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Some New Integral Inequalities for Convex Functions in (p, q)-Calculus

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Abstract

This paper presents Opial and Steffensen inequalities and also discussed q and (p, q)-calculus. Methods of convexity, (p, q)-differentiability and monotonicity of functions were employed in the analyses and new results related to the Opial's-type inequalities were established.

1 Introduction

Quantum calculus or q-calculus is the study of calculus without limits. This study was first discovered by the famous mathematician Leonhard Euler (1707–1783) in the way of Newton's work for infinite series. However, the study is well attributed to F. H. Jackson who discovered the q-derivative and q-integral in [6]. The q-calculus is widely studied by many researchers and it has applications in modern mathematical analysis, orthogonal polynomials, combinatorics, number theory, quantum theory, basic hypergeometric functions, mechanics, and special theory of relativity [12].

Many researchers have developed the theory of quantum calculus based on two-parameter (p, q)-integer that is used efficiently in many fields as in difference equations, Lie group, hypergeometric series, physical sciences, and so

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on. The (p,q)-calculus is known as two parameter quantum calculus which is a generalization of q-calculus and whose applications play important roles in physics, chemistry, orthogonal polynomials and number theory [7].

The classical Opial's integral inequality [13] established in 1960 is an inequality involving the integral of a function and its derivative as

$$\int_0^\lambda |\varphi(t)\varphi'(t)|dt \leq \frac{\lambda}{4}\int_0^\lambda |\varphi'(t)|^2 dt, \tag{1}$$

where $\varphi \in C^1[0,\lambda]$, $\varphi(x) > 0$ for $x \in [0,\lambda]$ and $\varphi(0) = \varphi(\lambda) = 0$. The coefficient $\lambda/4$ is the best constant possible.

Further generalizations of the classical Opial's inequality were established in [16] as

$$\int_a^b |\varphi(t)\varphi'(t)|dt \leq \frac{b-a}{2}\int_a^b |\varphi'(t)|^2 dt \tag{2}$$

and

$$\int_a^b |\varphi(t)\varphi'(t)|dt \leq \frac{b-a}{4}\int_a^b |\varphi'(t)|^2 dt, \tag{3}$$

where the coefficients $(b-a)/2$ and $(b-a)/4$ are their respective best constants possible.

The q-analogue of a generalized type Opial integral inequality was also established in [9] as

$$\int_a^b |D_q\varphi(t)||\varphi(t)|^p d_q t \leq (b-a)^p \int_a^b |D_q\varphi(t)|^{p+1} d_q t, \tag{4}$$

where $\varphi \in C^1[0,1]$ is a q-decreasing function with $\varphi(bq^0) = 0$ and $p \geq 0$.

It was found in [18] that J.F. Steffensen discovered the inequality

$$\int_{b-\lambda}^b f(u)du \leq \int_a^b g(u)f(u)du \leq \int_a^{a+\lambda} f(u)du, \tag{5}$$

where $\lambda = \int_a^b g(u)du$, f and g are integrable functions defined on (a,b), f is decreasing and for each $u \in (a,b)$, $0 \leq g(u) \leq 1$ (see also [10], [11]), [14] and [19]).

This inequality (5) was further refined in [5] to

$$\phi\left(\int_0^1 g(u)du\right) \leq \int_0^1 g(u)\phi'(u)du \tag{6}$$

where ϕ is convex and differentiable function with $\phi(0) = 0$.

In [4], the q-version of (6) was also established as

$$\phi\left(\int_0^1 g(u)d_qu\right) \leq \int_0^1 g(u)(D_q\phi)(u)d_qu, \tag{7}$$

where ϕ is a convex and q-differentiable function with $\phi(0) = 0$, $0 \leq g(u) \leq 1$ for each $u \in [0,1]$.

The aim of this paper is to establish new results involving the Opial's inequality and Steffensen's inequality and their q and (p,q)-analogues.

2 Methodology

The preliminaries and methods for the understanding of this paper are presented in this section. Let us begin with convex functions.

