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Certain Integrals Associated with Hypergeometric Functions of Four Variables

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Abstract

The main objective of this paper is to present integral representations of Euler type and Laplace type for five new hypergeometric series of four variables.

1. Introduction

In diverse areas in engineering and mathematical physics, integral representations play an important role in the view point of application. A number of integral representations involving various hypergeometric functions have been investigated by many authors (see [1, 2, 3, 4, 5, 6, 8, 15, 17]). Very recently, Bin-Saad et al. [2, 3] have introduced and studied ten quadruple hypergeometric functions $X_1^{(4)}, X_2^{(4)}, \ldots, X_{10}^{(4)}$. In [17] Younis and Bin-Saad gave integral representations for twenty new hypergeometric functions of four variables $X_{31}^{(4)}, X_{32}^{(4)}, \ldots, X_{50}^{(4)}$. Motivated mainly by the works [2, 3, 17], we further derive the following five new hypergeometric functions of four variables:

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$$X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u)$$

$$= \sum_{m, n, p, q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.1}$$

$$X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u)$$

$$= \sum_{m, n, p, q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.2}$$

$$X_{23}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u)$$

$$= \sum_{m, n, p, q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.3}$$

$$X_{24}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_3; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \sum_{m, n, p, q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{n+p}(a_3)_{p+q}}{(c_1)_{m+p}(c_2)_n(a_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.4}$$

$$X_{25}^{(4)}(a_1, a_1, a_3, a_1, a_1, a_2, a_4, a_2; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \sum_{m, n, p, q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{n+q}(a_3)_p(a_4)_p}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.5}$$

where $(a)_m$ denotes the Pochhammer symbol given by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\cdots(a+m-1) \quad (m \in N := \{1, 2, 3\}) \text{ and } (a)_0 = 1.$$

In the present paper, we will introduce and study several integral representations for the new quadruple functions defined above. In Section 2, we have obtained some Euler type integrals involving quadruple functions $X_i^{(4)}$ $(i = 21, 22, 23, 24, 25)$. Then we aim to present integral representations of Laplace-type are for each functions $X_i^{(4)}$ $(i = 21, 22, 23, 24, 25)$.

2. Integral Representations of Euler-Type

Here, in order to present our main finding, we recall the definition of some well-known hypergeometric functions as follows:

The Gaussian hypergeometric function is defined by [15]

$$ {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (|x| < 1). $$

Appell's hypergeometric functions F_2, F_3 and F_4 of two variables and the Horn's series H_4 of two variables are given by

$$ F_2(a, b, c; d, e; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n}(b)_m (c)_n}{(d)_m (e)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (|x| + |y| < 1), $$

$$ F_3(a, b, c, d; e; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_m (b)_n (c)_m (d)_n}{(e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (\max\{|x|, |y|\} < 1), $$

$$ F_4(a, b; c, d; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (\sqrt{|x|} + \sqrt{|y|} < 1) $$

and

$$ H_4(a, b; c, d; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (|x| < r, |y| < s, 4r = (s-1)^2) $$

respectively (see [15]). The Exton's triple functions X_2, X_3, X_4, X_{13}, X_{14}, X_{15}, X_{16} and X_{20} [9] are defined by as follows:

$$ X_2(a, b; c, d, e; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+2n+p}(b)_p}{(c)_m (d)_n (e)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, $$

$$ X_3(a, b; c, d; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n+p}(b)_{n+p}}{(c)_{m+n}(d)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, $$

$$X_4(a, b; c, d, e; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n+p}(b)_{n+p}}{(c)_m(d)_n(e)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$$X_{13}(a, b, c; d; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$$X_{14}(a, b, c; d, e; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+n}(e)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$$X_{15}(a, b, c; d, e; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_m(e)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},$$

$$X_{16}(a, b, c; d, e; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+p}(e)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}$$

and

$$X_{20}(a, b, c, d; e, f; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n}(b)_n(c)_p(d)_p}{(e)_{m+p}(f)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}.$$

The Lauricella's triple functions $F_C^{(3)}$ and F_F (see [11]) are given by

$$F_C^{(3)}(a, b; c_1, c_2, c_3; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{m+n+p}(b)_{m+n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}$$

and

$$F_F(a, a, a, b, c, b; d, e, e; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{m+n+p}(b)_{m+p}(c)_n}{(d)_m(e)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}.$$

