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On Some Boundary Value Methods

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Abstract

Circumventing order restrictions on numerical methods designed for the integration of stiff initial value problem is the concern here via Boundary Value Method. The attainable order $p = k + v$ and linear stability properties of the methods are discussed. The numerical test on some stiff problems shows that the new methods developed, compare favourably with existing methods, with ODE15s of MATLAB used as reference numerical solution.

1. Introduction

The Dahlquist second order barrier [9] places stringent requirement for numerical methods designed for the approximation of the solutions of stiff initial value problems. The stringent requirement is that LMF for integrating stiff IVPs must possess A-stability. The highest order the LMF can attain is two and this is attainable by implicit subclass of the LMF. The conventional linear multi step formula (LMF) considered in [9] was regarded as unstable initial value methods and which was later formulated as a boundary value method (BVM) by Amodio et al. [2]. However, there is a need to look for a polynomial with best stability properties for all step number k of the LMF. It is well-known that BVMs are efficient for solving ODEs since it can avoid the Dahlquist order barrier of classical initial methods for ODEs see [2-7, 18-22]. The BVMs and block BVMs have been used to approximate the solutions of delay differential equations [26, 27] and differential algebraic equations in [18]. Recently, [28] studied Hamiltonian problems. Some other recent classes of BVM that has being analyzed by some authors includes the generalized second derivative linear multistep method based on Enright [20]

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where methods developed limitations for all step number $k \geq 1$. In [19] a new family of second derivative generalized Adams-type methods (SDGAMs) for the numerical solution of stiff IVPs in ODEs is proposed. The new formulas are found to be $0_{v,k-v}$-stable and $A_{v,k-v}$-stable with $(v, k-v)$-boundary conditions for all values of $k \geq 1$ and are of order $p = 2k + 2$. Boundary value techniques on IVP are considered in [24, 25]. The aim of this paper is to develop new boundary value method for stiff initial value problem. The organization is as follows; in Section 2 a brief review of Boundary Value Method and its stability is discussed, Section 3 contains the construction of the new scheme, while the experiment and implementation are documented in Section 4. The conclusion is in Section 5

2. Boundary Value Method (BVM)

Consider the initial value problems (IVPs) of ODEs

$$y' = f(x, y), \;\; y(a) = \eta. \;\; f\colon \Re \times \Re^m \to \Re^m, y\colon \Re \to \Re^m, \; x\epsilon[a, b]. \tag{1}$$

A k-step linear multistep formula for solving the IVPs in (1) is given as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j,} \tag{2}$$

where y_n signifies the discrete approximation of the solution $y(x)$ at $x = x_n$ and $h = \frac{(x - x_0)}{N}$ and $f_n = f(x_n, y_n)$. The method (2) required k initial solution values $y_0, y_1, \dots, y_{k-1}$ to obtain an output y_k. That is the continuous solution $y(x)$ in (1) is approximated by means sequence discrete of discrete values y_n. Nevertheless, due to order and A-stability barrier in LMF, a new approach consists of fixing the first $k_1 (\leq k)$ values of the discrete solution $y_0, y_1, \dots, y_{k_1-1}$ and the last $k_2 = k - k_1$ values $y_{N-k_2+1}, \dots, y_N$ yielding the discrete problem,

$$\sum_{i=-k_1}^{k_2} \alpha_{i+k_1} y_{n+i} = h \sum_{i=k_1}^{k_2} \beta_{i+k_1} f_{n+i}\,, \; n - k_1, \dots, n = k_1, k_1 + k_2 = k \tag{3}$$

$$y_0, y_1, \dots, y_{k_1-1}, \quad y_{N-k_2+1}, \dots, y_N \quad \text{fixed}$$

where the continuous IVPs (1) is approximated by means of discrete BVPs. The methods obtained are referred as Boundary Value Methods (BVMs), with (k_1, k_2)-boundary conditions. It can be observed that for $(k_1 = k)$ and $k_2 = 0$ in (3), one has the initial value methods (IVMs). So, the class of IVMs is a subclass of BVMs for ODEs based on LMF. Nevertheless, the continuous problem (1) provides only the initial value y_0 where the extra solution values need to be provided. In the sense of [6], the k additional values $y_0, y_1, \dots, y_{k_1-1}, \; y_{N-k_2+1}, \dots, y_N$ are obtained by introducing a set of k additional equations.