Definition 2.1. Let I be an interval in $\Re$. Then $\phi : I \longrightarrow \Re$ is said to be *convex* if for all $u_1, u_2 \in I$ and for all positive m and n satisfying $m + n = 1$, we have

$$\phi(mu_1 + nu_2) \leq m\phi(u_1) + n\phi(u_2). \tag{8}$$

A function ϕ is said to be *concave* if $-\phi$ is convex (i.e. if the inequality (8) is reversed). ϕ is *strictly concave* if it is strict for all $u_1 \neq u_2$. Some examples of convex functions are: $|u|$, u^k for $k > 1$ and $-u^k$ for $0 < k < 1$, e^u, $u(\log u)^k$ for $k \geq 1$, $-\log u$, etc. Concave functions are u^k for $0 < k < 1$, $\log u$, $\sqrt{u}$ for $u \geq 0$ etc.

The notion of q-calculus for the purpose of this paper is discussed. In the following, q is a real number that satisfies $0 < q < 1$ and which can be found in [2].

Definition 2.2. The *q-differential* of a function f is defined as

$$(d_q f)(t) = f(qt) - f(t).$$

In particular,

$$d_q t = (q-1)t.$$

Definition 2.3. The *q-derivative* of a function f is defined as

$$(D_q f)(t) = \frac{(d_q f)(t)}{d_q t} = \frac{f(qt) - f(t)}{(q-1)t}.$$

Clearly, $(D_q f)(t) \to \frac{(df)(t)}{dt}$ as $q \to 1$.

Remark 2.4. The *q-analogue* of the Leibniz rule is given as

$$(D_q fg)(t) = g(t) D_q f(t) + f(qt) D_q g(t)$$

(See [2, 17] and the references cited therein).

Definition 2.5. Suppose $0 < a < b$. The *definite q-integral* also known as the *q-Jackson integral* is defined as (see [2, 17])

$$\int_0^b f(t) d_q(t) = (1-q) b \sum_{j=0}^{\infty} q^j f(q^j b) \tag{9}$$

provided the series converges.

Note that

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q(t) - \int_0^a f(t) d_q t. \tag{10}$$

The values of such defined q-integrals of the polynomials form have very similar form to those in the standard integral calculus. For example

$$\int_a^b t^n d_q t = \frac{b^{n+1} - a^{n+1}}{[n+1]_q}. \tag{11}$$

(See [8]).

By [2], if $f(t) \geq 0$, it is not necessarily true that $\int_a^b f(t) d_q t \geq 0$.

Definition 2.6. The q-integration by parts for suitable functions f and g is given as ([1, 2])

$$\int_a^b f(t)(D_q g)(t) d_q t = f(b)g(b) - f(a)g(a) - \int_a^b g(qt)(D_q f)(t) d_q t. \tag{12}$$

Theorem 2.7. *Let f be a continuous functions on a segment $[a,b]$. Then there exists $c \in (a,b)$ such that*

$$\int_a^b f(t) d_q t = f(c)(b-a) \tag{13}$$

for every $q \in (0,1)$.

Theorem 2.8. *Let f and g be some continuous functions on a segment $[a,b]$. Then there exists $c \in (a,b)$ such that*

$$\int_a^b f(t)g(t) d_q t = f(c) \int_a^b g(t) d_q t \tag{14}$$

for every $q \in (0,1)$.

The q-version of the Opials inequality was established in [9] as follows:

Theorem 2.9. *Let $f \in C'[0,1]$ be q-decreasing function with $f(bq^0) = 0$. Then, for any $p \geq 0$,*

$$\int_a^b |D_q f(t)||f(t)|^p d_q t \leq (b-a)^p \int_a^b |D_q f(t)|^{p+1} d_q t. \tag{15}$$

Let us now present some well-known facts on (p,q)-calculus that can be found in ([7], [15]). See also the references cited therein. Throughout this paper, let $[a,b] \subset \Re$ be an interval with $a < b$ and p, q be two real numbers satisfying $0 < q < p \leq 1$ and (p,q)-bracket is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + qp^{n-2} + \cdots + q^{n-1}, \quad n \in \mathcal{N}.$$

$$[n]_{p,q}! = \begin{cases} [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q} = \prod_{i=1}^{n} \frac{p^i - q^i}{p-q}, & \text{if} \quad n \in \mathcal{N} \\ 1, \quad n = 0 & . \end{cases}$$

Definition 2.10. ([7],[15]) The (p,q)-*derivative* of a function f is defined as

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \quad t \neq 0,$$

and $(D_{p,q}f)(0) = f'(0)$, provided f is differentiable at 0.

