**Earthline Journal of Mathematical Sciences**
ISSN (Online): 2581-8147
Volume 9, Number 2, 2022, Pages 179-199
https://doi.org/10.34198/ejms.9222.179199

# The Transmuted Marshall-Olkin Extended Topp-Leone Distribution

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**Abstract**

In this paper, we introduced an extension of the Marshall-Olkin Extended Topp-Leone distribution using the quadratic rank transmutation map (QRTM). Statistical properties of the proposed distribution are examined and its parameter estimates are obtained using the maximum likelihood method. A real data set defined on a unit interval is employed to illustrate the usefulness of the proposed distribution among existing distributions with bounded support.

## 1 Introduction

[6] introduced a method of adding a parameter to a family of distributions and called it "The Marshall-Olkin Extended Family of distributions". The authors defined the survival function of the Marshall-Olkin Extended family of distributions as

$$\bar{G}(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}, \qquad -\infty \leq x \leq \infty, \quad 0 < \alpha < \infty \tag{1}$$

Received: March 3, 2022; Accepted: April 5, 2022
2020 Mathematics Subject Classification: Primary 60E05; Secondary 62E99.
Keywords and phrases: Topp-Leone distribution, Marshall-Olkin method, transmutation.
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with the corresponding density function $g(x)$ defined as

$$g(x) = \frac{\alpha f(x)}{\left(1 - \bar{\alpha}\bar{F}(x)\right)^2}, \tag{2}$$

given that $G(x)$ is the cumulative distribution function of the family of distributions.

Many researchers have introduced generalizations of some well-known distributions using this method of generalization. [4] proposed the Marshall-Olkin Kumaraswamy distribution, [5] introduced the Marshall-Olkin Extended Power Lindley distribution, [14] proposed the Marshall-Olkin Half Logistic distribution, etc.

Recently, [10] developed a new bounded distribution by considering the one-parameter Topp-Leone distribution reported in [9] and [13] as the baseline distribution in equation (1), yielding a survival function of the form

$$\bar{G}(x) = \frac{\alpha\{1 - \left(1 - (1-x)^2\right)^\beta\}}{1 - \bar{\alpha}\{1 - \left(1 - (1-x)^2\right)^\beta\}}, \tag{3}$$

with corresponding cdf and pdf respectively defined as

$$G(x) = \frac{\left(1 - (1-x)^2\right)^\beta}{1 - \bar{\alpha}\{1 - \left(1 - (1-x)^2\right)^\beta\}} = \frac{\left(1 - (1-x)^2\right)^\beta}{\alpha + \bar{\alpha}\left(1 - (1-x)^2\right)^\beta}, \tag{4}$$

and

$$g(x) = \frac{2\alpha\beta\,(1-x)\left(1 - (1-x)^2\right)^{\beta-1}}{\left[1 - \bar{\alpha}\{1 - \left(1 - (1-x)^2\right)^\beta\}\right]^2}, \quad 0 \le x \le 1,\ \alpha, \beta > 0,\ \bar{\alpha} = 1 - \alpha. \tag{5}$$

Equations (4) and (5) respectively define the cumulative distribution function and density function of the Marshall-Olkin Extended Topp-Leone distribution.

Another interesting method of adding parameters to existing distributions to obtain a more flexible new family of distributions is the one based on the quadratic rank transmutation map (QRTM) proposed by [12]. This generated family of distributions is called the Transmuted Extended distributions.

Let $F_1$ and $F_2$ be two distribution functions with a common sample space. [12] defined the quadratic rank transmutation as

$$G_{R12}(u) = F_2(F_1^{-1}(u))$$

$$G_{R21}(u) = F_1(F_2^{-1}(u))$$

where $F^{-1}(\psi) = inf_{x \in R}\{F(x) \geq \psi\}, \quad \psi \in [0,1]$, that is,

$$G_{R12}(u) = u + \lambda u(1-u), \quad |\lambda| < 1, \quad u \in (0,1).$$

It follows that $F_1$ and $F_2$ satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1^2(x). \tag{6}$$

Equation (6) generates a transmuted distribution $F_2$ of $F_1$. More explicitly, the cumulative distribution function of the transmuted extended distribution with $G(x)$ as the baseline distribution is defined as

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2 = G(x)\left[1+\lambda-\lambda G(x)\right], \quad -1 < \lambda < 1, \tag{7}$$

so that differentiating equation (7) yields the transmuted density function as

$$f(x) = g(x)\left[(1+\lambda) - 2\lambda G(x)\right]. \tag{8}$$

A unique characterization of the Marshall-Olkin Extended and the Transmuted Extended method of generalization is the property of allowing the random variable of the proposed distribution to maintain the same support with the baseline distribution.