To better understand the BVM, the following definitions shall be considered in the next subsection

2.1. Generalization of IVM to BVM

The LMF (2) are generalized to BVM by introducing the following definitions;

Definition 1. Consider a polynomial $p(z)$ such that p is a function of a complex variable *z*, calculated by the formula:

$$p(z) = \sum_{j=0}^{k} \alpha_j z^{k-j}, \qquad \alpha_k \neq 0. \tag{4}$$

The zeros of the polynomial $p(z)$ in (4) are denoted by $z_i, i = 1, \dots, k$. If the zeros z_i are simple for all values of *i*.

1. The polynomial $p(z)$ in (4) is called the *Schur polynomial* that is $S_{k_1 k_2}$ if for all values of $i = 1, \dots, k$ the inequality $|z_i| < 1$ holds.
2. The polynomial $p(z)$ in (4) is called the *Von Neumann polynomial* that is $N_{k_1 k_2}$ if for all values of $i = 1, \dots, k$ the inequality $|z_i| \leq 1$ is satisfied.

Definition 3. A BVM with (k_1, k_2)-boundary conditions are $0_{k_1 k_2}$-*stable* if $\rho(z)$ is a $N_{k_1 k_2}$-polynomial.

From Definition 3, $0_{k_1 k_2}$ -stability reduces to the usual zero stability for LMF when $k_1 = k$ and $k_2 = 0$.

Definition 4.

(a) For a given $q \in C$, a BVM with (k_1, k_2)-boundry conditions is (k_1, k_2)-*absolutely stable* if $\prod(z, q)$ is a $S_{k_1 k_2}$-polynomial, again, (k_1, k_2)-absolute stability reduces to the usual notion of absolute stability when $k_1 = k$ and $k_2 = 0$ for LMF.

(b) The region of (k_1, k_2)-absolute stability of the method, $D_{k_1 k_2} = \{q \in C : \prod(z, q)\}$ is a $S_{k_1 k_2}$ *polynomial*. Here $\prod(z, q)$ is a polynomial of type $(k_1, 0, k_2)$.

3. Construction of the Proposed New Boundary Value Methods (NBVM)

The k-step LMF (2) was considered in [9] is an unstable initial value method, this was formulated as a boundary value method in [2]. The BVM based on LMF in (2) is known as Top Order Methods (TOM) and is zero-stable for only odd step number k, while for even step number of k are of less interesting due to their nonzero-stable nature. However, there is need to search for polynomial with best stability properties for all step number k of the LMF in (2). Consider the Linear Multistep Formula

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{v} \beta_j f_{n+j}; \qquad \beta_v = 1, \qquad k_1 = v, \qquad k_2 = k - v \tag{5}$$

$$\underset{\mathrm{m}(1)}{y_1, y_2, \dots, y_{v-1}} \qquad \underset{\mathrm{m}(2)}{y_v, \dots, y_{N-k-v-1}} \qquad \underset{\mathrm{m}(3)}{y_{N-k-v}, \dots, y_N} \quad ; n = 0$$

where the solution values m(1) and m(3) are given by the low order LMF. The additional formula that couples with (5) are

$$y_i - y_{i-1} = \sum_{j=0}^{k+v-1} \beta_{ji} f_i, \qquad i = 1, \dots, v-2 \tag{6}$$

and

$$y_{N+i-1} - y_{N+i-2} = \sum_{j=0}^{k+v-1} \beta_{ji} f_{N+v-2-j}, \qquad i = \begin{cases} 0, \dots, v-1 & \text{odd} \\ 1, \dots, v-1 & \text{even} \end{cases} \tag{7}$$

(6) and (7) are the initial and final formula.

$$v = \begin{cases} \dfrac{k+1}{2}, & for\ odd\ k, \\ \dfrac{k+2}{2} & for\ even\ k, \end{cases} \qquad k = 1,2,3, \dots \tag{8}$$

The method (5) is a boundary value method with $v \neq k$. The values of v is selected such that it gives the best stability properties for all $k \geq 1$.

Here $k = k_1 + k_2$ and $k_1 = v - 1$, $k_2 = v$ and y_{n+j} signifies the discrete approximation of the analytic solution $y(x_{n+j})$, $f_{n+j} = f(x_{n+j}, y_{n+j})$, $j = 0(1)k$ and $h = \frac{X - x_0}{s}$, $s = N$. The $k + v$ parameters $\{\alpha_j\}_{j=0}^{k}$ and $\{\beta_j\}_{j=0}^{v}$ allow the construction of the methods (5) of highest order $p = k + v$.