Sharma and Parihar hypergeometric function of four variables $F_{14}^{(4)}$ is as follows (see [12]):

$$F_{14}^{(4)}(a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_2; c_1, c_2, c_3, c_1; x, y, z, u)$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+p}(b_2)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}.$$

Lauricella hypergeometric function of four variables $F_C^{(4)}$ is as below (see [10])

$$F_C^{(4)}(a, b; c_1, c_2, c_3, c_4; x, y, z, u)$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}$$

$$(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|} + \sqrt{|u|} < 1).$$

Now, we evaluate five integrals representations for each functions $X_i^{(4)}$ $(i = 21, 22, 23, 24, 25)$ involving the Gauss hypergeometric function ${}_2F_1$, Appell hypergeometric functions F_2, F_3 and F_4, Horn's function H_4 of two variables, the Exton's triple series X_2, X_3, X_4, X_{13}, X_{14}, X_{15}, X_{16} and X_{20}, the Lauricella's triple series $F_C^{(3)}$ and F_F, and the quadruple series $F_{14}^{(4)}$, $X_{21}^{(4)}$ and $F_C^{(4)}$, as follows:

$$X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u)$$

$$= \frac{\Gamma(a_1 + a_2 + a_3)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a)\Gamma(c_1 - a)}$$

$$\times \int_0^1 \int_0^1 \int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}\gamma^{a-1}(1-\gamma)^{c_1-a-1}$$

$$\times F_C^{(4)}\left(\frac{a_1+a_2+a_3}{2}, \frac{a_1+a_2+a_3+1}{2}; a, c_1 - a, c_2, c_3; \lambda_1 x, \lambda_2 y, \lambda_3 z, \lambda_4 u\right) d\alpha d\beta d\gamma$$

$$(\lambda_1 = 4\alpha^2\beta^2\gamma,\ \lambda_2 = 4\alpha(1-\alpha)\beta^2(1-\gamma),\ \lambda_3 = 4(1-\alpha)\beta(1-\beta),\ \lambda_4 = 4\alpha(1-\alpha)\beta^2),$$

$$(\mathrm{Re}(a_i) > 0,\ i = (1, 2, 3),\ \mathrm{Re}(a) > 0,\ \mathrm{Re}(c_1 - a) > 0); \tag{2.1}$$

$$X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u)$$

$$= \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_2-1}$$

$$\times X_2(a_1 + a_2, a_3; c_1, c_3, c_2; \alpha^2 x + \alpha(1-\alpha)y, \alpha(1-\alpha)u, (1-\alpha)z)d\alpha$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(a_2) > 0); \tag{2.2}$$

$$X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u)$$

$$= \frac{2\Gamma(c_3)}{\Gamma(a_1)\Gamma(c_3 - a_1)} \int_0^{\frac{\pi}{2}} (\sin^2 \alpha)^{a_1 - \frac{1}{2}} (\cos^2 \alpha)^{c_3 - a_1 - \frac{1}{2}} (1 - u\sin^2 \alpha)^{-a_2}$$

$$\times X_{14}\left(1 + a_1 - c_3, a_2, a_3; c_1, c_2; x\tan^4 \alpha, -\frac{y\tan^2 \alpha}{(1 - u\sin^2 \alpha)}, \frac{z}{(1 - u\sin^2 \alpha)}\right)d\alpha$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(c_3 - a_1) > 0); \tag{2.3}$$

$$X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_1)(S-T)^{a_1}(R-T)}{\Gamma(a_1)\Gamma(c_1 - a_1)(S-R)^{2c_1 - a_1 - a_2 - 2}} \int_R^S (\alpha - R)^{a_1 - 1}(\alpha - T)^{1 + a_1 + a_2 - 2c_1}$$

$$\times [(R-T)(S-R)(S-\alpha)(\alpha - T) + (S-T)^2(\alpha - R)^2 x]^{c_1 - a_1 - 1}$$

$$\times [(S-R)(\alpha - T) - (S-T)(\alpha - R)y]^{-a_1}$$

$$\times F_2(a_2, a_3, 1 + a_1 - c_1; c_2, c_3; \lambda_1 z, \lambda_2 u)d\alpha$$