Several generalizations of some well-known distributions using this method have been explored by many researchers. [1] proposed the Transmuted Weibull distribution, [8] developed the Transmuted Lindley distribution, [3] introduced the Transmuted Generalized Lindley distribution, [2] presented the Transmuted

Generalized Quasi Lindley distribution, etc. In this paper, we have introduced the Transmuted Marshall-Olkin Extended Topp-Leone distribution.

The rest of the paper is organized as follows: The derivation of the Transmuted Marshall-Olkin Extended Topp-Leone (TMOETL) distribution is presented in Section 2. The Reliability analysis which include; the survival, hazard and reversed hazard functions are presented in Section 3. In Section 4, some statistical properties of the TMOETL distribution and the estimation of the unknown parameters of the TMOETL distribution are explored. The applicability of the proposed model along side with some existing bounded distributions using a real data set is illustrated in Section 5. Finally, in Section 6, we give a concluding remark.

## 2 The Transmuted Marshall-Olkin Extended Topp-Leone (TMOETL) Distribution

Let $X$ be a random variable of a continuous probability distribution with cumulative distribution function $G(x)$ and density function $g(x)$ defined in equations (4) and (5) respectively, then substituting equation (4) into (7), we obtain the cumulative distribution function of the Transmuted Marshall-Olkin Extended Topp-Leone (TMOETL) Distribution as

$$F_{TMOETL}(x,\alpha,\beta,\lambda) = \frac{\left(1-(1-x)^2\right)^\beta}{\alpha+\bar{\alpha}\left(1-(1-x)^2\right)^\beta}\left[1+\lambda-\lambda\left(\frac{\left(1-(1-x)^2\right)^\beta}{\alpha+\bar{\alpha}\left(1-(1-x)^2\right)^\beta}\right)\right], \quad (9)$$

with the corresponding density function obtained by differentiating equation (9) as

$$f_{TMOETL}(x) = \frac{2\alpha\beta(1-x)\left(1-(1-x)^2\right)^{\beta-1}}{\left[1-\bar{\alpha}\left(1-\left(1-(1-x)^2\right)^\beta\right)\right]^2}\left[1+\lambda-2\lambda\left(\frac{\left(1-(1-x)^2\right)^\beta}{1-\bar{\alpha}\left(1-\left(1-(1-x)^2\right)^\beta\right)}\right)\right]. \quad (10)$$

Using the generalized binomial expansion for any real number and $|q| < 1$, defined by

$$(1-q)^{-z} = \sum_{k=0}^{\infty} \binom{z+k-1}{k} q^k, \tag{11}$$

we can express the density function of the TMOETL distribution given in equation (10) as

$$f_{TMOETL}(x, \alpha, \beta, \lambda) = 2\alpha\beta\,(1-\lambda) \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\beta(2j+1)-1} \sum_{m=0}^{2k+1} \binom{2+i}{i} \binom{i}{j} \times \binom{\beta(2j+1)-1}{k} \binom{2k+1}{m} (-1)^{i+j+k+m}\, \bar{\alpha}^i x^m. \tag{12}$$

***Remark***: Note that the Transmuted Marshall-Olkin Extended Topp-Leone distribution reduces to the baseline (Marshall-Olkin Extended Topp-Leone) distribution when $\lambda = 0$ and reduces to the one-parameter Topp-Leone distribution when $\lambda = 0$ and $\alpha = 1$.

Figure 1 presents some possible shapes of the density function of the Transmuted Marshall-Olkin Extended Topp-Leone distribution for varying values of the parameters.

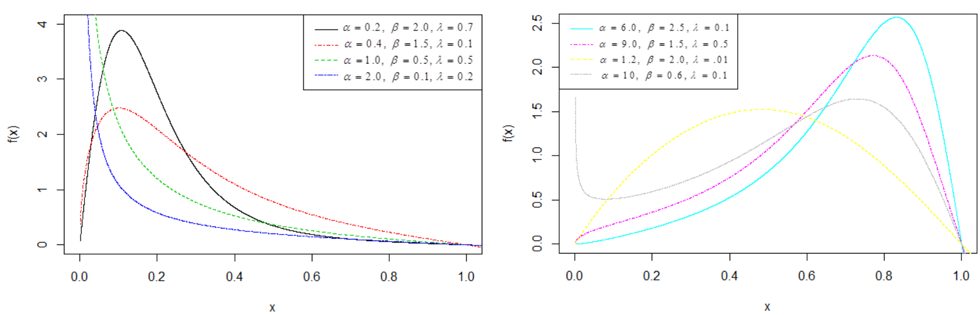


Figure 1: The pdf of the TMOETL distribution for varying values of the parameters.