Some of the step number k in the method (5) is characterized as reversed methods. Thus, the reverse method of (5) is given as

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = -h \sum_{j=0}^{v} \beta_j f_{n-j}. \tag{9}$$

The (9) can conveniently be written as

$$\sum_{j=0}^{k} \alpha_{k-j} y_{n+k} = -h \sum_{j=k-v}^{k} \beta_{j-k+v} f_{n+k}, \tag{10}$$

with $\beta_0 = \beta_1 = \beta_{k-v+1} = 0$. The method (9) shall be referred to as Reversed New Boundary Value Methods (RNBVM). The coefficient of the reversed form in (5) and (9) are presented in Table 1 and are obtained from the transformation.

$$\alpha_j \rightarrow \alpha_j, \beta_j \rightarrow \beta_j, y_{n+j} \rightarrow y_{n-j}, h \rightarrow -h \tag{11}$$

Then the transformation (9) to (10) is

$$\alpha_j \rightarrow \alpha_{k-j}, \quad \beta_j \rightarrow \beta_{j-k-v}, \qquad y_n \rightarrow y_{n+k}, \qquad f_n \rightarrow f_{n+k}, \qquad h \rightarrow -h. \tag{12}$$

The coefficients for the method in (5), for $k = 1(1)6$ are presented in Table 1. The methods are found to be $A_{k_1k_2}$-stable and are used along with (k_1, k_2) boundary conditions.

Table 1: The coefficients of new schemes in (5) and (9) for $k = 1, 2, \ldots, 5$

k	α_0	α_1	α_2	α_3	α_4	α_5	α_6	β_0	β_1	β_2	β_3
1	-2	2						1			
2	-3	0	3					1	4		
3	$-\frac{10}{9}$	-1	2	$\frac{1}{9}$				$\frac{1}{3}$	2		
4	$\frac{47}{48}$	-4	$\frac{9}{4}$	$\frac{8}{3}$	$\frac{1}{16}$			$\frac{1}{4}$	3	$\frac{9}{2}$	
5	$\frac{247}{600}$	$\frac{19}{18}$	$\frac{1}{2}$	$\frac{13}{6}$	$\frac{1}{8}$	$\frac{1}{200}$		$\frac{1}{10}$	$\frac{3}{2}$	3	
6	$-\frac{68}{225}$	$\frac{-245}{75}$	$-\frac{7}{2}$	$\frac{40}{9}$	$\frac{8}{3}$	$\frac{2}{25}$	$-\frac{1}{450}$	$\frac{1}{15}$	$\frac{8}{5}$	6	$\frac{16}{3}$

3.1. The order of NBVM

The linear difference operator $\mathcal{L}[y(x),h]$ connected to the NBVM of (5) is given by,

$$\mathcal{L}[y(x),h] = \sum_{j=0}^{k} \alpha_j\, y(x+jh) = h \sum_{j=0}^{v-1} \beta_j\, y'(x+jh) \tag{13}$$

where $y(x)$ is a continuously differentiable function in $[x_0, X]$, [11, 17]. The equation (11) gives the local truncation error of the method in (5), since $y(x)$ is a supposed solution of (2). Let $y(x)$ have higher derivatives $y'(x+jh)$, $y''(x+jh)$,…, then by Taylor series expansion, (5) becomes

$$\mathcal{L}[y(x),h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \cdots + c_q h^q y^q(x) + \cdots, \tag{14}$$

where

$$c_0 = \sum_{j=0}^{k} \alpha_j$$

$$c_1 = \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{v} \beta_j$$

$$c_2 = \sum_{j=0}^{k} \frac{j^2}{2}\alpha_j - \sum_{j=0}^{v} j\beta_j \tag{15}$$

$$\vdots$$

$$c_p = \sum_{j=0}^{k} \frac{j^p}{p!}\alpha_j - \sum_{j=0}^{v} \frac{j^{p-1}}{(p-1)!}\beta_j \qquad p = 0,1,\ldots$$

Thus, the following definition holds

Definition 5. The NBVM (5) has an order of order *p*, if

$$c_j = 0, \qquad j = 0(1)p, \qquad c_{p+1} \neq 0 \tag{16}$$

where $c_{p+1} \neq 0$ is the error constant and its principal local truncation error can be given as

$$lte = c_{p+1} h^{p+1} y^{(p+1)}(x) + O(h^{p+1}), \tag{17}$$

from equation (14).

