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Entropy of the Earth / Sun Planetary System: The Earth's Temperature in a Millennial Precession Cycle

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Abstract

We will use the definition of entropy to calculate the Earth annual and millennial temperature profile having the highest probability among all probable temperature profiles. We will achieve this through the development of an original procedure for identifying the physical parameters of a paleo-climatic model of the Earth. This investigation will allow us to answer two questions of contemporary experimental and theoretical research: The first concerns the increase in sea temperature and the Earth accumulation of heat, measured in the last sixty years. The second concerns the probability of two events of capital interest for humanity: the onset of an ice age and the onset of a climatic optimum. This paleo-climatic model provides the answer that the probability of the onset of an ice age is higher than the advent of a climatic optimum within an interglacial period.

Introduction

Entropy is a measure of the degree of probability of a physical state: if a certain state is very probable then its entropy is high; conversely, an unlikely state has low entropy. If the parameters that determine the Earth's temperature are the result of a physical state to which a certain level of entropy can be associated, then the value assumed by the parameters has a more or less high level of probability, depending on the level of entropy to which they are associated. The approach of the concept of entropy to that of the identification of the parameters warns the reader that this process, like any other process

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of identification, faces problems of non-uniqueness of the solutions. In other words, the Earth's temperature profile is subject to probabilistic events, albeit in a modeling context which, at least initially, is strictly deterministic. In this work, we will determine the most probable temperature profile of the Earth, for the duration of a precession cycle of the equinoxes. This investigation will allow us to answer two questions of contemporary experimental and theoretical research. The first concerns the increase in sea temperature, determined by von Schuckmann et al., [1], [2]. A group of over 30 researchers from scientific institutions around the world have been tracking and quantifying the global heat distribution for nearly 60 years, (1960-2018). In this interval of years the accumulated heat, in the various compartments of the Earth, was equal to $(358 \pm 37)\cdot 10^{21}$J, equivalent to a positive accumulation of (0.47 ± 0.1) W/m^2. If we read these experimental data in the perspective of a precession cycle, then the paleo-climatic model interprets them as a millennial heat accumulation of the Earth. The second concerns the probability level of two events of capital interest for humanity: the onset of an ice age or the onset of a climatic optimum. The paleo-climatic model provides the answer that the probability of the onset of an ice age is higher than the advent of a climatic optimum within an interglacial period.

The paleo-climatic model

Any astronomical theory of the Earth's paleo-climate aims to establish a link between the flux of solar energy and the Earth's climate, on a global scale. The Serbian mathematician Milankovitch, [3], fully developed such an approach in the previous century: indeed, he was the first to calculate the solar radiation received from the Earth, (insolation), as a function of the geographic coordinates of the Earth and the orbital parameters, eccentricity and inclination, both variables over the millennia. For a critical and updated review of paleo-climatic models, see Berger, [3] and Loutre, [4]. On the contrary, the paleo-climatic model used in this work calculates the insolation by means of a modified Lambert extinction function, referred to the ecliptic plane, thus regardless of the geographic coordinates of the Earth, [5]. As regards the more strictly astronomical problems, the model takes into account the daily scale of rotation of the Earth, the annual scale of revolution, the millennial scale of variation of eccentricity, [6] and the inclination of the rotation axis, [7], during a precession cycle of equinoxes. In order to understand the procedure for identifying the model parameters, it is sufficient here to summarize its essential features. From a geometric point of view, we placed ourselves in the planetary perspective, of a distant observer, who sees the Earth as a small sphere, substantially blue due to the prevalence of the water that covers it, surrounded by a tenuous and thin gaseous atmosphere. There are three constitutive hypotheses: 1- The planets describe an elliptical orbit around the Sun which occupies one of the two foci (Kepler); 2- The planets are considered material points, (Newton); 3- Each planet (material point) is associated, in the same spirit of Newton, with an energy balance.

The energy balance consists of a three terms ordinary differential equation given by a source term, (radiative forcing), constantly equal to the sum of a loss term and an accumulation term. The Stefan-Boltzmann radiation equation for an opaque body describes the loss term. In subsequent developments, in order to obtain an analytical solution, we will use the linearized Stefan-Boltzmann equation, in the form of Newton's linear flow equation, where T_0 is a parameter corresponding to a suitable temperature of the troposphere:

$$\underbrace{-V_1 \cdot \frac{dT}{d\beta}}_{\substack{Millennial \\ accumulation}} + \underbrace{V \cdot (T - T_0)}_{Loss} = \underbrace{F(\alpha, \beta)}_{Solar\ forcing} \quad . \tag{1}$$

The initial condition is $T(0) = T_i$. Three free parameters appear in the energy balance, (V, V_1, T_0) and two independent variables, (α, β) where, (α) is the angle of revolution and (β) the angle of total precession. This model is able to describe the millennial temperature profile of the Earth, as a function of the precession angle (β) at each given value of the angle of revolution (α) considered as a parameter. The theory of ordinary differential equations teaches how to construct a solution to linear equations of this type for given initial conditions and assigned free parameters (V, V_1, T_0). In our case, on the contrary, the experimental value of the Earth's annual temperature at a given precession angle is known and we want to identify the free parameters and the initial condition that determine at best the experimental data, through the analytical solution. The process of parameters identification is therefore an inverse problem, as described, for example, by Bellman, [8].

Theoretical part: Identification of the parameters

The identification of the parameters is the most delicate part of the modeling work and, perhaps, the most irritating for the reader, due to the discretion of the choices made by the author of the mathematical model. In this regard, it is worth recalling the criticisms raised to the least squares method developed by C.F. Gauss as an analytical approximation criterion for experimental data. Gauss's answer was "why not? This method guarantees the uniqueness of the solution!". We will deal with a uniqueness problem also in the case of the paleo-climatic model of this work.

To determine the value of the three free parameters (V, V_1, T_0) it is necessary to have (at least) a system of three equations in the three free parameters, considered as unknowns. Two equations arise, very spontaneously, from the assignment of the value of the two isothermal average temperatures of January and July that roughly correspond to the temperatures at the two solstices, in winter and in summer. The third equation comes from a consideration of a physical nature: - Since the energy balance equation of the model must represent a real phenomenon, we impose as the third equation, the constraint that the three free parameters (V, V_1, T_0) generate the most probable temperature profile. In thermodynamic terms, this constraint translates into requiring that the entropy variation between the two solstices be the maximum. Recalling that the entropy of a system is, by definition, the ratio between the change in heat dQ per unit of surface area and the temperature T at which this transformation occurs, the third condition reads:

$$\int_{T_1}^{T_2} \frac{dQ}{T} = Max. \tag{2}$$

In the case of the paleo-climatic model, the change in heat over time is described by the accumulation term $-V_1 dT/d\beta$ therefore the entropy variation between the two solstices explicitly becomes:

$$\int_{T_1}^{T_2} -V_1 \frac{dT}{T} = -V_1 [lnT]_{T_1}^{T_2} \cong -V_1 \frac{T_2 - T_1}{T_1}\ . \tag{3}$$

In other words, the factor $-V_1(T_2 - T_1)$ represents the entropy variation of a half-cycle of revolution of the Earth around to the Sun, between the winter solstice and the summer solstice. By evaluating in (T_1, T_2) the analytical solution, (Appendix, equation A9), and rearranging, after placing $R = -V/V_1$ one obtains:

$$-V_1(T_2 - T_1)\frac{2a_0}{\varphi F_0} R + a_1 cos\beta_0 = -a_2 \frac{R^2}{R^2 + 1}. \tag{4}$$

By rationalizing, we obtain a fraction with a cubic equation in the numerator. The Cardano's formula for the roots of third degree polynomials, [9] provides its analytical solution. The graphical representation of the two terms of the equation (4), given in

Figure 1, very briefly highlights which problems can arise solving this equation. Graphically, the function to the left of the equality sign is a straight line in the variable R, (dashed line in blue) while the equation on the right is a positive even function, (continuous curve in red). The physical sense requires that the parameter $R = -V/V_1$ be negative, since the two thermal transmittances (V, V_1) are both positive. As the slope of the line varies, rotating around the point of intersection with the ordinate axis, three cases can occur: 1) there are two coincident and negative solutions if the line is tangent, (as in Figure 1). In this case, the slope of the line is the maximum. 2) Two distinct and negative solutions if the line is secant. The slope of the line is less than the maximum. 3) No negative solution if the slope of the line is greater than the maximum.