Figure 1 clearly shows that for varying values of the parameters, the density function of the TMOETL distribution accommodates a decreasing (reversed-J), decreasing-increasing-decreasing, left-skewed, right-skewed and symmetric shapes.

# 3 Reliability Analysis

In this section, we introduce some characteristics of reliability analysis used in describing the length of life (working time) as well as the rate of failure of a component in a device (system). These include the survival, hazard and reversed hazard functions.

## 3.1 Survival Function

Let $T$ be a non-negative valued random variable with density function $f(t)$ and cumulative distribution function $F(t)$, then $T$ is called the survival time, if it determines the working time of a component in a system. The function is defined by

$$S(x) = 1 - F(x).$$

By this definition, the survival function of the Transmuted Marshall-Olkin Extended Topp-Leone distribution is given by

$$\begin{aligned} S_{TMOETL}(x, \alpha, \beta, \lambda) &= 1 - F_{TMOETL}(x, \alpha, \beta, \lambda) \\ &= \frac{\alpha\left(1 - \left(1-(1-x)^2\right)^{\beta}\right)}{\alpha + \bar{\alpha}\left(1-(1-x)^2\right)^{\beta}} \\ &\quad \times \left[1 - \lambda\left(\frac{\left(1-(1-x)^2\right)^{\beta}}{\alpha + \bar{\alpha}\left(1-(1-x)^2\right)^{\beta}}\right)\right]. \end{aligned} \tag{13}$$

## 3.2 Hazard and Reverse Hazard Function

Given that $T$ is the length of life of a device, then the rate at which failure occurs in the device after working satisfactorily up to time $t$ is referred to as the hazard rate function. The function is defined as

$$h(x) = \frac{f(x)}{1 - F(x)}. \tag{14}$$

From Equation (14), we can define the hazard rate function of the TMOETL distribution as

$$\begin{aligned} h_{TMOETL}(x, \alpha, \beta, \lambda) &= \frac{f_{TMOETL}(x, \alpha, \beta, \lambda)}{1 - F_{TMOETL}(x, \alpha, \beta, \lambda)} \\ &= \frac{\left[2\beta(1-x)\left(1-(1-x)^2\right)^{\beta-1}\right]\left[(1+\lambda)\left(\alpha+\bar{\alpha}\left(1-(1-x)^2\right)^{\beta}\right) - 2\lambda\left(1-(1-x)^2\right)^{\beta}\right]}{\left[1-\left(1-(1-x)^2\right)^{\beta}\right]\left[\alpha+\bar{\alpha}\left(1-(1-x)^2\right)^{\beta}\right]\left[\alpha+(\bar{\alpha}-\lambda)\left(1-(1-x)^2\right)^{\beta}\right]}. \end{aligned} \tag{15}$$

The reversed hazard rate function of the TMOETL distribution is given by

$$\begin{aligned} H_{TMOETL}(x, \alpha, \beta, \lambda) &= \frac{f_{TMOETL}(x, \alpha, \beta, \lambda)}{F_{TMOETL}(x, \alpha, \beta, \lambda)} \\ &= \frac{2\beta(1-x)\left[(1+\lambda)\left(\alpha+\bar{\alpha}\left(1-(1-x)^2\right)^{\beta}\right) - 2\lambda\left(1-(1-x)^2\right)^{\beta}\right]}{\left(1-(1-x)^2\right)\left[(1+\lambda)\left(\alpha+\bar{\alpha}\left(1-(1-x)^2\right)^{\beta}\right) - \lambda\left(1-(1-x)^2\right)^{\beta}\right]}. \end{aligned} \tag{16}$$

Figure 2 presents the graphical plots of the TMOETL distribution for different values of the parameters.