$$\left(\lambda_1 = \frac{(S-R)(\alpha - T)}{[(S-R)(\alpha - T) - (S-T)(\alpha - R)y]},\right.$$

$$\left.\lambda_2 = -\frac{(S-R)^2(S-T)(\alpha - R)(\alpha - T)^2}{\begin{array}{c}[(R-T)(S-R)(S-\alpha)(\alpha - T) + (S-T)^2(\alpha - R)^2 x] \\ \times [(S-R)(\alpha - T) - (S-T)(\alpha - R)y]\end{array}}\right),$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(c_1 - a_1) > 0, T < R < S); \tag{2.4}$$

$$X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1 - a_1)\Gamma(c_2 - a_3)}$$

$$\times \int_0^\infty \int_0^\infty (e^{-\alpha})^{a_1} (e^{-\beta})^{a_3} (1 - e^{-\beta})^{c_2 - a_3 - 1}$$

$$\times [(1 - e^{-\alpha}) + xe^{-2\alpha}]^{c_1 - a_1 - 1} (1 - ye^{-\alpha} - ze^{-\beta})^{-a_2}$$

$$\times {}_2F_1\left(1 + a_1 - c_1, a_2; c_3; -\frac{ue^{-\alpha}}{[(1 - e^{-\alpha}) + xe^{-2\alpha}](1 - ye^{-\alpha} - ze^{-\beta})}\right) d\alpha d\beta$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(a_3) > 0, \mathrm{Re}(c_1 - a_1) > 0, \mathrm{Re}(c_2 - a_3) > 0); \quad (2.5)$$

$$X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(a_1 + a_3)}{\Gamma(a_1)\Gamma(a_3)} \int_0^\infty (e^{-\alpha})^{a_1} (1 - e^{-\alpha})^{a_3 - 1}$$

$$\times X_4(a_1 + a_3, a_2; c_2, c_1, c_3; xe^{-2\alpha}, ye^{-\alpha} + z(1 - e^{-\alpha}), ue^{-\alpha}) d\alpha$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(a_3) > 0); \quad (2.6)$$

$$X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_3)}{\Gamma(a_1)\Gamma(c_3 - a_1)} \int_0^1 \alpha^{a_1 - 1} (1 - \alpha)^{c_3 - a_1 - 1} (1 - \alpha u)^{-a_2}$$

$$\times X_{15}\left(1 + a_1 - c_3, a_2, a_3; c_2, c_1; \frac{\alpha^2 x}{(1 - \alpha)^2}, \frac{\alpha y}{(1 - \alpha)(1 - \alpha u)}, \frac{z}{(1 - \alpha u)}\right) d\alpha$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(c_3 - a_1) > 0); \quad (2.7)$$

$$X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2 - a_1)} \int_0^\infty \alpha^{a_1 - 1} (1 + \alpha)^{1 + a_1 - 2c_2} [(1 + \alpha) + \alpha^2 x]^{c_2 - a_1 - 1}$$

$$\times F_F(a_2, a_2, a_2, 1+a_1-c_2, a_3, 1+a_1-c_2; c_3, c_1, c_1; \lambda u, z, \lambda y)\,d\alpha$$

$$\left(\lambda = -\frac{\alpha(1+\alpha)}{[(1+\alpha)+\alpha^2 x]}\right),$$

$$(\mathrm{Re}(a_1) > 0,\ \mathrm{Re}(c_2 - a_1) > 0); \tag{2.8}$$

$$X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u)$$

$$= \frac{2\Gamma(c_1)}{\Gamma(a_2)\Gamma(c_1-a_2)}$$

$$\times \int_0^{\frac{\pi}{2}} (\sin^2\alpha)^{a_2-\frac{1}{2}}(\cos^2\alpha)^{c_1-a_2-\frac{1}{2}}(1-y\sin^2\alpha)^{-a_1}(1-z\sin^2\alpha)^{-a_3}$$

$$\times H_4\left(a_1, 1+a_2-c_1; c_2, c_3; \frac{x}{(1-y\sin^2\alpha)^2}, -\frac{u\tan^2\alpha}{(1-y\sin^2\alpha)}\right)d\alpha$$

$$(\mathrm{Re}(a_2) > 0,\ \mathrm{Re}(c_1 - a_2) > 0); \tag{2.9}$$

$$X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u)$$

$$= \frac{4\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1-a_2)\Gamma(c_2-a_1)}$$