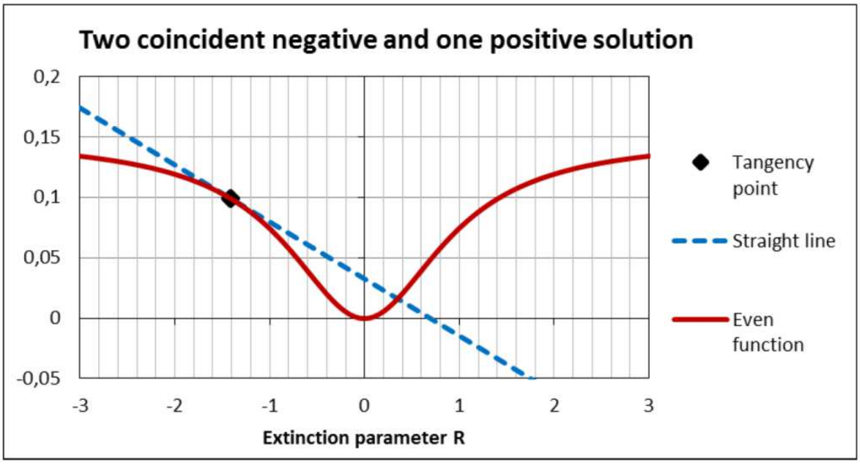

Figure 1: Identification diagram of the negative parameter R as the point of tangency between the two curves.

The maximum variation in entropy between the two solstices is associated with the point of tangency between the two curves, corresponding to the maximum slope of the line. The coordinates of the point of tangency identify an astronomical invariant of the Earth / Sun system since it depends only on astronomical parameters.

When the change in entropy is greatest, the Earth / Sun system thermodynamically has the highest probability. The value of the transmittance (V_1) thus identified is unique and is the maximum of the possible values that determine the solutions.

Results: 1. Reconstruction of an annual seasonal cycle with the identified parameters

Historically, the most important and popular experimental data for Earth's temperature are the January and July isothermal lines reported as thematic maps in geographic atlases. Isotherms are appropriate lines for the paleo-climatic model because they are long-term spatial and temporal averages produced by National Geographic Institutes. Currently, the American government agency NASA collects and makes available to the public the spatial and temporal average temperature measurements around the world. In this regard, it is worth mentioning that the calculation procedures of the averages constitute a thorny problem to such an extent that NASA itself has decided not to publish the actual values of the average temperature anymore but to provide the anomaly of the Earth’s temperature. Figure 2 shows, without adaptations, the temperature anomaly since 1880, with respect to the average value of 1975, [10].

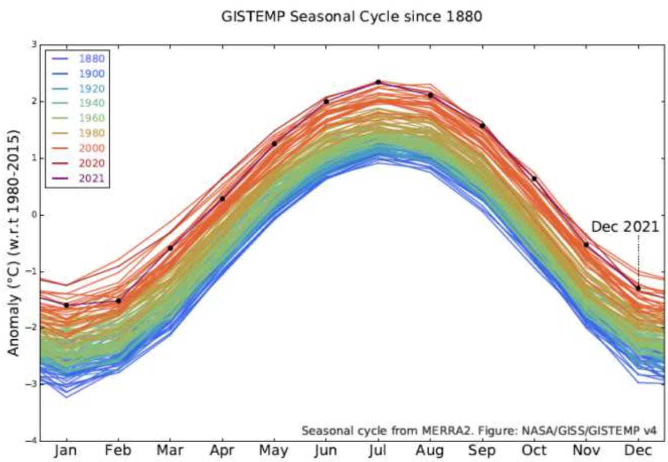

Figure 2: Earth temperature anomaly, starting from 1880, compared to the average value of 1975, [10].

The line in black color represents the seasonal temperature of the year 2021. From this figure, the trend towards increasing the average seasonal temperature of the Earth over the last 140 years appears clearly. However, to a closer inspection, it appears also that the increase has not been monotonous over the years, as it is the case for the year 2021 temperature profile. Figure 3 allows you to make a visual comparison between the NASA experimental data of the monthly temperature and the theoretical curve of the daily temperature, (Appendix, equation A9), after the identification of the three free parameters (V, V_1, T_0), as of the reference date 1975, [10]. Table 1 collects the identified environmental parameters and the astronomical parameters.

Table 1: Identified environmental factors and astronomical orbital factors that determined the average daily temperature of the Earth in 1975.

Tropospheric environmental parameters			Geographical parameters		
T_{0NL}	-9.364	°C	H_{0NL} (Height s.l.)	+3.716	Km
T_0	-6.662	°C	H_0 (Height s.l.)	+3.287	Km
T_1	+12.45	°C	Astronomical orbital parameters		
T_2	+15.25	°C	β	-0.22362	precession
t_0	8.766 10^4	s	δ	+0.40928	inclination
t_1	1.852 10^9	s	e	+0.0167	eccentricity
φ	+0.633	dimensionless	$a_0 = (1 - e^2)^2$	+0.999442	
F_0	+1361.25	W/m^2	$a_1 = 2e$	+0.03340	
V	+10.206	W/(m^2·°C)	$a_2 = -(1 - 2/\pi)\delta$	-0.148724	
V_1	+7.248	W/(m^2·°C)	$b_0 = e^2$	+0.000279	
R	- 1.408	dimensionless	$b_1 = -2e(1 - 2/\pi)\delta$	-0.004967	
T_M	+14.012	°C	$b_2 = -2/\pi(1 - 2/\pi)\delta^2$	-0.038751	

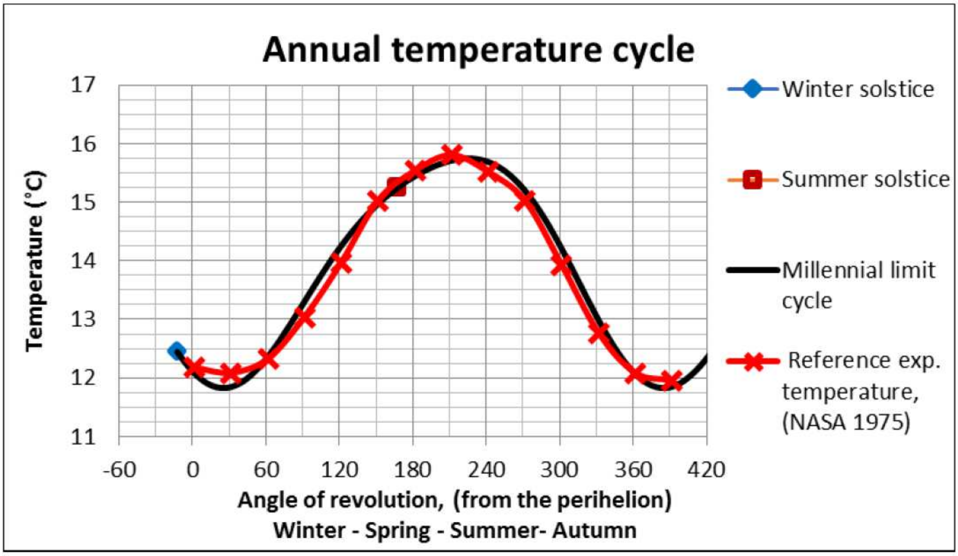

Figure 3a: Comparison between the theoretical curve, (Appendix, equation A9; $\beta_0 = 12°.8125$) and the experimental data of seasonal Earth temperature in 1975, [10].