Clearly,

$$D_{p,q}t^n = [n]_{p,q}t^{n-1}.$$

Note that for $p = 1$, the (p,q)-derivative reduces to the Hahn derivative given as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0.$$

The twin-basic number or the so-called (p,q)-bracket is a natural generalisation of the q-number, that is

$$\lim_{p \to 1}[n]_{p,q} = [n]_q = \frac{1-q^n}{1-q}, \quad q \neq 1.$$

The (p,q)-derivative satisfies the product rule as follows:

$$D_{p,q}(f(t)g(t)) = g(pt)D_{p,q}f(t) + f(qt)D_{p,q}g(t),$$

$$D_{p,q}(f(t)g(t)) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t).$$

Now the turn of (p,q)-integral which is presented as follows:

Definition 2.11. ([7],[15]) Let f be an arbitrary function and $x \in \Re^+$, the (p,q)-*integral* of f from 0 to x is defined as

$$\int_0^x f(t)d_{p,q}t = (p-q)x\sum_{i=0}^{\infty} \frac{q^i}{p^{i+1}} f\left(\frac{q^i}{p^{i+1}}x\right). \tag{16}$$

For two nonnegative numbers $a < b$, we have

$$\int_a^b f(t)d_{p,q}t = \int_0^b f(t)d_{p,q}t - \int_0^a f(t)d_{p,q}t.$$

Definition 2.12. [9] The real function f defined on $[a,b]$ is called *q-increasing* (*q-decreasing*) on $[a,b]$ if $f(qt) \leq f(t)$ $(f(qt) \geq f(t))$ for $t, qt \in [a,b]$. It is not difficult to see that if the function f is increasing (decreasing), then it is q-increasing (q-decreasing) too.

The fundamental theorem of (p,q)-calculus is stated as follows.

Theorem 2.13. *If $F(t)$ is an antiderivative of $f(t)$ and $F(t)$ continuous at $t=0$, then*

$$\int_a^b f(t) d_{p,q} t = F(b) - F(a), \tag{17}$$

where $0 \leq a < b \leq \infty$.

Definition 2.14. The (p,q)*-integration by parts* for suitable functions f and g is given as [3]

$$\int_a^b f(pt) D_{p,q} g(t) d_{p,q} t = f(b)g(b) - f(a)g(a) - \int_a^b g(qt) D_{p,q} f(t) d_{p,q} t. \tag{18}$$

Theorem 2.15. *[3] Let g be absolutely continuous and non-increasing function on $[0,a]_{p,q}$ with $g(0) = g(a) = 0$ and $0 < q < p \leq 1$. Then*

$$\int_0^a g(pt) D_{p,q} g(t) d_{p,q} t \leq \frac{a}{4} \int_0^a |D_{p,q} g(t)|^2 d_{p,q} t. \tag{19}$$

3 Results and Discussion

Let us begin this section with a lemma.

Lemma 3.1. *Let $\phi \in C'[0,1]$ be convex with $\phi(0) = 0$ and $0 \leq \phi(t) \leq 1$ for each $t \in [0,1]$, then*

$$\phi\left(\int_0^1 \phi(t) dt\right) \leq \frac{1}{2} \int_0^1 |\phi'(t)|^2 dt, \tag{20}$$

and

$$\phi\left(\int_0^1 \phi(t) dt\right) \leq \frac{1}{4} \int_0^1 |\phi'(t)|^2 dt. \tag{21}$$

where the constants $\frac{1}{2}$ and $\frac{1}{4}$ are the best possible.