$$\times \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (\sin^2\alpha)^{a_1-\frac{1}{2}}(\cos^2\alpha)^{c_2-a_1-\frac{1}{2}}(\sin^2\beta)^{a_2-\frac{1}{2}}(\cos^2\beta)^{c_1-a_2-\frac{1}{2}}$$

$$\times (1+x\sin^2\alpha\tan^2\alpha + y\tan^2\alpha\sin^2\beta)^{c_2-a_1-1}(1-z\sin^2\beta)^{-a_3}$$

$$\times {}_2F_1\left(1+a_1-c_2, 1+a_2-c_1; c_3; \frac{u\tan^2\alpha\tan^2\beta}{(1+x\sin^2\alpha\tan^2\alpha + y\tan^2\alpha\sin^2\beta)}\right)d\alpha d\beta$$

$$(\mathrm{Re}(a_1) > 0,\ \mathrm{Re}(a_2) > 0,\ \mathrm{Re}(c_1 - a_2) > 0,\ \mathrm{Re}(c_2 - a_1) > 0); \tag{2.10}$$

$$X_{23}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u)$$

$$= \frac{2\Gamma(c_1)}{\Gamma(a)\Gamma(c_1-a)}\int_0^{\frac{\pi}{2}} (\sin^2\alpha)^{a-\frac{1}{2}}(\cos^2\alpha)^{c_1-a-\frac{1}{2}}$$

$$\times X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; a, a, c_1 - a, c_2;$$

$$x \sin^2 \alpha, y \sin^2 \alpha, z \cos^2 \alpha, u) d\alpha$$

$$(\mathrm{Re}(a) > 0, \mathrm{Re}(c_1 - a) > 0); \tag{2.11}$$

$$X_{23}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u)$$

$$= \frac{\Gamma(c_1)(1+M)^{a_3}}{\Gamma(a_3)\Gamma(c_1 - a_3)} \int_0^1 \alpha^{a_1 - 1}(1-\alpha)^{c_1 - a_3 - 1}(1 + M\alpha)^{a_2 - c_1}[(1 + M\alpha) - (1 + M)\alpha z]^{-a_2}$$

$$\times X_3\left(a_1, a_2; c_1 - a_3, c_2; \frac{(1-\alpha)x}{(1+M\alpha)}, \frac{(1-\alpha)y}{[(1+M\alpha)-(1+M)\alpha z]}, \frac{(1+M\alpha)u}{[(1+M\alpha)-(1+M)\alpha z]}\right) d\alpha$$

$$(\mathrm{Re}(a_3) > 0, \mathrm{Re}(c_1 - a_3) > 0, M > -1); \tag{2.12}$$

$$X_{23}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u)$$

$$= \frac{\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2 - a_2)} \int_0^\infty (e^{-\alpha})^{a_2}(1 - e^{-\alpha})^{c_2 - a_2 - 1}(1 - ue^{-\alpha})^{-a_1}$$

$$\times X_{13}\left(a_1, 1 + a_2 - c_2, a_3; c_1; \frac{x}{(1 - ue^{-\alpha})^2}, -\frac{ye^{-\alpha}}{(1 - e^{-\alpha})(1 - ue^{-\alpha})}, -\frac{ze^{-\alpha}}{(1 - e^{-\alpha})}\right) d\alpha$$

$$(\mathrm{Re}(a_2) > 0, \mathrm{Re}(c_2 - a_2) > 0); \tag{2.13}$$

$$X_{23}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u)$$

$$= \frac{2\Gamma(c_2)M^{a_1}}{\Gamma(a_1)\Gamma(c_2 - a_1)} \int_0^{\frac{\pi}{2}} (\sin^2 \alpha)^{a_1 - \frac{1}{2}}(\cos^2 \alpha)^{c_2 - a_1 - \frac{1}{2}}(\cos^2 \alpha + M \sin^2 \alpha)^{a_2 - c_2}$$

$$\times [(\cos^2 \alpha + M \sin^2 \alpha) - Mu \sin^2 \alpha]^{-a_2}$$

$$\times X_{13}(1 + a_1 - c_2, a_2, a_3; c_1; M^2 x \tan^4 \alpha, \lambda_1 y, \lambda_2 z) d\alpha$$

