\lambda+1)\frac{a_1}{s}z+(2\lambda+1)\frac{a_2}{s}z^2+(3\lambda+1)\frac{a_3}{s}z^3+\cdots\right)^{-\beta}$ further we obtain,

$$1-\beta(\lambda+1)\frac{a_1}{s}z+\left(\frac{\beta(\beta+1)}{2}(\lambda+1)^2\frac{a_1^2}{s^2}-\beta(2\lambda+1)\frac{a_2}{s}\right)z^2+ \left(-\beta(3\lambda+1)\frac{a_3}{s}+\beta(\beta+1)[(\lambda+1)(2\lambda+1)]\frac{a_1a_2}{s^2}-\frac{\beta(\beta+1)(\beta+2)}{6}(\lambda+1)^3\frac{a_1^3}{s^3}\right)z^3+\cdots \quad (2.4)$$

then (2.3) becomes

$$1+\left((\lambda+1)(2-\beta)\frac{a_1}{s}\right)z+\left((\lambda+1)^2\left[\frac{\beta(\beta+1)}{2}-2\beta\right]\frac{a_1^2}{s^2}+(2\lambda+1)(3-\beta)\frac{a_2}{s}\right)z^2 +\left((3\lambda+1)(4-\beta)\frac{a_3}{s}+(2\lambda+1)(\lambda+1)(\beta^2-4\beta)\frac{a_1a_2}{s^2}+(\lambda+1)^3(\beta^2+\beta)\left(1-\frac{(\beta+2)}{6}\right)\frac{a_1^3}{s^3}\right)z^3+\cdots \quad (2.5)$$

then

$$\varphi(\omega(z))=1+(B_1w_1)z+(B_1w_2+B_2w_1^2)z^2+(B_1w_3+2B_2w_1w_2+B_3w_1^3)z^3+\cdots \quad (2.6)$$

Equating (2.5) and (2.6), compare the coefficient of z, z^2 and z^3, we get

$$\frac{a_1}{s}=\frac{B_1w_1}{(\lambda+1)(2-\beta)} \quad (2.7)$$

$$\frac{a_2}{s} = \frac{B_1}{(2\lambda+1)(3-\beta)}\left[w_2 + Yw_1^2\right] \tag{2.8}$$

$$\frac{a_3}{s} = \frac{1}{(1+3\lambda)(4-\beta)}\left(B_1w_3 + w_1w_2X + w_1^3\left(B_3 + \frac{\beta B_1^3(\beta+1)(\beta-4)}{6(2-\beta)^3} - \frac{B_1^2}{(2-\beta)(3-\beta)}Y\right)\right) \tag{2.9}$$

where $X = \left(2B_2 - \frac{B_1^2(\beta^2-4\beta)}{(2-\beta)(3-\beta)}\right)$ and $Y = \left(\frac{B_2}{B_1} - \frac{B_1(\beta^2-3\beta)}{2(2-\beta)^2}\right)$. □

Theorem 2.2. *Let* $0 < \beta \leq 1, |w_n| = 2, |B_k| \leq 2$ *and* $0 \leq \lambda < 1$. *If* $K_\phi \in S(\beta, \lambda, z)$, *where* $X = \left(2B_2 - \frac{B_1^2(\beta^2-4\beta)}{(2-\beta)(3-\beta)}\right)$ *and* $Y = \left(\frac{B_2}{B_1} - \frac{B_1(\beta^2-3\beta)}{2(2-\beta)^2}\right)$, *then*

$$\left|\frac{a_1}{s}\right| \leq \frac{2|B_1|}{(\lambda+1)(2-\beta)}$$

$$\left|\frac{a_2}{s}\right| \leq \frac{|B_1|(2+4Y)}{(2\lambda+1)(3-\beta)}$$

$$\left|\frac{a_3}{s}\right| \leq \frac{1}{(1+3\lambda)(4-\beta)}\left(2|B_1| + 4X + 8\left(|B_3| + \frac{\beta|B_1|^3(\beta+1)(\beta-4)}{6(2-\beta)^3} - \frac{|B_1|^2}{(2-\beta)(3-\beta)}Y\right)\right).$$