Proof. Substituting $g = \phi$ into (6) and applying (2) and (3) yields

$$\begin{aligned}\phi\left(\int_0^1 \phi(t)dt\right) &\leq \int_0^1 \phi(t)\phi'(t)dt \\ &\leq \frac{1}{2}\int_0^1 |\phi'(t)|^2 dt\end{aligned}$$

and

$$\begin{aligned}\phi\left(\int_0^1 \phi(t)dt\right) &\leq \int_0^1 \phi(t)\phi'(t)dt \\ &\leq \frac{1}{4}\int_0^1 |\phi'(t)|^2 dt\end{aligned}$$

respectively as required. □

Example 1. Let $\phi(t) = t^p$ for $p > 1$. Then by (20), we have

$$2\left(\frac{1}{p+1}\right)^p \leq \frac{p^2}{2p-1}.$$

Theorem 3.2. *Let $\phi : [0,1] \to \Re$ be convex and q-differentiable function with $\phi(0) = 0$ and $0 \leq \phi(t) \leq 1$ for each $t \in [0,1]$, then*

$$\phi\left(\int_0^1 \phi(t)d_qt\right) \leq \int_0^1 |D_q\phi(t)|^2 d_qt. \tag{22}$$

Proof. Substituting $g = \phi$ into (7) and applying (4) for $p = 1$ yields

$$\begin{aligned}\phi\left(\int_0^1 \phi(t)d_qt\right) &\leq \int_0^1 |D_q\phi(t)||\phi(t)|d_qt \\ &\leq \int_0^1 |D_q\phi(t)|^2 d_qt\end{aligned}$$

as required. □

Theorem 3.3. *Let $\phi : [0,1] \to \Re$ be convex and (p,q)-differentiable function with $\phi(0) = 0$ and $0 \leq f(t) \leq 1$ for each $t \in [0,1]$, then*

$$\phi\left(\int_0^1 f(t)d_{p,q}t\right) \leq \int_0^1 f(pt)D_{p,q}\phi(t)d_{p,q}t. \tag{23}$$

Proof. Let
$$H(x)=\int_0^x f(t)d_{p,q}t\leq x$$
and
$$\omega(x)=\phi[H(x)]=\int_0^x f(t)d_{p,q}t.$$
Since
$$D_{p,q}H(x)=f(x),$$
then by the chain rule we have
$$D_{p,q}\omega(x)=D_{p,q}H(X).D_{p,q}\phi[H(x)]\leq f(x)D_{p,q}\phi(x)$$
Thus
$$\int_0^1 D_{p,q}\omega(x)d_{p,q}x\leq\int_0^1 f(px)D_{p,q}\phi(x)d_{p,q}x.$$
Since $\phi(0)=0$, then by (17) we have
$$\phi\left(\int_0^1 f(t)d_{p,q}t\right)\leq\int_0^1 f(pt)D_{p,q}\phi(t)d_{p,q}t$$
as required. □

Remark 3.4. By taking $p=1$, the inequality (23) reduces to the inequality (7).

Theorem 3.5. *Let $\phi:[0,1]_{p,q}\to\Re$ be convex, (p,q)-differentiable and decreasing function with $\phi(0)=0$ then*
$$\phi\left(\int_0^1\phi(t)d_{p,q}t\right)\leq\frac{1}{4}\int_0^1|D_{p,q}\phi(t)|^2d_{p,q}t. \tag{24}$$

Proof. Substituting $f(t)=\phi(t)$ into (23) and applying (19) yields
$$\begin{aligned}\phi\left(\int_0^1\phi(t)d_{p,q}t\right) &\leq \int_0^1\phi(pt)D_{p,q}\phi(t)d_{p,q}t\\ &\leq \frac{1}{4}\int_0^1|D_{p,q}\phi(t)|^2d_{p,q}t\end{aligned}$$
as required. □

4 Conclusion

In this paper, a refinement of the new integral inequality related to the Steffensen's inequality was presented. Results on q and (p,q) calculus were established. Further results were obtained through the Opial's-type inequalities.

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