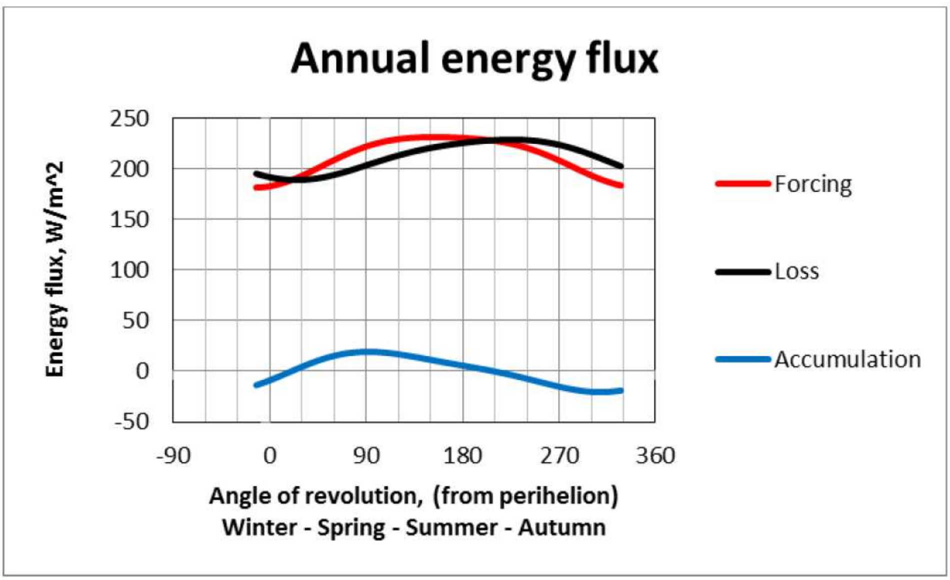

Figure 3b: Corresponding energy fluxes, forcing, loss and accumulation terms, (Appendix, equations A3, A8 and A9).

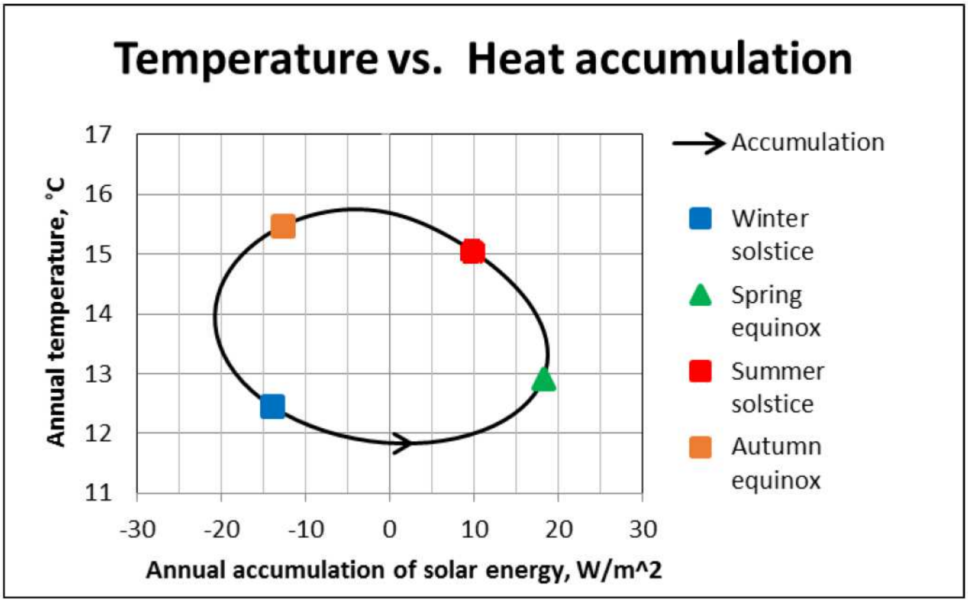

Figure 3c: Combination of the two diagrams, by elimination of the variable of angular revolution: Note the cyclical character of the annual accumulation of energy.

Results: 2. Annual average reconstruction of the millennial energy balance, (Northern Hemisphere)

The annual average of the balance equation (1) is particularly useful for comparison with recently published experimental data. It cancels from the balance the explicit dependence on the angle of revolution (α) and provides another important balance, depending only on the precession angle (β). After applying the definition of mean and the differentiation rule under the integral sign, to the accumulation term of balance equation (1) one obtains:

$$\underbrace{-V_1 \cdot \frac{dT_M}{d\beta}}_{\substack{Average \\ accumulation}} + \underbrace{V \cdot (T_M - T_0)}_{Average\ Loss} = \underbrace{F_M(\beta, e, \delta)}_{Average\ forcing} \quad . \tag{5}$$

The instantaneous balance, equation (1), and the annual average balance, equation (5), are formally identical. The one transforms into the other after substitution of the current temperature T with its average T_M defined as follows:

$$2\pi\, T_M = \int_0^{2\pi} T(\alpha, \beta)\, d\alpha. \tag{6}$$

The average solar forcing term is the annual average of the actual forcing term, (Appendix, equation A8):

$$F_M(\beta, e, \delta) = \frac{\varphi F_0}{4 a_0}\left(1 + \frac{b_0 + b_1 cos\beta + b_2}{2}\right). \tag{7}$$

By integrating the millennial balance equation (5), one obtains the average millennial temperature of the Earth:

$$T_M = T_0 + \frac{\varphi\, F_0}{4\, a_0\, V}\left\{1 + \frac{b_0 + b_2}{2} + \frac{b_1}{2}\frac{R}{R^2 + 1}[R\, cos\beta + sin\beta]\right\}. \tag{8}$$

The free parameters, (V, V_1, T_0) have already been identified through the procedure of maximum entropy variation between the two solstices at the reference year 1975. Figure 4a represents the three terms of the millennial balance equation, accumulation, loss and forcing:

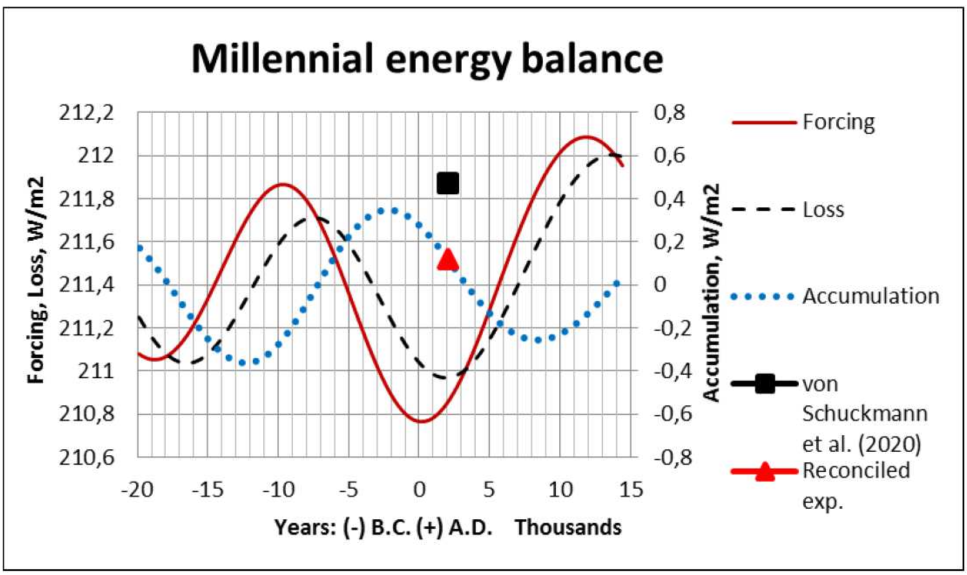

Figure 4a: Reconstruction of the millennial energy balance, forcing, loss and accumulation, (Equations 7 and 5).

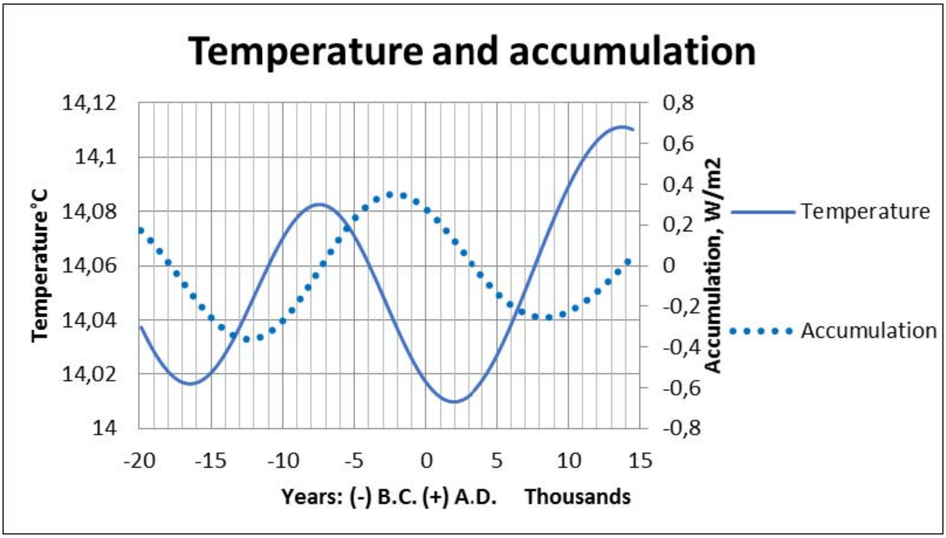

Figure 4b: Reconstruction of the millennial temperature and energy accumulation, (Equations 8 and 5).