Proof. From(2.7), (2.8) and (2.9), let $|w_n| = 2$

$$\left|\frac{a_1}{s}\right| \leq \frac{2|B_1|}{(\lambda+1)(2-\beta)} \tag{2.10}$$

$$\left|\frac{a_2}{s}\right| \leq \frac{|B_1|(2+4Y)}{(2\lambda+1)(3-\beta)} \tag{2.11}$$

$$\left|\frac{a_3}{s}\right| \leq \frac{1}{(1+3\lambda)(4-\beta)}\left(2|B_1| + 4X + 8\left(|B_3| + \frac{\beta|B_1|^3(\beta+1)(\beta-4)}{6(2-\beta)^3} - \frac{|B_1|^2}{(2-\beta)(3-\beta)}Y\right)\right) \tag{2.12}$$

where $X = \left(2B_2 - \frac{B_1^2(\beta^2-4\beta)}{(2-\beta)(3-\beta)}\right)$ and $Y = \left(\frac{B_2}{B_1} - \frac{B_1(\beta^2-3\beta)}{2(2-\beta)^2}\right)$. □

Remark 1. If $|B_n| \leq 2$, then using Lemma 1

$$\left|\frac{a_1}{s}\right| \leq \frac{4}{(\lambda+1)(2-\beta)} \tag{2.13}$$

$$\left|\frac{a_2}{s}\right| \leq \frac{2(2+4Y)}{(2\lambda+1)(3-\beta)} \tag{2.14}$$

$$\left|\frac{a_3}{s}\right| \leq \frac{1}{(1+3\lambda)(4-\beta)}\left(4+4X+\left(\frac{1}{4}+\frac{\beta(\beta+1)(\beta-4)}{6(2-\beta)^3}-\frac{1}{2(2-\beta)(3-\beta)}Y\right)\right). \tag{2.15}$$

Theorem 2.3. *Let the function $P(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be a member of a class $S(\alpha, \lambda, z)$. Then*

$$\left|w_2 - v w_1^2\right| \leq \frac{2B_1}{(2\lambda+1)(3-\beta)} max\left\{1, \left|\frac{2\mu(2\lambda+1)(3-\beta)+B_1\beta(\beta-3)(\lambda+1)^2}{(\lambda+1)^2(2-\beta)^2} - \frac{2B_2+B_1}{B_1}\right|\right\}.$$

Proof. From (2.7) and (2.8), we need to show that $\left|\frac{a_2}{s} - \mu\frac{a_1^2}{s^2}\right| = |w_2 - v w_1^2|$. Using Lemma 4,

$$\left|\frac{a_2}{s} - \mu\frac{a_1^2}{s^2}\right| = \left|\frac{B_1}{(2\lambda+1)(3-\beta)}\left(w_2+\left(\frac{B_2}{B_1}-\frac{B_1(\beta^2-3\beta)}{2(2-\beta)^2}\right)w_1^2\right)-\frac{\mu B_1^2 w_1^2}{(\lambda+1)^2(2-\beta)^2}\right| \tag{2.16}$$

$$\left|\frac{a_2}{s} - \mu\frac{a_1^2}{s^2}\right| = \frac{B_1}{(2\lambda+1)(3-\beta)}\left|w_2 - \left(\frac{\mu B_1(2\lambda+1)(3-\beta)}{(\lambda+1)^2(2-\beta)^2} - \left(\frac{B_2}{B_1} - \frac{B_1(\beta^2-3\beta)}{2(2-\beta)^2}\right)\right)w_1^2\right| \tag{2.17}$$