$$\left(\lambda_1 = -\frac{M(\cos^2 \alpha + M \sin^2 \alpha)\tan^2 \alpha}{[(\cos^2 \alpha + M \sin^2 \alpha) - Mu \sin^2 \alpha]}, \lambda_2 = \frac{(\cos^2 \alpha + M \sin^2 \alpha)}{[(\cos^2 \alpha + M \sin^2 \alpha) - Mu \sin^2 \alpha]}\right),$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(c_2 - a_1) > 0, M > 0); \tag{2.14}$$

$$X_{23}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u)$$

$$= \frac{\Gamma(a_1 + a_2 + a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^1 \int_0^1 \alpha^{a_1 - 1}(1 - \alpha)^{a_2 - 1}\beta^{a_1 + a_2 - 1}(1 - \beta)^{a_3 - 1}$$

$$\times F_4\left(\frac{a_1 + a_2 + a_3}{2}, \frac{a_1 + a_2 + a_3 + 1}{2}, c_1, c_2;\right.$$

$$\left. 4\alpha^2\beta^2 x + 4\alpha(1 - \alpha)\beta^2 y + 4(1 - \alpha)\beta(1 - \beta) z, 4\alpha(1 - \alpha)\beta^2 u \right) d\alpha d\beta$$

$$(\mathrm{Re}(a_1) > 0, i = (1, 2, 3)); \tag{2.15}$$

$$X_{24}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_3; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{2\Gamma(c_3)}{\Gamma(a_1)\Gamma(c_3 - a_1)} \int_0^{\frac{\pi}{2}} (\sin^2 \alpha)^{a_1 - \frac{1}{2}} (\cos^2 \alpha)^{c_3 - a_1 - \frac{1}{2}} (1 - u \sin^2 \alpha)^{-a_3}$$

$$\times X_{16}\left(1 + a_1 - c_3, a_2, a_3; c_1, c_2; x \tan^4 \alpha, - y \tan^2 \alpha, \frac{z}{(1 - u \sin^2 \alpha)}\right) d\alpha$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(c_3 - a_1) > 0); \tag{2.16}$$

$$X_{24}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_3; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_3)(S - T)^{a_3}(R - T)^{c_3 - a_3}}{\Gamma(a_3)\Gamma(c_3 - a_3)(S - R)^{c_3 - a_1 - 1}}$$

$$\times \int_R^S (\alpha - R)^{a_3 - 1}(S - \alpha)^{c_3 - a_3 - 1}(\alpha - T)^{a_1 - c_3}[(S - R)(\alpha - T) - (S - T)(\alpha - R)u]^{-a_1}$$

$$\times X_{16}(a_1, a_2, 1 + a_3 - c_3; c_1, c_2; \lambda_1 x, \lambda_2 y, \lambda_3 z) d\alpha$$

$$\left(\lambda_1 = \frac{(S-R)^2(\alpha-T)^2}{[(S-R)(\alpha-T)-(S-T)(\alpha-R)u]^2}, \lambda_2 = \frac{(S-R)(\alpha-T)}{[(S-R)(\alpha-T)-(S-T)(\alpha-R)u]},\right.$$

$$\left.\lambda_3 = -\frac{(R - T)(S - \alpha)}{(S - R)(\alpha - T)}\right),$$

$$(\mathrm{Re}(a_3) > 0, \mathrm{Re}(c_3 - a_3) > 0, T < R < S); \tag{2.17}$$

$$X_{24}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_3; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(a_1 + a_2 + a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty (e^{-\alpha})^{a_1}(1 - e^{-\alpha})^{a_2 - 1}(e^{-\beta})^{a_1 + a_2}(1 - e^{-\beta})^{a_3 - 1}$$

$$\times F_C^{(3)}\left(\frac{a_1 + a_2 + a_3}{2}, \frac{a_1 + a_2 + a_3 + 1}{2}; c_1, c_2, c_3; \lambda_1 x + \lambda_2 z, \lambda_3 y, \lambda_4 u\right) d\alpha d\beta$$

$$(\lambda_1 = 4e^{-2(\alpha+\beta)}, \lambda_2 = 4(1 - e^{-\alpha})e^{-\beta}(1 - e^{-\beta}),$$