Note that the variation of the reconstructed temperature over the millennia is very limited, having the order of magnitude of 0.1°C. This observation confirms that phenomena specifically associated to the industrial revolution and the contemporary age are causing the observed variation of the annual temperature of the Earth in the last two centuries. From a mathematical point of view, such phenomena are perturbations of the governing millennial energy balance. Note in Figure 4b that the min. /Max points of the accumulation curve follow the min. /Max points of the temperature curve with a delay of about 6,000 years. Figure 4a is of particular practical interest because experimental data have recently been published, [1], [2], which could be interpreted as the experimental value assumed by the accumulation term in the millennial balance equation. J. Hansen et al., [2], defines these experimental data with the phrase: Earth's Energy Imbalance, (EEI). Such data describe the accumulation of heat in the oceans, on land, in polar ice and in the atmosphere over the years (1971–2018) summing up to (358 ± 37) 10^{21} J and equivalent to a global warming rate of (0.473 ± 0.1) W/m^2. We observe that the reconstruction of the millennial energy balance is amazing for the fact that it captures the order of magnitude of the experimental data: in fact, the theoretical prediction of the model is 0.118 W/m^2. A first possible interpretation of the discrepancy between the theoretical datum and the experimental one could be that the theoretical prediction constitutes a portion of the experimental datum and that the difference between the two is due to a secular disturbance of heat accumulation. A second interpretative possibility could concern the reconciliation of the two data, reducing them to a unique number. We can achieve the reconciliation normalizing the experimental data with respect to the average depth of the oceans. In fact, the theoretical data refers to the average depth of the oceans of 4,000 m while the experimental data refers to the average depth of 1,000 m. Normalizing experiments with the average depth of oceans produce immediate reconciliation: 0.473 $W/m^2 \cdot 1{,}000\ m/4{,}000\ m = 0.118\ W/m^2$. A third completely different interpretation is that offered by the well-known climatologist J. Hansen as co-author of the article by von Schuckmann et al., [2]: his interpretation is that the positive accumulation of terrestrial heat of $(0.47 \pm 0.1)\ W/m^2$ does not have a millennial character but is to be completely attributed to the contemporary phenomenon of Global Warming.

Results: 3. Reconstruction of the millennial seasonal cycle, (Northern Hemisphere)

In the solution formula for the earth's temperature, (Appendix, equation A9), the angle of revolution (α) is the assigned parameter while the precession angle $\beta < 0$ is the independent variable. The sequence of the annual seasons: winter, spring, summer and autumn takes place when the angle of revolution takes on the values in an orderly fashion: $\alpha = (\beta;\ \beta + \frac{\pi}{2};\ \beta + \pi;\ \beta + \frac{3\pi}{2})$. With this expedient of assigning the parameter (α), we obtain the most probable millennial entropic profile of the temperature and of the solar forcing, at the four seasons of each single year, shown here in Figures 5a and 5b.

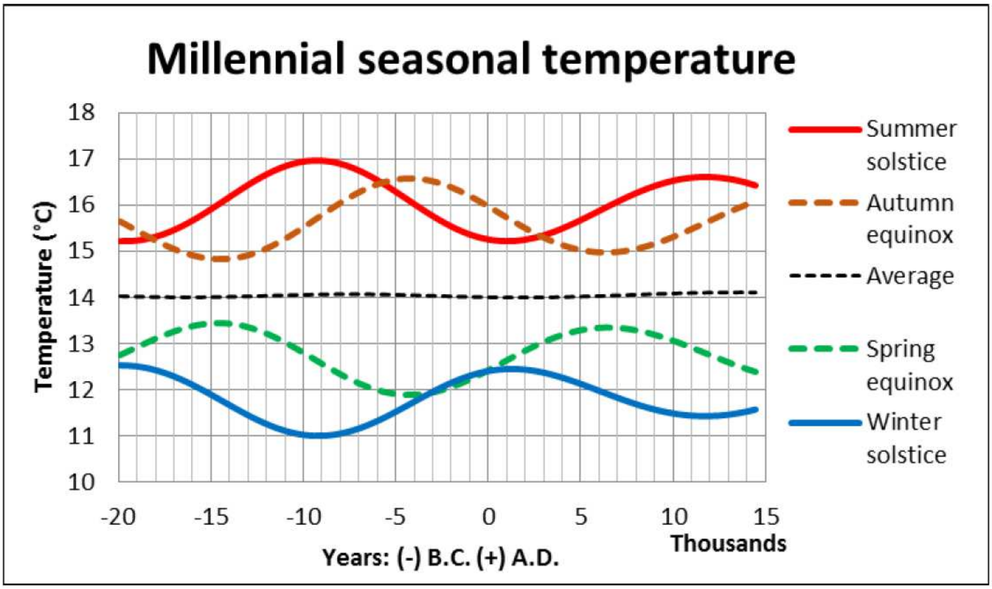

Figure 5a: Entropic reconstruction of the most probable profile of millennial seasonal temperatures, (Appendix, equation A9).

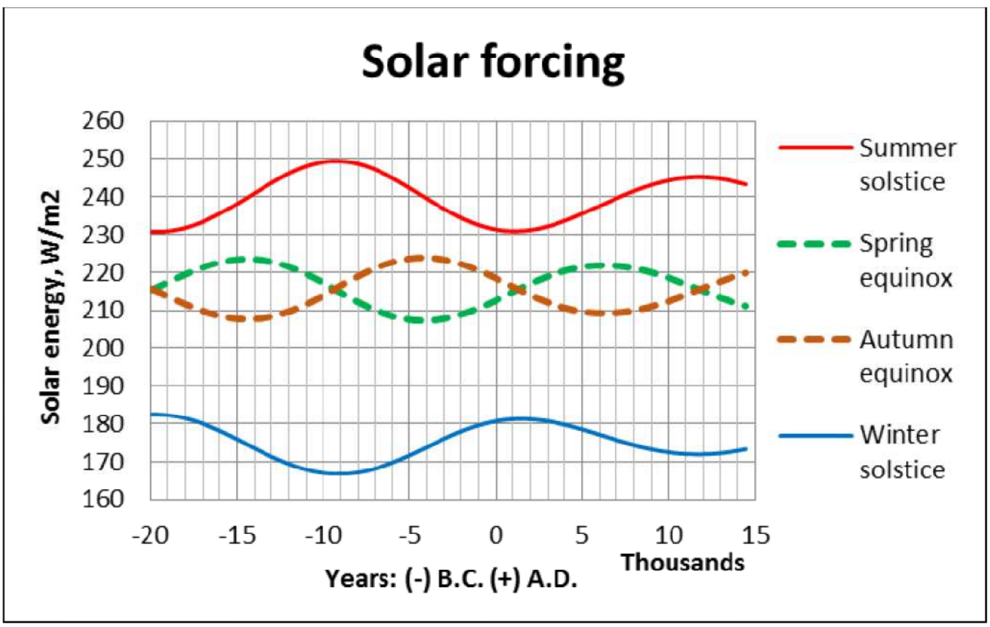

Figure 5b: Reconstruction of the solar forcing, (Appendix, equation A8).

Seasonal temperatures and seasonal solar forcing are almost exactly in phase. Figure 5c better highlights this fact: it combines the two variables thus representing temperatures as a function of the respective forcing, almost exactly, as a line segment.