$$v = \frac{\mu B_1(2\lambda+1)(3-\beta)}{(\lambda+1)^2(2-\beta)^2} - \left(\frac{B_2}{B_1} - \frac{B_1\beta(\beta-3)}{2(2-\beta)^2}\right) \tag{2.18}$$

$$\therefore \left|\frac{a_2}{s} - \mu\frac{a_1^2}{s^2}\right| = \frac{B_1}{(2\lambda+1)(3-\beta)}\left|w_2 - v w_1^2\right|$$

hence

$$\left|w_2-vw_1^2\right| \leq \frac{2B_1}{(2\lambda+1)(3-\beta)} max\left\{1, \left|\frac{2\mu(2\lambda+1)(3-\beta)+B_1\beta(\beta-3)(\lambda+1)^2}{(\lambda+1)^2(2-\beta)^2} - \frac{2B_2+B_1}{B_1}\right|\right\}. \tag{2.19}$$

□

Theorem 2.4. *Let $\nu(z)$ be of the form (1.2) belongs to the class $S(\alpha, \lambda, z)$. Then,*

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left|\frac{4}{(2\lambda+1)^2(3-\beta)^2}(B_1+2B_2)^2 - \frac{4B_1^2}{(2-\beta)^2}\left(\frac{1}{(\lambda+1)^2} + \frac{4B_2(\beta^2-3\beta)}{(2\lambda+1)^2(3-\beta)^2}\right) - \frac{4B_1^3(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}\left(2 - \frac{B_1(\beta^2-3\beta)}{(2-\beta)}\right)\right|$$

$0 \leq \lambda < 1$, $0 < \beta \leq 1$ *and* $z \in \xi$.

Proof. Using the coefficients in Theorem 2.1, $\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right|$ becomes

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| = \left|\frac{B_1^2w_2^2}{(2\lambda+1)^2(3-\beta)^2} + \frac{2B_1B_2w_1^2w_2}{(2\lambda+1)^2(3-\beta)^2} - \frac{B_1^3w_1^2w^2(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^2(3-\beta)^2} + \frac{B_2^2w_1^4}{(2\lambda+1)^2(3-\beta)^2} - \frac{B_1^2B_2w_1^4(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^2(3-\beta)^2} + \frac{B_1^4w_1^4(\beta^2-3\beta)^2}{4(2\lambda+1)^2(2-\beta)^4(3-\beta)^2} - \frac{B_1^2w_1^2}{(\lambda+1)^2(2-\beta)^2}\right| \tag{2.20}$$

using Lemma 3 in (2.20)

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| = \left|\frac{B_1^2(w_1^4+2w_1^2xX+x^2X^2)}{4(2\lambda+1)^2(3-\beta)^2} + \frac{2B_1B_2w_1^2(w_1^2+(4-w_1^2)x)}{(2\lambda+1)^2(3-\beta)^2} - \frac{B_1^3w_1^2(\beta^2-3\beta)(w_1^2+(4-w_1^2)x}{2(2\lambda+1)^2(2-\beta)^2(3-\beta)^2} + \frac{B_2^2w_1^4}{(2\lambda+1)^2(3-\beta)^2} - \frac{B_1^2B_2w_1^4(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^2(3-\beta)^2} + \frac{B_1^4w_1^4(\beta^2-3\beta)^2}{4(2\lambda+1)^2(2-\beta)^4(3-\beta)^2} - \frac{B_1^2w_1^2}{(\lambda+1)^2(2-\beta)^2}\right| \tag{2.21}$$