$$\lambda_3 = 4e^{-(\alpha+2\beta)}(1 - e^{-\alpha}), \lambda_4 = 4e^{-(\alpha+\beta)}(1 - e^{-\beta})),$$

$$(\mathrm{Re}(a_i) > 0, i = (1, 2, 3)); \tag{2.18}$$

$$X_{24}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_3; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_2)\Gamma(c_3)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_2 - a_2)\Gamma(c_3 - a_3)}$$

$$\times \int_0^1 \int_0^1 \alpha^{a_2 - 1}(1 - \alpha)^{c_2 - a_2 - 1}\beta^{a_3 - 1}(1 - \beta)^{c_3 - a_3 - 1}(1 - \alpha y - \beta u)^{-a_1}$$

$$\times F_3\left(\frac{a_1}{2}, 1 + a_2 - c_2, \frac{a_1 + 1}{2}, 1 + a_3 - c_3; c_1; \frac{4x}{(1 - \alpha y - \beta u)^2}, \frac{\alpha\beta z}{(1 - \alpha)(1 - \beta)}\right) d\alpha d\beta$$

$$(\mathrm{Re}(a_2) > 0, \mathrm{Re}(a_3) > 0, \mathrm{Re}(c_2 - a_2) > 0, \mathrm{Re}(c_3 - a_3) > 0); \tag{2.19}$$

$$X_{24}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_3; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_2)\Gamma(c_3)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_2 - a_2)\Gamma(c_3 - a_3)}$$

$$\times \int_0^\infty \int_0^\infty \alpha^{a_2 - 1}(1 + \alpha)^{a_1 - c_2}\beta^{a_3 - 1}(1 + \beta)^{a_1 - c_3}[(1 + \alpha)(1 + \beta) - \alpha(1 + \beta)y - (1 + \alpha)\beta u]^{-a_1}$$

$$\times F_3\left(\frac{a_1}{2}, 1 + a_2 - c_2, \frac{a_1 + 1}{2}, 1 + a_3 - c_3; c_1; \frac{4(1 + \alpha)^2(1 + \beta)^2 x}{[(1 + \alpha)(1 + \beta) - \alpha(1 + \beta)y - (1 + \alpha)\beta u]^2}, \alpha\beta z\right)$$

$$\times d\alpha d\beta$$

$$(\mathrm{Re}(a_2) > 0, \mathrm{Re}(a_3) > 0, \mathrm{Re}(c_2 - a_2) > 0, \mathrm{Re}(c_3 - a_3) > 0); \tag{2.20}$$

$$X_{25}^{(4)}(a_1, a_1, a_3, a_1, a_1, a_2, a_4, a_2; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{2\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^{\frac{\pi}{2}} (\sin^2 \alpha)^{a_1 - \frac{1}{2}} (\cos^2 \alpha)^{a_2 - \frac{1}{2}}$$

$$\times F_{14}^{(4)}\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, \frac{a_1 + a_2 + 1}{2}, \right.$$

$$\left. \frac{a_1 + a_2 + 1}{2}, \frac{a_1 + a_2 + 1}{2}, a_4; c_1, c_2, c_3, c_1; \lambda_1 x, \lambda_2 y, \lambda_2 u, z \right) d\alpha$$

$$(\lambda_1 = 4\sin^4 \alpha, \lambda_2 = \sin^2 2\alpha),$$

$$(\mathrm{Re}(a_1) > 0, \mathrm{Re}(a_2) > 0); \tag{2.21}$$

$$X_{25}^{(4)}(a_1, a_1, a_3, a_1, a_1, a_2, a_4, a_2; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_1)(S - T)^{a_3}(R - T)^{c_1 - a_3}}{\Gamma(a_3)\Gamma(c_1 - a_3)(S - R)^{c_1 - a_4 - 1}}$$

$$\times \int_R^S (\alpha - R)^{a_3 - 1}(S - \alpha)^{c_1 - a_3 - 1}(\alpha - T)^{a_4 - c_1}[(S - R)(\alpha - T) - (S - T)(\alpha - R)z]^{-a_4}$$

$$\times X_4\left(a_1, a_2; c_1 - a_3, c_2, c_3; \frac{(R - T)(S - \alpha)x}{(S - R)(\alpha - T)}, y, u \right) d\alpha$$