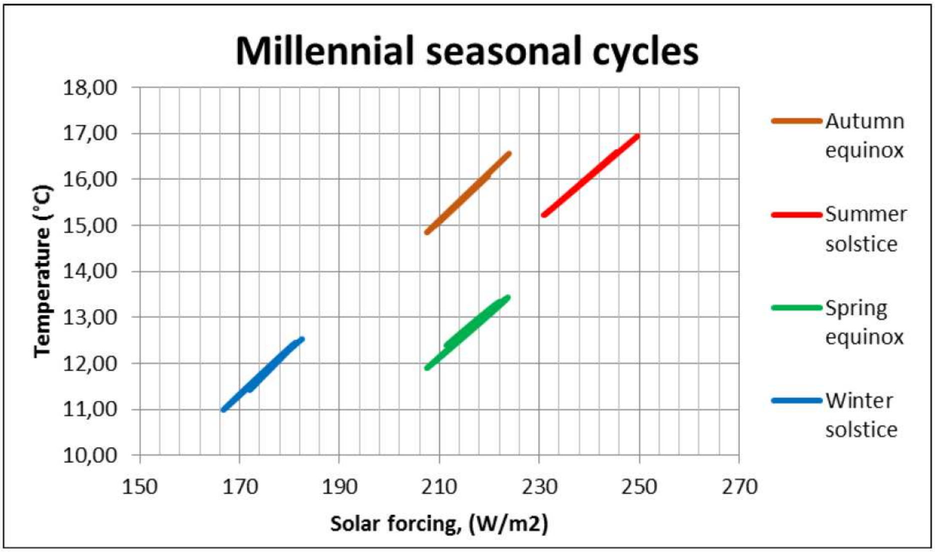

Figure 5c: Millennial cycles of seasonal temperatures as a function of seasonal forcing.

These diagrams of the four millennial seasons of the Earth are exciting because they shed light on the past and provide a possible interpretation of the climate changes affecting our planet. First, note that in Figure 5a, summer and winter temperatures are almost exactly in phase opposition. Second, note that if we let the winter solstice temperature guide us then it is natural to attribute a pejorative climate meaning when winter temperatures drop towards a minimum and vice versa when they tend to rise towards a maximum. Combining these two notes, we can attribute a meaning of climatic "*pessimum*" to the maximum excursion between the solstices, (9,200 BC) and, vice versa, the meaning of climatic "*optimum*" to the minimum excursion between them, and (1,200 AD). This type of analysis can also be applied to Figure 5b, reaching the same conclusions: The summer solar forcing is in phase opposition with the winter forcing thus clearly identifying a maximum and a minimum excursion in correspondence with the same couple of years as in Figure 5a. On the other hand, Figure 5c confirms the substantial variation, in phase, between temperature and solar forcing. In this regard, we observe that Milankovic based his paleo-climatic considerations precisely on the analysis of the solar forcing alone, [4]. The considerations made by means of this paleo-climatic model fully confirm Milankovic's choice, specifying the scope of validity: not all the solar forcing but only the excursion between the summer forcing and the winter forcing has full rights to describe and identify climatic events of geological interest, that is, climatic optimum and pessimum events. Finally, we underline a further extraordinary meaning underlying the maximum entropy variation highlighted in Figure 1 and expressed by equation (4) which identifies the free parameters of the paleo-climatic model. The coordinates of the point of maximum slope of the straight line, in Figure 1, identify an astronomical invariant of the Earth / Sun system, as they depend only on the astronomical parameters: eccentricity, axis inclination and (mainly) precession angle. As the precession angle varies the temperature difference between the two solstice changes and therefore the associated probability level. We have just attributed a climatic significance to the temperature excursion between the summer solstice and the winter solstice: there is an Optimum climate when the excursion is the minimum and a Pessimum climate when the excursion is the maximum. Now we are also able to associate a different level of probability to the two meanings: the Pessimum climate event has the highest probability, being associated with the maximum temperature excursion at the solstices, that is, with the maximum variation in entropy. Conversely, for the Optimum it has the lowest probability. Ultimately, the climatic Pessimum event is much more likely than the climatic Optimum event. This entropic consideration is of extraordinary interest. It provides a possible interpretative key to the experimental evidence that glaciations have been so frequent, in the geological history of the Earth, compared to the interglacial periods of more favorable climate.

Results: 4. Significance of the identified parameters

It is crucial that the free parameters (V, V_1, T_0) of the linearized paleo-climatic astronomical model, equation (1), identified with the procedure described above, have a sensible value. Only in this case, will it make sense to trust the projections drawn from the model, because of the value attributed to them. The analysis of the accumulation and of the loss term will help to answer this question positively. The identified parameters and their predicted values will result coherent the one another.

The term of accumulation. A geometric model of the Earth's heat storage, represented as a core / shell-1 / shell-2 composite system, will help describe the thermal behavior of the various compartments (oceans, land, Antarctic ice caps and atmosphere). The core represents the interior of the Earth but does not contribute, with the endogenous heat flow, to the solar energy balance, [11]. The shell-1 represents the earth's crust, made up of oceans, land and ice of the Antarctic polar cap. The shell-2 represents the atmosphere. As a first approximation, the Earth's crust and the atmosphere have equal weight in determining the thermal profile of the Earth. The crust, in turn, exerts its weighted effect through the contribution of the surface of the oceans ($\phi_W = 70.8\ \%$), the mainland ($\phi_T = 26.7\ \%$) and the Antarctic polar ice cap ($\phi_G = 2.5\ \%$). Consequently, the identified value of the accumulation transmittance $V_1 = 7.248$ [W/(m^{2}K)] can be decomposed as the sum of four terms, (A = Atmosphere, W = Oceans, T = Land and G = Antarctic Ice,): $V_1 = V_1^A + (\varphi_W V_1^W + \varphi_T V_1^T + \varphi_G V_1^G)$. Now we will evaluate the congruence of the identified value V_1 with the calculated value for each compartment of the Earth, just mentioned. For this purpose it is necessary to estimate the thickness of each single compartment involved in the phenomenon of heat accumulation, at a given reference time. Consistent with the duration of the millennial cycle, the reference time is $t_1 = 1.852\ 10^9$ s (amounting to 58.696 years). It is the time necessary to travel the arc of one degree of precession of the Earth. Table 2 collects the physical properties of interest of these compartments of the Earth.

Table 2: Physical properties of some substances, [11].

	ρ	C_p	K	$D = K/(\rho\ C_p)$
	$\mathrm{g/m^3}$	J/(g K)	J/(m K)	$\mathrm{m^2/s}$
Atmosphere, (A)	$1.29\ 10^3$	1.00	$2.423\ 10^{-2}$	$1.87\ 10^{-5}$
Land, (T)	$2.50\ 10^6$	0.837	0.963	$4.60\ 10^{-7}$
Antarct. ice, (G)	$0.92\ 10^6$	2.101	2.219	$1.15\ 10^{-6}$
Oceans, (W)	$1.00\ 10^6$	4.186	0.603	$1.44\ 10^{-7}$

1 - Atmosphere. Atmospheric photochemical studies indicate that not all of the atmosphere but only greenhouse gases (GHG) participate in the accumulation of solar energy. The prevalent greenhouse gas is water vapor, H_2O, while the triatomic gases, CO_2, NO_2, SO_2 and polyatomic, CH_4, NH_3 … contribute to a lesser extent. A simple dimensional analysis suggests writing down the accumulation transmittance of the atmosphere as:

$$V_1^A = \varepsilon \frac{\gamma Cp}{t_1} = 9.426\ 10^{-5}\ \ [\mathrm{W/(m^2K)}]. \tag{9}$$

where the factor $\varepsilon = v/V$ is the volumetric fraction of greenhouse gases contained in the atmosphere. Its value is $\varepsilon \cong 16{,}900$ ppmv at the average temperature of 14°C and water vapor saturation. The factor $\gamma = m/S$ is the surface density, that is the ratio between the mass of the atmosphere and the Earth's surface. The Cp factor is the specific heat of dry air.