$$\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right|=\left|\frac{B_1^2w_1^4}{4(2\lambda+1)^2(3-\beta)^2}+\frac{B_1^2w_1^2xX}{2(2\lambda+1)^2(3-\beta)^2}+\frac{B_1^2x^2X^2}{4(2\lambda+1)^2(3-\beta)^2}\right.$$
$$+\frac{B_1B_2w_1^4}{(2\lambda+1)^2(3-\beta)^2}+\frac{4B_1B_2w_1^2x}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1B_2w_1^4x}{(2\lambda+1)^2(3-\beta)^2}$$
$$-\frac{B_1^3w_1^4(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}-\frac{2B_1^3w_1^2(\beta^2-3\beta)x}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}$$
$$+\frac{B_1^3w_1^4(\beta^2-3\beta)x}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}+\frac{B_2^2w_1^4}{(2\lambda+1)^2(3-\beta)^2}$$
$$-\frac{B_1^2B_2w_1^4(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(2-\beta)^2}$$
$$\left.+\frac{B_1^4w_1^4(\beta^2-3\beta)^2}{4(2\lambda+1)^2(2-\beta)^4(3-\beta)^2}-\frac{B_1^2w_1^2}{(\lambda+1)^2(2-\beta)^2}\right|. \tag{2.22}$$

We assume that $X=(4-w_1^2)$. Applying Triangular Inequality and let $w_1=w$,

$$\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right|=\left|\frac{B_1^2w_1^4}{4(2\lambda+1)^2(3-\beta)^2}+\frac{B_1B_2w_1^4}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1^3w_1^4(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}\right.$$
$$+\frac{B_2^2w_1^4}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1^2B_2w_1^4(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(2-\beta)^2}$$
$$+\frac{B_1^4w_1^4(\beta^2-3\beta)^2}{4(2\lambda+1)^2(2-\beta)^4(3-\beta)^2}-\frac{B_1^2w_1^2}{(\lambda+1)^2(2-\beta)^2}$$
$$+\frac{B_1^2w_1^2(4-w_1^2)x}{2(2\lambda+1)^2(3-\beta)^2}+\frac{B_1^2x^2(4-w_1^2)^2}{4(2\lambda+1)^2(3-\beta)^2}$$
$$+\frac{4B_1B_2w_1^2x}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1B_2w_1^4x}{(2\lambda+1)^2(3-\beta)^2}$$
$$-\frac{2B_1^3w_1^2x(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}$$
$$\left.+\frac{B_1^3w_1^4x(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^4(3-\beta)^2}\right| \tag{2.23}$$

$$\begin{aligned}\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right| \leq \Bigg| & \frac{B_1^2 w_1^4}{4(2\lambda+1)^2(3-\beta)^2}+\frac{B_1 B_2 w_1^4}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1^3 w_1^4(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2} \\ & +\frac{B_2^2 w_1^4}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1^2 B_2 w_1^4(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(2-\beta)^2} \\ & +\frac{B_1^4 w_1^4(\beta^2-3\beta)^2}{4(2\lambda+1)^2(2-\beta)^4(3-\beta)^2}-\frac{B_1^2 w_1^2}{(\lambda+1)^2(2-\beta)^2}\Bigg| \\ & +\frac{2B_1^2 w_1^2|x|}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1^2 w_1^4|x|}{2(2\lambda+1)^2(3-\beta)^2}+\frac{4B_1^2|x|^2}{(2\lambda+1)^2(3-\beta)^2} \\ & -\frac{2B_1^2 w_1^2|x|^2}{(2\lambda+1)^2(3-\beta)^2}+\frac{B_1^2 w_1^4|x|^2}{4(2\lambda+1)^2(3-\beta)^2}+\frac{4B_1 B_2 w_1^2|x|}{(2\lambda+1)^2(3-\beta)^2} \\ & -\frac{B_1 B_2 w_1^4|x|}{(2\lambda+1)^2(3-\beta)^2}-\frac{2B_1^3 w_1^2|x|(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2} \\ & +\frac{B_1^3 w_1^4|x|(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}=\upsilon(w,|x|). \end{aligned} \tag{2.24}$$