$$(\mathrm{Re}(a_3) > 0, \mathrm{Re}(c_1 - a_3) > 0, T < R < S); \tag{2.22}$$

$$X_{25}^{(4)}(a_1, a_1, a_3, a_1, a_1, a_2, a_4, a_2; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{\Gamma(c_3)}{\Gamma(a_2)\Gamma(c_3 - a_2)} \int_0^{\infty} \alpha^{a_2 - 1}(1 + \alpha)^{a_1 - c_3}[(1 + \alpha) - \alpha u]^{-a_1}$$

$$\times X_{20}\left(a_1, 1 + a_2 - c_3, a_3, a_4; c_1, c_2; \frac{(1 + \alpha)^2 x}{[(1 + \alpha) - \alpha u]^2}, -\frac{\alpha(1 + \alpha)y}{[(1 + \alpha) - \alpha u]}, z \right) d\alpha$$

$$(\mathrm{Re}(a_2) > 0, \mathrm{Re}(c_3 - a_2) > 0); \tag{2.23}$$

$$X_{25}^{(4)}(a_1,\ a_1,\ a_3,\ a_1,\ a_1,\ a_2,\ a_4,\ a_2;\ c_1,\ c_2,\ c_1,\ c_3;\ x,\ y,\ z,\ u)$$

$$= \frac{\Gamma(a_1 + a_2 + a_3 + a_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}$$

$$\times \int_0^1 \int_0^1 \int_0^1 \alpha^{a_1 - 1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}\gamma^{a_1+a_2+a_3-1}(1-\gamma)^{a_4-1}$$

$$\times F_C^{(3)}\left(\frac{a_1+a_2+a_3+a_4}{2}, \frac{a_1+a_2+a_3+a_4+1}{2}; c_1, c_2, c_3; \lambda_1 x + \lambda_2 z, \lambda_3 y, \lambda_3 u\right) d\alpha d\beta d\gamma$$

$$(\lambda_1 = 4\alpha^2\beta^2\gamma^2,\ \lambda_2 = 4(1-\beta)\gamma(1-\gamma),\ \lambda_3 = 4\alpha(1-\alpha)\beta^2\gamma^2),$$

$$(\mathrm{Re}(a_i) > 0,\ i = (1,\ 2,\ 3,\ 4)); \tag{2.24}$$

$$X_{25}^{(4)}(a_1,\ a_1,\ a_3,\ a_1,\ a_1,\ a_2,\ a_4,\ a_2;\ c_1,\ c_2,\ c_1,\ c_3;\ x,\ y,\ z,\ u)$$

$$= \frac{\Gamma(c_1)\Gamma(c_3)}{\Gamma(a_2)\Gamma(a_4)\Gamma(c_1 - a_4)\Gamma(c_3 - a_2)}$$

$$\times \int_0^\infty \int_0^\infty e^{-(a_4\alpha + a_2\beta)}(1-e^{-\alpha})^{c_1-a_4-1}(1-e^{-\beta})^{c_3-a_2-1}(1-ze^{-\alpha})^{-a_3}(1-ue^{-\beta})^{-a_1}$$

$$\times H_4\left(a_1,\ 1 + a_2 - c_3;\ c_1 - a_4,\ c_2;\ \frac{x(1-e^{-\alpha})}{(1-ue^{-\beta})^2}, -\frac{ye^{-\beta}}{(1-e^{-\beta})(1-ue^{-\beta})}\right) d\alpha d\beta$$

$$(\mathrm{Re}(a_2) > 0,\ \mathrm{Re}(a_4) > 0,\ \mathrm{Re}(c_1 - a_4) > 0,\ \mathrm{Re}(c_3 - a_2) > 0). \tag{2.25}$$

Proof of the integral representations of Euler-type

Once substituting the series definition of the special function in each integrand and then, changing the order of the integral and the summation, and finally taking into account the following integral representations of the Beta function and their various associated Eulerian integrals (see, for example, [7, 13, 14, 16]), we derive each of the integral representations from (2.1) to (2.25).