Definition 6. The NBVM in (5) is said to be *consistent* if it has an order $p \geq 1$. To get the stability of this method in (5), it is applied on the test problem

$$y' = \lambda y, \qquad Re(\lambda) < 0. \tag{18}$$

This will help to obtain the boundary locus plot of the stability polynomial of the method in (5). Thus,

$$\lambda \sum_{j=0}^{k} \alpha_j r^j y_n - \lambda h \sum_{j=0}^{v} \beta_j r^j y_n = 0, \tag{19}$$

here $r^j y_n = y_{n+j}$. The stability polynomial associated with the method in (5) is given as

$$\prod(r, z) = \rho(r) - z\,\sigma(r) \tag{20}$$

with

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j, \qquad \sigma(r) = \sum_{j=0}^{v-1} \beta_j r^j \tag{21}$$

which are the first and second characteristics polynomial associated with (5). The necessary condition for having the boundary loci as a regular curve is for the form first characteristics polynomial $\rho(r)$ to have only one root of unit modulus. The stability plot of the method in (5) is given in Figures 1-5.

The composite scheme of (5), (6) and (7) is conveniently written in one block form as,

$$A_N Y_{n+1} + A_0 Y_n = h(B_N F_{n+1} + B_0 F_n); n = 0,1, \ldots \tag{22}$$

where the matrix $A = [a|A_N] \in R^{N\times(N+1)}$ and is given as

$$A = \begin{pmatrix} \alpha_0^{(1)} & | & \alpha_1^{(1)} & \alpha_2^{(1)} & \cdots & \cdots & \alpha_k^{(1)} \\ \vdots & | & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_0^{(v-1)} & | & \alpha_1^{(v-1)} & \cdots & \cdots & \alpha_2^{(1)} & \alpha_k^{(v-1)} \\ 0 & | & \alpha_0 & \cdots & \cdots & \alpha_k & 0 \\ 0 & | & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & | & 0 & \alpha_0 & \ddots & \ddots & \alpha_k \\ 0 & | & 0 & \alpha_0^{(k)} & \cdots & \cdots & \alpha_k^{(k)} \\ 0 & | & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & | & 0 & \alpha_0^{(k-v-1)} & \cdots & \cdots & \alpha_k^{(1)} \end{pmatrix}. \tag{23}$$

The matrix $B = [b|B_N] \in R^{N\times(N+1)}$ can be similarly defined with β_j instead of α_j and $\beta_j^{(i)}$ instead of $\alpha_j^{(i)}$ in A, the matrices A_0,B_0 are assumed to be $A_0 := ae_s^T \in \mathbb{R}^{s\times s}$ and $B_0 := be_s^T \in \mathbb{R}^{s\times s}$ with $e_s := (0,0,\dots,0,1)^T \in \mathbb{R}^s$. Hereafter, both matrices A and B are can be shown to be nonsingular.

$$Y_{n+1} = (y_{n+1}, y_{n+2}, \dots, y_{n+N})^T, Y_n = (y_{n-N+1}, y_{n-N+2}, \dots, y_n)^T \quad (24)$$

$$F_{n+1} = (f_{n+1}, f_{n+2}, \dots, f_{n+N})^T, F_n = (f_{n-N+1}, f_{n-N+2}, \dots, f_n)^T. \quad (25)$$

Definition 7. A block method is said to be *pre-stable* if the spectrum of the corresponding matrix pencil $(A_N - zB_N)$ is contained in $\mathbb{C}^+$.

The case of solving linear problem in (1), Gaussian elimination with partial pivoting is required, while Modified Newton approach is needed for non-linear problem. The stability plot of the methods in (5) and (9) are given in Figures 1-4. For step number $k = 1,2,\dots,20$.