Differentiating $\upsilon(w,|x|)$ partially with respect to $|x|$ and clearly $\varphi'(|x|)>0$ on $[0,1]$ which implies $\varphi(|x|,w)\leq\varphi(1,w)$

$$\begin{aligned}\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right| \leq \Bigg| & \frac{B_1^2 w_1^4}{4(2\lambda+1)^2(3-\beta)^2}+\frac{B_1 B_2 w_1^4}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1^3 w_1^4(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2} \\ & +\frac{B_2^2 w_1^4}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1^2 B_2 w_1^4(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(2-\beta)^2} \\ & +\frac{B_1^4 w_1^4(\beta^2-3\beta)^2}{4(2\lambda+1)^2(2-\beta)^4(3-\beta)^2}-\frac{B_1^2 w_1^2}{(\lambda+1)^2(2-\beta)^2}\Bigg| \\ & +\frac{2B_1^2 w_1^2}{(2\lambda+1)^2(3-\beta)^2}-\frac{B_1^2 w_1^4}{2(2\lambda+1)^2(3-\beta)^2}+\frac{8B_1^2|x|}{(2\lambda+1)^2(3-\beta)^2} \\ & -\frac{4B_1^2 w_1^2|x|}{(2\lambda+1)^2(3-\beta)^2}+\frac{B_1^2 w_1^4|x|}{2(2\lambda+1)^2(3-\beta)^2}+\frac{4B_1 B_2 w_1^2}{(2\lambda+1)^2(3-\beta)^2} \\ & -\frac{B_1 B_2 w_1^4}{(2\lambda+1)^2(3-\beta)^2}-\frac{2B_1^3 w_1^2(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2} \\ & +\frac{B_1^3 w_1^4(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2} \end{aligned} \tag{2.25}$$

since $|x| = 1$ and $w \in [0, 2]$, (2.25) becomes

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left|\frac{B_1^2 w^4}{4(2\lambda+1)^2(3-\beta)^2} + \frac{B_1 B_2 w^4}{(2\lambda+1)^2(3-\beta)^2} - \frac{B_1^3 w^4(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}\right.$$
$$+ \frac{B_2^2 w^4}{(2\lambda+1)^2(3-\beta)^2} - \frac{B_1^2 B_2 w^4(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(2-\beta)^2}$$
$$\left. + \frac{B_1^4 w^4(\beta^2-3\beta)^2}{4(2\lambda+1)^2(2-\beta)^4(3-\beta)^2} - \frac{B_1^2 w^2}{(\lambda+1)^2(2-\beta)^2}\right|$$
$$+ \frac{2B_1^2 w^2}{(2\lambda+1)^2(3-\beta)^2} - \frac{B_1^2 w^4}{2(2\lambda+1)^2(3-\beta)^2} + \frac{8B_1^2}{(2\lambda+1)^2(3-\beta)^2}$$
$$- \frac{4B_1^2 w^2}{(2\lambda+1)^2(3-\beta)^2} + \frac{B_1^2 w^4}{2(2\lambda+1)^2(3-\beta)^2} + \frac{4B_1 B_2 w^2}{(2\lambda+1)^2(3-\beta)^2}$$
$$- \frac{B_1 B_2 w^4}{(2\lambda+1)^2(3-\beta)^2} - \frac{2B_1^3 w^2(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}$$
$$+ \frac{B_1^3 w^4(\beta^2-3\beta)}{2(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}. \tag{2.26}$$

Obtaining the maximum value for $|T_2(2)|$ at $w[0, 2]$, i.e., $w \leq 2$, we have