$$B(a, b) = \begin{cases} \int_0^1 t^{a-1}(1-t)^{b-1}dt, & (\mathrm{Re}(a) > 0,\ \mathrm{Re}(b) > 0), \\ \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, & (a, b \in C \backslash Z_0^-), \end{cases} \tag{2.26}$$

$$B(a, b) = \int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}d\alpha = \int_0^\infty (e^{-\alpha})^a (1-e^{-\alpha})^{b-1}d\alpha$$

$$(\mathrm{Re}(a) > 0,\ \mathrm{Re}(b) > 0), \tag{2.27}$$

$$B(a, b) = 2\int_0^{\frac{\pi}{2}} (\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1}d\alpha = \int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}d\alpha$$

$$(\mathrm{Re}(a) > 0,\ \mathrm{Re}(b) > 0), \tag{2.28}$$

$$B(a, b) = \frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}}\int_R^S \frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}d\alpha\,(T < R < S)$$

$$= (1+M)^a \int_0^1 \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)^{a+b}}d\alpha \qquad (M > -1)$$

$$(\mathrm{Re}(a) > 0,\ \mathrm{Re}(b) > 0). \tag{2.29}$$

3. Integrals Representations of Laplace-Type

Here we present certain integral representations of Laplace-Type for the functions in (1.1) to (1.5).

$$X_{21}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u)$$

$$= \frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1}$$

$$\times\, {}_0F_1(-; c_1; s^2x + sty)\, {}_1F_1(a_3; c_2; tz)\, {}_0F_1(-; c_3; stu)dsdt,$$

$$(\mathrm{Re}(a_1) > 0,\ \mathrm{Re}(a_2) > 0); \tag{3.1}$$

$$X_{22}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_2, c_1, c_1, c_3; x, y, z, u)$$

$$= \frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1}$$

$$\times \Psi_2(a_2; c_1, c_3; sy + tz, su)\, {}_0F_1(-; c_2; s^2x)dsdt,$$

$$(\text{Re}(a_1) > 0, \text{Re}(a_2) > 0); \tag{3.2}$$

$$X_{23}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u)$$

$$= \frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(s+t+v)} s^{a_1-1} t^{a_2-1} v^{a_3-1}$$

$$\times\, {}_0F_1(-; c_1; s^2x + sty + tvz)\, {}_0F_1(-; c_2; stu)dsdtdv,$$

$$(\text{Re}(a_1) > 0, \text{Re}(a_2) > 0, \text{Re}(a_3) > 0); \tag{3.3}$$

$$X_{24}^{(4)}(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_3; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(s+t+v)} s^{a_1-1} t^{a_2-1} v^{a_3-1}$$

$$\times\, {}_0F_1(-; c_1; s^2x + tvz)\, {}_0F_1(-; c_2; sty)\, {}_0F_1(-; c_3; svu)dsdtdv,$$

$$(\text{Re}(a_1) > 0, \text{Re}(a_2) > 0, \text{Re}(a_3) > 0); \tag{3.4}$$

$$X_{25}^{(4)}(a_1, a_1, a_3, a_1, a_1, a_2, a_4, a_2; c_1, c_2, c_1, c_3; x, y, z, u)$$

$$= \frac{1}{\Gamma(a_1)} \int_0^\infty e^{-s} s^{a_1-1}$$

$$\times \Xi_2(a_3, a_4; c_1; z, s^2x)\Psi_2(a_2; c_2, c_3; sy, su)ds, \ \ (\text{Re}(a_1) > 0); \tag{3.5}$$

where $({}_0F_1, {}_1F_1)$, Ψ_2 and Ξ_2 denote the confluent hypergeometric functions and the Humbert functions defined, respectively, by

$${}_0F_1(-; c; x) = \sum_{m=0}^{\infty} \frac{1}{(c)_m} \frac{x^m}{m!},$$

$${}_1F_1(a; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \frac{x^m}{m!},$$

$$\Psi_2(a; b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}}{(b)_m (c)_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

and

$$\Xi_2(a, b; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_m}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}.$$

Proof of the integral representations of Laplace-type

It is noted that each of the integral representations (3.1) to (3.5) can be proved mainly by expressing the series definition of the involved special functions in each integrand and changing the order of the integral sign and the summation, and finally using the following well-known integral formula [1]:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

4. Concluding Remarks

Integral representations for most of the special functions of mathematical physics and applied mathematics have been investigated in the existing literature. Here we have presented some integral representations for five new quadruple hypergeometric series.

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