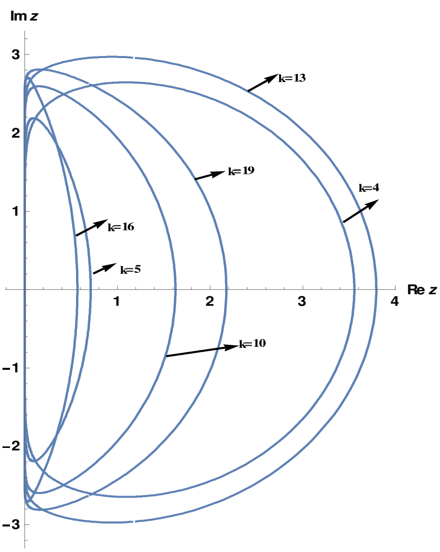

Fig. 1. Plots show boundary loci for $k =$ 4,5,10, 16 and 19.

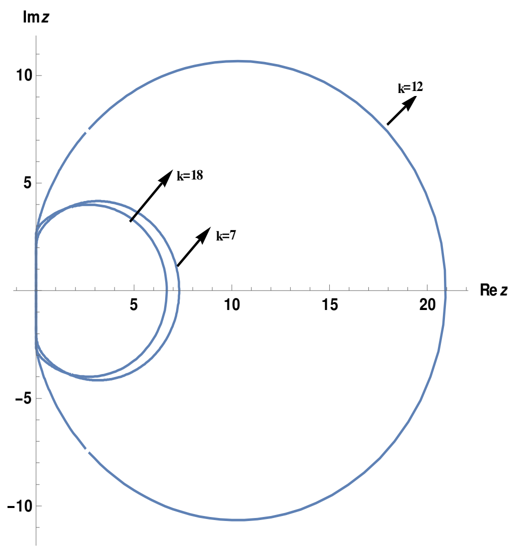

Fig. 2. Reserved plots showing boundary loci for $k = 7$,12 and 18.

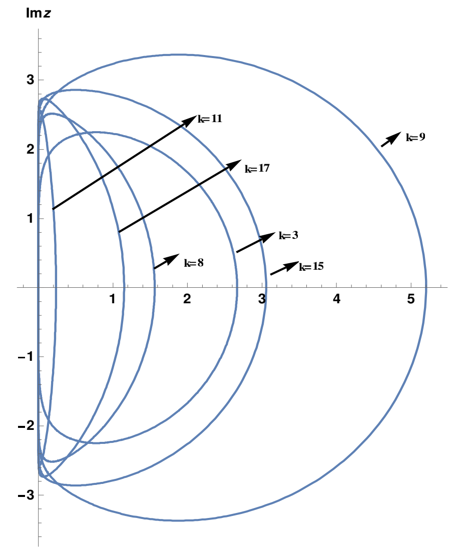

Fig. 3. Plot of Boundary loci of $k = 3, 8, 9, 11, 15$ and 17.

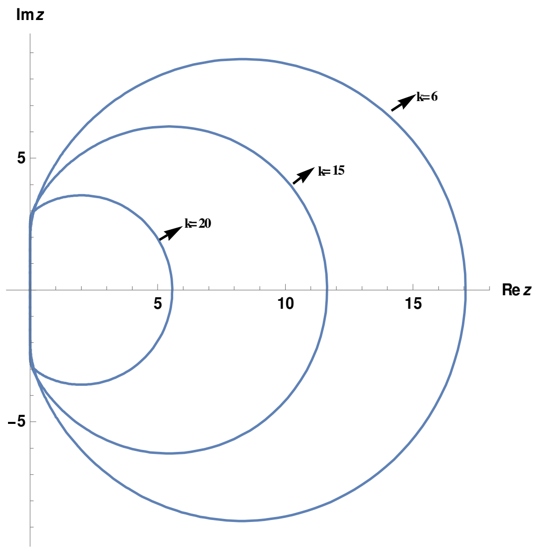

Fig. 4. Reversed Plot of Boundary loci of $k = 6, 15$ and 20.

Consider the example, when $k = 2$ in (5) with the main method

$$3y_{n+2} - 3y_n = h(f_n + 4f_{n+1}) + f_{n+2}\,;\; n = 0,1,\ldots,N-3 \quad C_5 = -\frac{1}{30}. \tag{26}$$

The method (27) is $A_{1,1}$-stable as shown in Figure 1 applied on (1) with one final condition

$$y_N - y_{N-1} = h\left(\frac{3}{8}f_N + \frac{19}{24}f_{N-1} - \frac{5}{24}f_{N-2} + \frac{1}{24}f_{N-3}\right) \tag{27}$$

for $k = 3$ the method in (5) is given as

$$-\frac{10}{9}y_n - y_{n+1} + 2\,y_{n+2} + \frac{1}{9}y_{n+3} = h\left(\frac{1}{3}f_n + 2f_{n+1}\right); n = 0,1,\ldots,N-4 \quad C_6 = \frac{1}{180}. \tag{28}$$

It is $A_{2,1}$-stable and applied on (1) with two initial conditions and one final condition.