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left|\frac{4B_1^2}{(2\lambda+1)^2(3-\beta)^2} + \frac{16B_1 B_2}{(2\lambda+1)^2(3-\beta)^2} - \frac{8B_1^3(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}\right.$$
$$+ \frac{16B_2^2}{(2\lambda+1)^2(3-\beta)^2} - \frac{16B_1^2 B_2(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^2(3-\beta)^2}$$
$$\left. - \frac{4B_1^2}{(\lambda+1)^2(2-\beta)^2} + \frac{4B_1^4(\beta^2-3\beta)^2}{(2\lambda+1)^2(2-\beta)^4(3-\beta)^2}\right|$$
$$+ \frac{8B_1^2}{(2\lambda+1)^2(3-\beta)^2} - \frac{8B_1^2}{(2\lambda+1)^2(3-\beta)^2}$$
$$+ \frac{8B_1^2}{(2\lambda+1)^2(3-\beta)^2} - \frac{16B_1^2}{(2\lambda+1)^2(3-\beta)^2}$$
$$+ \frac{8B_1^2}{(2\lambda+1)^2(3-\beta)^2} + \frac{16B_1 B_2}{(2\lambda+1)^2(3-\beta)^2}$$
$$- \frac{16B_1 B_2}{(2\lambda+1)^2(3-\beta)^2} - \frac{8B_1^3(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}$$
$$+ \frac{8B_1^3(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2} \tag{2.27}$$

$$\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right| \le \left|\frac{4B_1^2}{(2\lambda+1)^2(3-\beta)^2}+\frac{16B_1B_2}{(2\lambda+1)^2(3-\beta)^2}-\frac{8B_1^3(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}\right.$$
$$+\frac{16B_2^2}{(2\lambda+1)^2(3-\beta)^2}-\frac{16B_1^2B_2(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^2(3-\beta)^2}$$
$$\left.-\frac{4B_1^2}{(\lambda+1)^2(2-\beta)^2}+\frac{4B_1^4(\beta^2-3\beta)^2}{(2\lambda+1)^2(2-\beta)^4(3-\beta)^2}\right| \quad (2.28)$$

$$\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right| \le \left|\frac{4}{(2\lambda+1)^2(3-\beta)^2}(B_1+2B_2)^2\right.$$
$$-\frac{4B_1^2}{(2-\beta)^2}\left(\frac{1}{(\lambda+1)^2}+\frac{4B_2(\beta^2-3\beta)}{(2\lambda+1)^2(3-\beta)^2}\right)$$
$$\left.-\frac{4B_1^3(\beta^2-3\beta)}{(2\lambda+1)^2(2-\beta)^3(3-\beta)^2}\left(2-\frac{B_1(\beta^2-3\beta)}{(2-\beta)}\right)\right|. \quad (2.29)$$

□

Remark 2. If $\lambda = 1$ and $0 < \beta \le 1$, then by using Lemma 1,

$$\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right| \le 17.7.$$

3 Conclusion

In this research we obtained the coefficient bounds which was used to derived the Fekete-Szegö type inequality and second bound of Toeplitz determinant. All results obtained in this research are new to the best of our knowledge.

References

[1] Abiodun Tinuoye Oladipo, Generalized distribution associated with univalent functions in conical domain, *An. Univ. Oradea Fasc. Mat.* 26(1) (2019), 161-167.

[2] Abiodun Tinuoye Oladipo, Bounds for probabilities of the generalized distribution defined by generalized polylogarithm, *Punjab University Journal of Mathematics* 51(7) (2019), 19-26.

[3] D. K. Thomas and S. A. Halim, Toeplitz matrices whose elements are the coefficient of starlike and close-to-convex functions, *Bull. Malays. Math. Sci. Soc.* 40(4) (2017), 1781-1790.

[4] Ş. Altinkaya and S. Yalçin, Poisson distribution series for analytic univalent functions, *Complex Anal. Oper. Theory* 12 (2018), 1315-1319. https://doi.org/10.1007/s11785-018-0764-y

[5] Ş. Altinkaya and S. Yalçin, Poisson distribution series for certain subclasses of starlike functions with negative coefficients, *An. Univ. Oradea Fasc. Mat.* 24(2) (2017), 5-8.

[6] O. E. Opaleye, S. O. Fawale, P. O. Awoleye and P. O. Oyewo, Some geometry properties of a class of analytic function with Gegenbauer polynomial, *Global Scientific Journals* 10(3) (2022), 94-100.