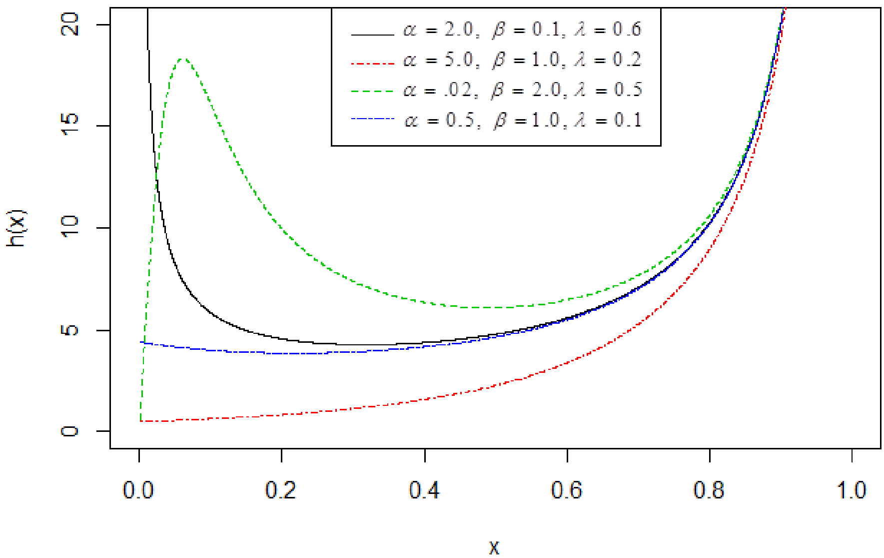


Figure 2: The hrf of the TMOETL distribution for varying values of the parameters.

Figure 2 clearly reveals that the hazard rate function of the TMOETL distribution accommodates an increasing, decreasing-increasing-decreasing, and bathtub shaped property.

## 4 Statistical Properties

In this section, we treat some statistical properties of the TMOETL distribution which include the quantile function, median, moments, moment generating and probability generating functions, Renyi entropy and the estimation of the unknown parameters of the TMOETL distribution using the maximum likelihood method.

## 4.1 Quantile Function

Let $X$ be a continuous random variable with cumulative distribution function $F(x)$. Then the quantile function of $X$ denoted by $x_q$ is defined as $x_q = F^{-1}(q), \quad q \in [0,1]$.

Hence, the quantile function of the TMOETL distribution can be obtained by solving for the root of the following system of non-linear equation:

from Equation (7),

$$G(x) = \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda},$$

that is,

$$\frac{\left(1-(1-x)^2\right)^\beta}{\alpha + \bar{\alpha}\left(1-(1-x)^2\right)^\beta} = \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda},$$

$$\left(1-(1-x)^2\right)^\beta = \frac{\alpha\left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right]}{1-\bar{\alpha}\left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right]},$$

$$x_q = 1 - \left[1 - \left(\frac{\alpha\left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right]}{1-\bar{\alpha}\left[\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}\right]}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{2}}. \quad (17)$$

The median of the TMOETL distribution is obtained from equation (17) by substituting $q = 0.5$, which yields

$$x_{0.5} = 1 - \left[ 1 - \left( \frac{\alpha \left[ \frac{(1+\lambda) - \sqrt{1+\lambda^2}}{2\lambda} \right]}{1 - \bar{\alpha} \left[ \frac{(1+\lambda) - \sqrt{1+\lambda^2}}{2\lambda} \right]} \right)^{\frac{1}{\beta}} \right]^{\frac{1}{2}}. \tag{18}$$

Random variates from the TMOETL distribution can be generated using Equation (17). Table 1 presents some quantiles from the TMOETL distribution at different values of the parameters.

Table 1: Some Quantiles from TMOETL Distribution for fixed value of $\lambda = 0.5$.

| $q$ | ($\alpha = 1$, $\beta = 2$) | ($\alpha = 1$,$\beta = 3$) | ($\alpha = 2$, $\beta = 1$) | ( $\alpha = 2$, $\beta = 2$) |
|---|---|---|---|---|
| 0.1 | 0.1405 | 0.2310 | 0.0660 | 0.1984 |
| 0.2 | 0.2088 | 0.3065 | 0.1313 | 0.2896 |
| 0.3 | 0.2680 | 0.3672 | 0.1966 | 0.3640 |
| 0.4 | 0.3247 | 0.4224 | 0.2628 | 0.4305 |
| 0.5 | 0.3820 | 0.4761 | 0.3313 | 0.4935 |
| 0.6 | 0.4427 | 0.5314 | 0.4036 | 0.5558 |
| 0.7 | 0.5104 | 0.5914 | 0.4829 | 0.6204 |
| 0.8 | 0.5913 | 0.6613 | 0.5748 | 0.6919 |
| 0.9 | 0.7010 | 0.7540 | 0.6944 | 0.7813 |