The initial condition is given by

$$y_1 - y_0 = h\left(\frac{11}{720}f_n - \frac{37}{360}f_{n-1} - \frac{19}{30}f_{n-2} + \frac{173}{360}f_{n-3} - \frac{19}{720}f_{n-4}\right) \tag{29}$$

and the final is given

$$y_{N-1} - y_{N-2} = h\left(-\frac{19}{720}f_N + \frac{173}{360}f_{N-1} - \frac{19}{30}f_{N-2} - \frac{37}{360}f_{N-3} + \frac{11}{720}f_{N-4}\right). \quad (30)$$

4. Numerical Experiments and Results

Some linear and non-linear stiff problems were carried out to examine the accuracy and performance of the NBVM. All computations were done in MATLAB R2015a (8.5.0.197613) software package.

Problem 1. Consider the linear problem

$$y' = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad (31)$$

$$y(0) = \begin{pmatrix} e^{-2x} + e^{-40x}\,(\cos(40x) + \sin(40x)) \\ e^{-2x} - e^{-40x}\,(\cos(40x) + \sin(40x)) \\ 2e^{-40x}\,(\cos(40x) - \sin(40x)) \end{pmatrix}$$

Table 2 contains the maximum relative error $\frac{max}{1<i<3} \mid y_i(x_X) - y, h| / (1 + |y_i, h\ |)$ at the interval $0 < x_X \leq 1$ using the NBVMs2 and RNBVMs2. The performance compared with Generalized Adams Methods (GAMs5) of order $p = 6$ in [5], and the variable-step boundary value methods based on Reverse Adams Method [2] of order $p = 6$ [3]. It is observed that the proposed method in (5) perform better than the GAMs5 order $p = 6$ with less function evaluation.

Table 2: The numerical solution of Problem 1.

Step size (h)	NBVM (rate) $k = 2, P = 4$	NBVM (rate) $k = 3, P = 5$	Amodio (rate) $P = 6$	GAMs5 (rate) $p = 6$
0.05	1.84x10^{-1}	1.09x10^{-1}	5.70x10^{-2}	2.25x10^{-1}
0.025	1.99x10^{-2}	1.30x10^{-2}	8.70x10^{-3}	4.41x10^{-2}
0.0125	1.99x10^{-3}	1.25x10^{-3}	4.9x10^{-4}	6.49x10^{-3}
0.00625	1.57x10^{-4}	3.68x10^{-5}	1.2x10^{-5}	8.86x10^{-4}
0.003125	7.75x10^{-6}	7.5x10^{-7}	2.20x10^{-7}	9.88x10^{-5}

Problem 2. Predator and corrector model

$$\frac{d}{dx}y(x) = a_1(x)y(x) - a_2(x)y(x)u(x) \tag{32}$$

$$\frac{d}{dt}y(x) = b(x)y(x) - b_2(x)y(t)u(x) \tag{33}$$

$a_1(t) = 4 + \tan(t),\ a_2(t) = \exp(2t),\ b_1(t) = -2,\ b_2(t) = \cos(t),\ y(0) = -4,\ u(0) = 4$

In Table 3, the proposed method was compared with ETR/BDF, GBDF/BDF, TOM/BDF in [16]. It was observed in Table 3 the proposed methods perform better with less function evaluation, LU decomposition and fewer Jacobian evaluations than the compared the ETR/BDF, GBDF/BDF, TOM/BDF in [16].

Table 3: Numerical Solution to problem 2.