2 - Land. The soil is a compartment in which the propagation of heat is mainly by conduction. The Fourier number set equal to one provides the estimate of the secular depth of conduction. We get $h = \sqrt{Dt_1} = 29.188$ m and therefore, the estimate of the soil transmittance:

$$V_1^T = \frac{\rho\, Cp\, h}{t_1} = 3.298\ \ 10^{-2}\ \ [\mathrm{W/(m^2K)}]. \tag{10}$$

Consequently, the contribution of the land to the millennial accumulation of energy is:

$$\varphi_T^* V_1^T\ =\ 8.805\ \ 10^{-3}\ \ [\mathrm{W/(m^2K)}]. \tag{11}$$

3 - Antarctic ice. Similarly, to the land, the Antarctic polar cap constitutes a compartment in which heat propagates, mainly, by conduction. Again, the Fourier number set equal to one provides the conduction thickness. We obtain $h = \sqrt{Dt_1} = 46.150$ m and therefore, the estimate of the accumulation transmittance of perennial ice is:

$$V_1^G\ =\ \frac{\rho\, Cp\, h}{t_0}\ =\ 4.817\ \ 10^{-2}\ \ [\mathrm{W/(m^2K)}]. \tag{12}$$

Consequently, the contribution of the Antarctic ice cap to the millennial accumulation of energy is:

$$\phi_G^* V_1^G\ =\ 1.204\ \ 10^{-3}\ \ [\mathrm{W/(m^2K)}]. \tag{13}$$

4 - Oceans. It is not easy determining the depth of heat accumulation, due to sea currents affecting surface waters and deep waters, with different transport mechanisms. However, we can assign an upper limit given by the depth of penetration of sunlight, equal to about 200 m; at greater depths, there is complete darkness. A lower limit is the average depth of the oceans of around 4,000 m. The depth of 200 meters is the experimental horizon of penetration of sunlight and therefore of the radiative transport of heat. However, the deep and abyssal surface sea currents are responsible for a complex of convective phenomena of additional heat transport, so that the average effective depth will be between the two limits indicated above. In conclusion, only the transmittance of the oceans is difficult to predict with adequate certainty. However, it can be estimated, a posteriori, after having carried out the identification of the parameters that provide the value of the total accumulation transmittance $V_1 = 7.248$ [W/(m^{2}K)] and subtracting from this value the estimated and substantially insignificant contribution of the mainland, ice and atmosphere compartments. The estimate of the contribution of the oceans to the millennial accumulation of energy thus results:

$$\phi_W * V_1^W = V_1 - V_1^A - \varphi_T V_1^T - \varphi_G V_1^G = 7.238 \text{ [W/(m}^2\text{K)]}. \tag{14}$$

From this identity, we obtain:

$$V_1^W = 10.223 \text{ [W/(m}^2\text{K)]}. \tag{15}$$

Having thus identified the value of V_1^W of the accumulation transmittance of the oceans, if we place ourselves in the Newtonian perspective of considering the Earth as a material point, then we can re-examine its definition in order to deduce the depth of the waters affected by the accumulation of the heat. On the millennial scale of the precession, this depth results:

$$h_1 = \frac{V_1^W * t_1}{\rho\, Cp} \cong 4{,}370 \text{ m}. \tag{16}$$

By comparing this identified depth value, $h_1 \cong 4{,}370$ m, with the estimated mean depth of oceanic waters, $H \cong 4{,}000$ m, we deduce that the Newtonian hypothesis is appropriate and provides consistent results. Therefore, the identified value of the accumulation transmittance is reliable. We conclude, see Figure 6a, that the entire mass of oceanic waters is interested in the entropic phenomenon of millennial heat accumulation / release and that the contribution of land, Antarctic ice and atmosphere is insignificant on a millennial scale.

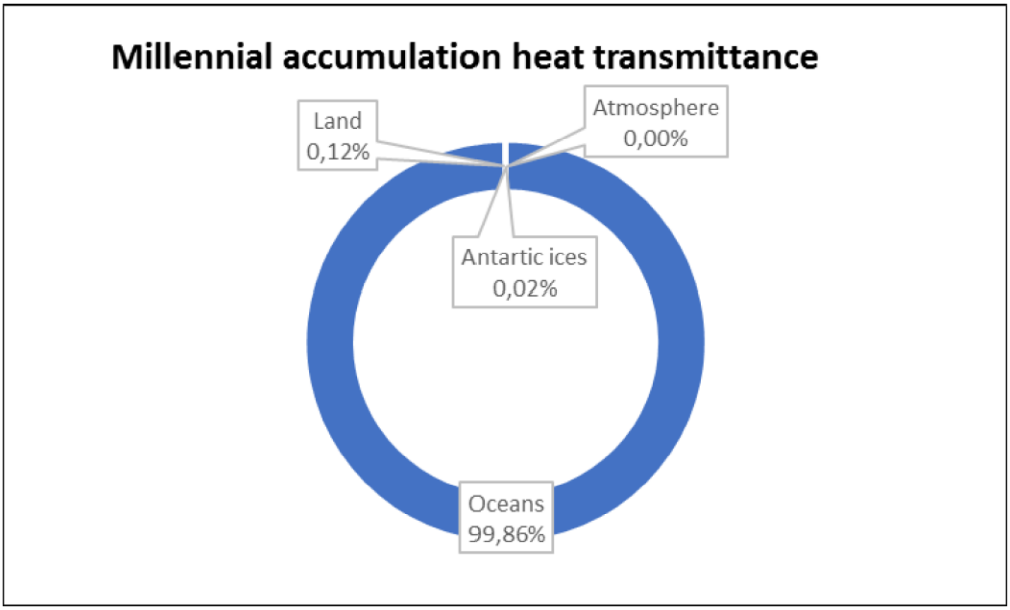

Figure 6a: Breakdown of the heath transmittances of millennial accumulation.

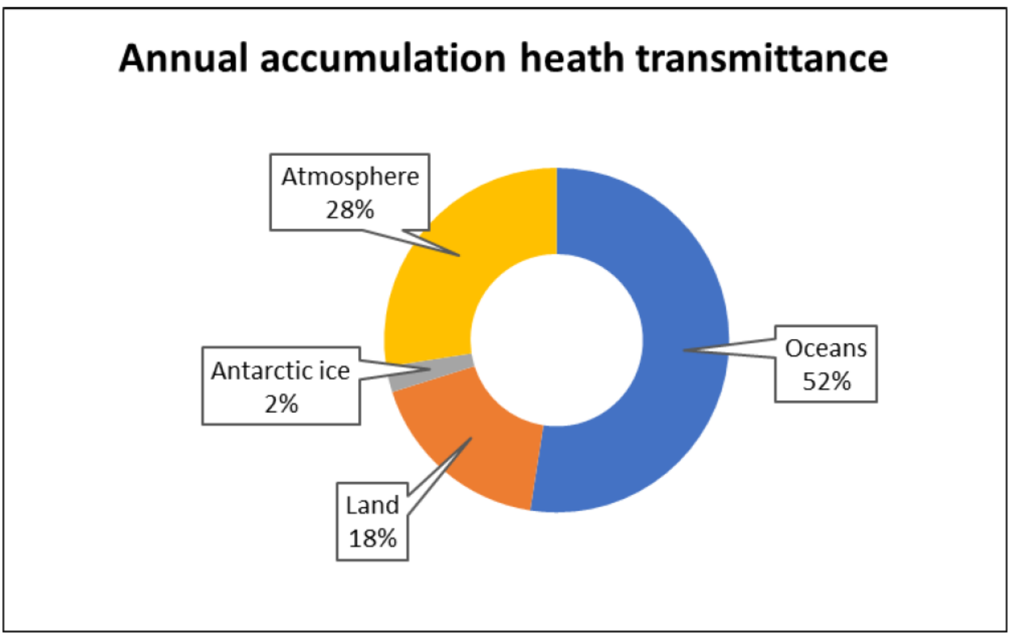

Figure 6b: Breakdown of the heath transmittances of annual accumulation.

The change of variable from (β) to (α) in the accumulation term of the model equation (1) gives the annual accumulation transmittance:

$$V_0 = V_1 \cdot \alpha/\beta. \tag{17}$$

By repeating the procedure indicated above by equations from (9) to (16), simply by changing the reference time from t_1 to $t_0 = 8.766\ 10^4$ s produces the breakdown of the annual accumulation transmittances. The new reference time is consistent with the duration of the annual temperature cycle. It corresponds to an average day and is the time needed to travel the arc of one degree of revolution of the Earth. Contrary to the millennial case, all compartments now have significant weight, because of the time inverse dependence of the heath transmittance. Figure 6b shows the breakdown of the annual heath transmittances of accumulation: Oceans, 52%; Atmosphere, 28%; Land, 18% and Antarctic ice, 2%. Table 3 collects the estimate of the accumulation transmittances of the compartments: atmosphere, land, Antarctic ice and oceans, for both the millennial and the annual case.