MTD_EMP	Tol	#funct_eva	#Reject	#LU_decomp	#Jacobian
ETR/BDF	10^{-2}	8000	0	400	100
	10^{-3}	9760	1	488	122
	10^{-4}	13520	1	680	169
	10^{-5}	18800	1	944	235
	10^{-6}	Failed test			
GDBF/BDF	10^{-2}	8000	0	400	100
	10^{-3}	9760	1	488	122
	10^{-4}	13520	1	680	169
	10^{-5}	18800	1	940	234
	10^{-6}	25920	1	1296	324
TOM/BDF	10^{-2}	8000	0	400	100
	10^{-3}	9760	1	488	122
	10^{-4}	13520	1	680	169
	10^{-5}	18800	1	944	235
	10^{-6}	Failed test			

k2	10^{-2}	560	0	70	34
	10^{-3}	632	0	79	34
	10^{-4}	808	0	101	34
	10^{-5}	848	0	106	34
	10^{-6}	4648	316	581	350
k3	10^{-2}	530	0	53	25
	10^{-3}	670	0	67	25
	10^{-4}	780	0	78	25
	10^{-5}	790	0	79	25
	10^{-6}	1020	0	102	25

Problem 3.

Vander Pol

$$y'' + \mu(y^2 - 1)y' + y = 0; \;\; y(0) = 2, \;\; y'(0) = 0, \;\; \mu > 0. \tag{34}$$

This is solved by transformation into a first-order system of two ODEs given by

$$y_1' = y_2 \tag{35}$$

$$y_2' = -y_1 + \mu y_2(1 - y_1^2); \;\; y_1(0) = 2, \;\; y_2(0) = 0. \tag{36}$$

The Vander Pol problem is used to demonstrate how robust the methods are in solving stiff nonlinear problems. The problem is solved for $\mu = 50$, using order $p = 5$ of the methods in (5) and step size $h = 0.0001$. The graph of the computed solution compared with that obtained using ODE15s is displayed in Figure 6.

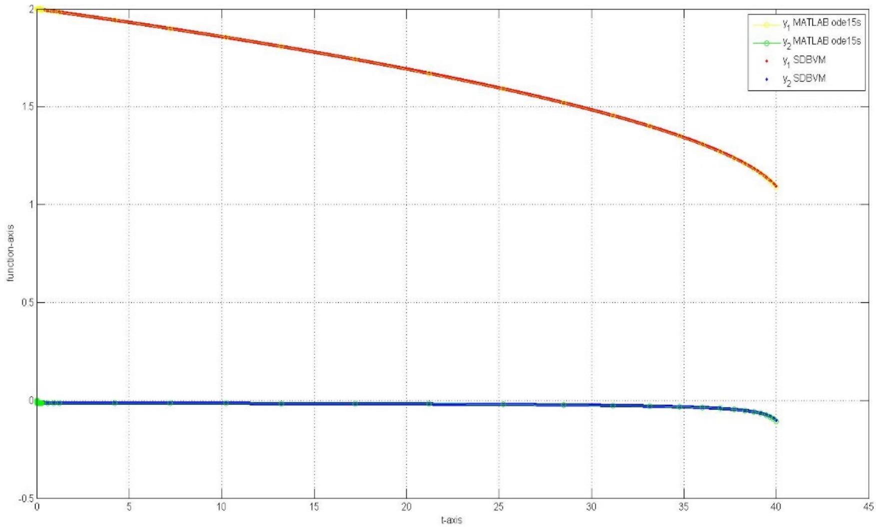

Figure 5: NBVM compared with ODE15s.

In Figure 5, the y_1 and y_2 are the numerical solution from the ODE15s from Problem 3.

The solution obtained from the NBVM in (5) coincides with the solution of ODE15s as shown in Figure 5.

5. Conclusion

This paper presents a NBVM with their stability properties. The NBVM in (5) has the stability at infinity for all k unlike the conventional LMF in (2) which stability is limited. The coefficients of the scheme are presented in Table 1. The methods are $A_{k_1k_2}$-stable and correctly implemented with $(k, k-v)$-boundary conditions. The NBVM of order $p = 5, k = 3$ and $p = 4, k = 2$ in (5) has been implemented in one block form (22) along with the extra initial method in (20) and final method in (21) on some stiff problems with their results shown in Tables 2, 3 and Figure 5. It is also seen that the proposed NBVM in (5) is well bounded. In Table 3, the proposed method was compared with ETR/BDF, GBDF/BDF, TOM/BDF in [16]. The new scheme performs better with less functions, LU decomposition and fewer Jacobian evaluations than the compared the ETR/BDF, GBDF/BDF, TOM/BDF as shown in Table 3.

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