Table 3: Estimation of the millennial and annual accumulation transmittances.

	Millennial case				Annual case		
	ε	$\gamma = m_A/S_T$	$V_1^A = \varepsilon\frac{\gamma Cp}{t_1}$		ε	$\gamma = m_A/S_T$	$V_0^A = \varepsilon\frac{\gamma Cp}{t_0}$
	ppmv	[g/(m^2)]	[W/(m^{2}K)]		ppmv	[g/(m^2)]	[W/(m^{2}K)]
Atmosphere, (A)	16,900	$1.033\ 10^7$	$9.426\ 10^{-5}$		16,900	$1.033\ 10^7$	1.990
index (i)	$h_1 = \sqrt{Dt_1}$	$V_1^i = \frac{\rho\ Cp\ h_1}{t_1}$	$\varphi_i * V_1^i$		$h_0 = \sqrt{Dt_0}$	$V_0^i = \frac{\rho\ Cp\ h_0}{t_0}$	$\varphi_i * V_0^i$
	m	[W/(m^{2}K)]	[W/(m^{2}K)]		m	[W/(m^{2}K)]	[W/(m^{2}K)]
Land, (T)	29.188	$3.298\ 10^{-2}$	$8.805\ 10^{-3}$		29.188	4.798	1.281
Antarct. ice, (G)	46.150	$4.817\ 10^{-2}$	$1.204\ 10^{-3}$		46.150	7.00	0.175
Oceans, (W)	4,370	10.223	7.238		0.112	5.535	3.919
Total estimate		V_1	7.248			V_0	7.365

The loss term. The empirical parameter V obtained from the linearization of the Stefan-Boltzmann radiation equation goes back to universal constants. In fact, its analytical expression and its identified numerical value are $V = 2 \cdot (4\sigma \cdot E \cdot T_M^3) = 10.206$ [W/(m^{2}K)], (being σ the Boltzmann constant, E the emissivity, T_M the average temperature). The factor 2 explains the fact that the atmosphere and the earth's crust contribute with equal weight to determining the thermal profile of the Earth. The identified value of the Earth's emissivity, $E = 0.95$, is consistent with the estimates of this magnitude available in the literature, [11].

Conclusions

Having become aware of an entropic principle underlying the parameters identification of the paleo-climatic model, allows us to grasp an aesthetic vision of the Universe, which is difficult to reach by any other way. In fact, the principle of maximum entropic variation of a thermal half-cycle allows to identify one and only one string of free parameters, (V, V_1, T_0) which is associated with the highest probability among all probable temperature profiles. Furthermore, we can classify the different temperature profiles, generated by the various existing climate models, based on the respective entropy variation presented between the two solstices and establish, among all, which is the most probable profile. The tabulation of the analytical solution of this model provided, easily, three results referred to the northern hemisphere. 1 - The reconstruction of an annual seasonal cycle. 2 - The reconstruction of the average millennial temperature of the Earth, 3 - the reconstruction of the millennial seasonal cycle. These three reconstructions constitute, from a theoretical point of view, the most probable temperature profile among all the possible profiles, as solutions of the paleo-climatic model. However, the three reconstructions also have a practical sense because they allow to validate the model and to answer two questions of contemporary experimental and theoretical research as outlined below:

1. The reconstruction of the annual seasonal cycle, Figure 3a, superimposes on the experimental temperature profile and therefore constitutes a timely validation of the model for the reference year, 1975.

2. The reconstruction of the millennial average temperature of the Earth, (Figure 4a), is of particular practical interest, because it adds a second validation over a period of about 60 years, therefore, already with a paleo-climatic character. Indeed, von Schuckmann et al., [1], [2] recently published experimental data concerning the increase of sea temperature: a group of over 30 researchers from scientific institutions around the

world have traced and quantified the global marine heat distribution for nearly 60 years, (1960-2018). In this interval of years the accumulated heat was equal to $(358 \pm 37)\cdot 10^{21}$J, equivalent to a positive accumulation of (0.47 ± 0.1) W/m^2. How to interpret this experimental result? If we read it in the perspective of a precession cycle, then the paleo-climatic model interprets it as a millennial accumulation of heat of the Earth. In this regard, we observe that the reconstruction of the millennial energy balance is amazing due to the fact that it captures the order of magnitude of the experimental data: in fact, the theoretical prediction of the model is: 0.118 W/m^2. The discrepancy between the theoretical datum and the experiments is apparent and can be reconciled, reducing them to a unique number. Consider that experiments refer to the average depth of 1,000 m while theoretical datum refer to the average depth of 4,000 m. Normalizing the experiments with the average depth of oceans produce immediate reconciliation: 0.473 W/m$^2\cdot$1,000 m/4,000 m = 0.118 W/m^2.

3. The reconstruction of the millennial seasonal cycle concerns the question of whether we can assign a level of probability to two events of capital interest for humanity: the onset of an ice age or the onset of a climatic optimum. The paleo-climatic model provides the answer that the probability of the onset of an ice age is higher than the advent of a climatic optimum, within an interglacial period.

APPENDIX: The paleo-climatic model of a precession cycle

Any astronomical theory of paleo-climate aims to establish a link between the flux of solar energy and the Earth's climate, on a global scale. Milankovitch, [4], fully developed in the previous century such an approach: he was the first to calculate the solar radiation received by the Earth (insolation), as a function of the geographic coordinates of the earth and astronomical parameters, eccentricity and inclination, variables on a millennial scale. For a critical and updated review of paleo-climatic models, see Loutre, [3] and Berger, [4]. The paleo-climatic model developed, for the first time, in this work calculates the insolation, in terms of the alt-azimuth coordinates of the Earth referred to the ecliptic plane, [5], thus regardless of the geographical coordinates. For the construction of this model, we placed ourselves in the planetary perspective, of a distant observer, who sees the Earth as a small sphere, substantially blue due to the prevalence of the water that covers it, surrounded by a tenuous and thin gaseous atmosphere. There are three constitutive hypotheses of the planetary model:

1- The planets describe elliptical orbits around the Sun, (Kepler).

2- The planets are material points, (Newton).

3- Each planet is associated, in the same spirit as Newton, with an energy balance.

In this perspective, the paleo-climatic model applies to all planets of the solar system. The daily energy balance takes into account the spherical geometry of the Earth. The solar energy incident on the outer surface of the atmosphere will be a function of the position on the spherical surface through a Lambert extinction function, specifically modified by the author, for the horizontal coordinates of this model, [5]. The time scales taken into account by the model are daily scale of rotation of the Earth, annual scale of revolution, millennial scale of variation of eccentricity, [6], inclination of the rotation axis, [7] and the combined scale of precession and rotation of the line of apsides. (The counterclockwise precession speed of -50.256" /year and the clockwise rotation speed of the apsis line of +11.077" /year generate an equivalent speed of -61.333" / year, so that the combined cycle of the two movements takes place in about 21,200 years). The model consists of an energy balance ordinary differential equation for each assigned daily value of the angle of revolution ($\alpha = +2\pi t/\tau;\ \tau = 360\ t_0$). The equation has three terms, a source term, (radiative forcing), a loss term and an accumulation term. The forcing term takes into account the inclination of the Earth's axis, (δ), the eccentricity of the orbit, (e), both variable on a millennial scale and the angle of precession, ($\beta = -2\pi t/\vartheta;\ \ \vartheta = 360\ t_1$). We can attribute to the model equation the character of a definition and consider it, in effect, as the constitutive equation of the Earth model considered in this work:

$$\underbrace{-V_1 \frac{dT}{d\beta}}_{Accumulation} + \underbrace{\sigma E(T^4 - T_{0NL}^4)}_{Loss} = \underbrace{F(\alpha, \beta, e, \delta)}_{Forcing}. \tag{A1}$$

With the initial condition:

$$T(0) = T_i. \tag{A2}$$

The loss term $\sigma E \cdot (T^4 - T_{0NL}^4)$ is described by the Stefan-Boltzmann radiation equation for an opaque body. In order to obtain an analytical solution, we linearized the radiation equation, by series development truncated to the first order. The linearized loss term has the form $V \cdot\ (T - T_0)$ of the Newton's linear flux equation, where we set $V =\ 2U$, being $U = 4\sigma \cdot E \cdot T_i^3$ the loss transmittance, while T_0 is a parameter corresponding to a suitable troposphere temperature:

$$\underbrace{-V_1 \frac{dT}{d\beta}}_{Accumulation} + \underbrace{V(T - T_0)}_{Loss} = \underbrace{F(\alpha, \beta, e, \delta)}_{Forcing}. \tag{A3}$$

Note that linearization was possible because of the favorable circumstance that we can choose the initial temperature so that it is not very different from the current temperature while the error made with linearization can be reduced imposing the constraint:

$$T_0 = T_i \left[\frac{3}{4} + \frac{1}{4}\left(\frac{T_{0NL}}{T_i}\right)^4\right]. \tag{A4}$$

All information relating to the incoming flow of solar energy is contained in the forcing term $F(\alpha, \beta, e, \delta)$, whose analytical formulation constitutes the qualifying and distinctive aspect of the model. It is written, [5], as the product of six factors $F(\alpha, \beta, e, \delta) = F_0 F_1 F_2 F_3 F_4 F_5$. The factor F_0 describes the solar constant. The factor F_1 describes the Sun / Earth distance, according to Kepler's law, variable with the angular position and eccentricity of the Earth's orbit:

$$F_1 = \left(\frac{1 + e \cdot \cos\alpha}{1 - e^2}\right)^2. \tag{A5}$$

The factor F_2 marks the turn of the seasons as a function of the angle of revolution (α) and the angle of precession (β) taking into account the orbital parameters of the Earth, inclination of the axis, (δ) and eccentricity of the orbit, (e):

$$F_2 = \frac{\pi}{4} \cdot \frac{[1 - \sin\psi]}{[1 - \frac{2}{\pi}\psi]}. \tag{A6}$$

The factor F_2 requires three steps: Step 1). We modify the classic Lambert polar extinction function $L(\theta) = \cos\theta$ by positively rotating the coordinates by 90° so that the pole (θ) lies in the plane of the ecliptic and coincides with the point of intersection of the terrestrial equator with the line of illumination that separates day from night, Figure 7.

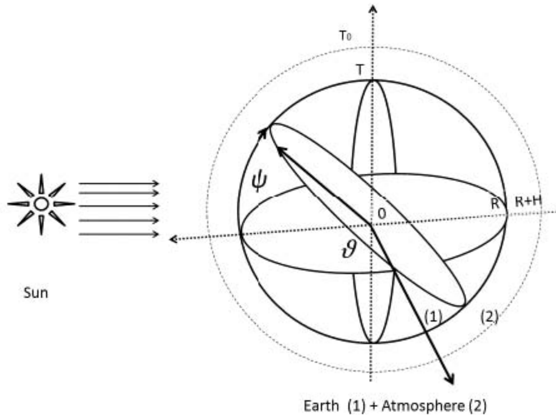

Figure 7: Polar coordinate system with the pole on the ecliptic plane.

The new extinction function is almost isotropic and is written $L^*(\theta, \psi) = \sin\theta \cos\psi$, where the coordinate ($\psi$) describes the azimuth angle of the earth's equator, starting from the ecliptic plane, at each days of the year. The function $L^*(\theta, \psi)$ assumes the maximum value on the ecliptic. The variation of the azimuth angle (ψ) can be approximated by the empirical function $\psi = \delta \cdot \cos(\alpha - \beta)$ which therefore constitutes the simplest equation of time. The sequence of the annual seasons: winter, spring, summer, autumn, takes place when the angle of revolution takes on the values: $\alpha = (\beta;\ \beta + \frac{\pi}{2};\ \beta + \pi;\ \beta + \frac{3\pi}{2})$. Step 2). We calculate the integral of the new Lambert function, on the Earth's illuminated surface over 12 hours, day by day. Step 3). We normalize the integral by dividing it by the surface of the illuminated area in 24 hours, equal to double the area corresponding to 12 hours.

The factor F_3 is the attenuation function of solar energy due to the albedo of the Earth. The factor F_4 incorporates the effect of greenhouse gases, (Green House Gases, GHG); finally, the factor F_5 describes high altitude aerosols, of natural or anthropogenic origin. In this model, the focus is exclusively on the millennial effects induced by the pair of factors $(F_1; F_2)$, assuming that the solar constant F_0 and the remaining factors remain constant and equal to a single factor $\varphi = F_3 F_4 F_5 \cong 0.633$. Ultimately, the millennial forcing term has the form:

$$F(\alpha, \beta) = \frac{\varphi\, F_0}{4\, a_0} \cdot \frac{[1 - \sin\psi]}{\left[1 - \frac{2}{\pi}\,\psi\right]}. \tag{A7}$$

In order to obtain an analytical solution of the paleo-climatic model, we linearized the "exact" relationship of the solar energy flux $F(\alpha, \beta, e, \delta)$, by means of series development of circular functions truncated to the second order, so obtaining the forcing term in the form:

$$F(\alpha,\beta) = \frac{\varphi\, F_0}{4\, a_0}\{1 + a_1 \cos\alpha + a_2 \cos(\alpha-\beta) + b_0 \cos^2\alpha$$

$$+ b_1 \cos\alpha \cdot \cos(\alpha-\beta) + b_2 \cos^2(\alpha-\beta)\}. \qquad \text{(A8)}$$

The solution of the linearized balance equation (A3) is the sum of two components: an infinitesimal component and a finite component, both with a periodic character. The first tends asymptotically to zero as the precession angle (β) increases and is a function of the initial condition. The second is independent of the initial condition and describes the stable, periodic behavior of the solution. This finite component of the solution is the asymptotic solution, that is, the limit cycle. The analytical solution of the paleo-climatic model implicitly takes into account the different distribution of land and oceans at various latitudes and the effect of this distribution in determining the temperature profile. It has the following expression, where $R = -V/V_1$ is the relaxation parameter:

$$T = T_0 + \frac{\varphi \cdot F_0}{8\, a_0\, U}\{1 + a_1 \cos\alpha + b_0 \cos^2\alpha$$

$$+ \frac{R}{R^2+1}(a_2 + b_1 \cos\alpha)[R\cos(\beta-\alpha) + \sin(\beta-\alpha)]$$

$$+ \frac{R}{R^2+4} b_2\{\cos(\beta-\alpha)\,[R\cos(\beta-\alpha) + 2\sin(\beta-\alpha)] + \frac{2}{R}\}\}. \qquad \text{(A9)}$$

In this solution formula, the angle of revolution (α) is an assigned parameter while the precession angle $\beta <$ zero is the independent variable. Table 1 collects the numerical values of the constants.

References

[1] K. von Schuckmann, F. Gaillard and P.-Y. Le Traon, *J. Geophys. Res.* 114 (2009), C09007. https://doi.org/10.1029/2008JC005237

[2] K. von Schuckmann, et al., *Earth Syst. Sci. Data* 12 (2020), 2013-2041. https://doi.org/10.5194/essd-12-2013-2020

[3] A. Berger, Milankovitch, the father of paleoclimate modelling, Climate of the Past: Discussions, Preprint. Discussion started: 18 March 2021. https://doi.org/10.5194/cp-2021-9

[4] M. F. Loutre, Ice ages (Milankovic Theory), *Encyclopedia of Atmospheric Sciences* (2003), 995-1003. https://doi.org/10.1016/B0-12-227090-8/00173-1

[5] S. Mazzullo, *La Chimica e l'Industria* 94(5) (2012), 75; 94(6) (2012), 94.

[6] A. Berger and M. F. Loutre, *Quaternary Science Reviews* 10 (1991), 297. https://doi.org/10.1016/0277-3791(91)90033-Q

[7] J. Laskar, *Astronomy and Astrophysics* 157(1) (1986), 59-70.

[8] R. Bellman and R. Roth, *Quasilinearization and the Identification Problem*, World Scientific, 1983. https://doi.org/10.1142/0029

[9] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, 1.8-3: Cubic equation: Cardano's solution, p. 23, McGraw Hill Book Company, 1968.

[10] http://data.giss.nasa.gov/gistemp/graphs_v4/: GISTEMP Seasonal Cycle since 1880.

[11] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Chapter 1: Black body radiation, p. 21, Oxford Univ. Press, 1980.