## 4.2 Moments

The $r^{th}$ moments of a continuous random variable $X$ with density function $f(x)$, is defined by

$$E[X^r] = \int_{-\infty}^{\infty} x^r f(x) dx, \tag{19}$$

substituting the series representation of the density function of TMOETL distribution defined in Equation (12) into (19), we have

$$E[X^r] = 2\alpha\beta(1-\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k} \times \binom{2k+1}{m}(-1)^{i+j+k+m}\,\bar{\alpha}^i\int_0^1 x^{r+m}dx, \tag{20}$$

$$= 2\alpha\beta(1-\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k} \times \binom{2k+1}{m}\frac{(-1)^{i+j+k+m}\,\bar{\alpha}^i}{r+m+1}. \tag{21}$$

The first four $r^{th}$ moments of the TMOETL distribution can be obtained by taking $r = 1, 2, 3,$ and 4 respectively in Equation (21) as

$$\mu_1^{'} = 2\alpha\beta(1-\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{i+j+k+m}\,\bar{\alpha}^i}{m+2},$$

$$\mu_2^{'} = 2\alpha\beta(1-\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{i+j+k+m}\,\bar{\alpha}^i}{m+3},$$

$$\mu_3^{'} = 2\alpha\beta(1-\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{i+j+k+m}\,\bar{\alpha}^i}{m+4},$$

$$\mu_4^{'} = 2\alpha\beta(1-\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{i+j+k+m}\,\bar{\alpha}^i}{m+5}.$$

The variance, measures of skewness and kurtosis of the TMOETL distribution can be derived by substituting the values of the $r^{th}$ moments into the expressions;

$$\text{Variance }(\sigma^2) = \mu_2^{'} - \mu^2, \qquad \text{Skewness}(S_k) = \frac{\mu_3^{'} - 3\mu_2^{'}\mu + 2\mu^3}{\left(\mu_2^{'} - \mu^2\right)^{\frac{3}{2}}},$$

$$\text{Kurtosis}(K_s) = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{\left(\mu_2' - \mu^2\right)^2}.$$

Table 2 presents some numerical computations of the theoretical moments of the TMOETL distribution for selected values of the parameters.

Table 2: Moments of TMOETL Distribution for selected values of the parameters.

| $\lambda$ | $\alpha$ | $\beta$ | $\mu_1'$ | $\mu_2'$ | $\mu_3'$ | $\mu_4'$ | $\sigma^2$ | $S_k$ | $K_s$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 1 | 2 | 0.4032 | 0.2063 | 0.1221 | 0.0796 | 0.0437 | 0.4002 | 2.4189 |
| | | 4 | 0.5402 | 0.3249 | 0.2115 | 0.1462 | 0.0331 | 0.0407 | 2.3614 |
| | | 6 | 0.6126 | 0.4010 | 0.2768 | 0.1994 | 0.0257 | -0.0889 | 2.3430 |
| | 3 | 2 | 0.5441 | 0.3419 | 0.2336 | 0.1688 | 0.0459 | -0.2367 | 2.2649 |
| | | 4 | 0.6582 | 0.4640 | 0.3431 | 0.2629 | 0.0308 | -0.5206 | 2.7760 |
| | | 6 | 0.7149 | 0.5340 | 0.4123 | 0.3268 | 0.0229 | -0.642 | 3.2021 |

We observe from Table 2, that the TMOETL distribution can be positively-skewed and negatively-skewed. This clearly affirms the graphical plots in Figure 1.

## 4.3 Moment Generating and Probability Generating Functions

The moment generating function, $M_x(t)$ of a continuous random variable $X$ is defined by

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx. \tag{22}$$

substituting Equation (12) into (22), we have

$$M_x(t) = 2\alpha\beta\,(1-\lambda)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k} \times \binom{2k+1}{m}(-1)^{i+j+k+m}\,\bar{\alpha}^i\int_0^1 e^{tx}x^m dx, \tag{23}$$

but

$$e^{tx} = \sum_{n=0}^{\infty}\frac{(tx)^n}{n!},$$

so that Equation (23) now becomes

$$M_x(t) = 2\alpha\beta\,(1-\lambda)\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k} \times \binom{2k+1}{m}\frac{(-1)^{i+j+k+m}\,\bar{\alpha}^i t^n}{n!(n+m+1)}. \tag{24}$$

Using similar approach, the probability generating function, $M_{[x]}(t)$ of the TMOETL distribution is defined as

$$M_{[x]}(t) = 2\alpha\beta\,(1-\lambda)\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}\binom{\beta(2j+1)-1}{k} \times \binom{2k+1}{m}(-1)^{i+j+k+m}\,\bar{\alpha}^i\int_0^1 t^x x^m dx, \tag{25}$$

$$= 2\alpha\beta(1-\lambda)\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{\beta(2j+1)-1}\sum_{m=0}^{2k+1}\binom{2+i}{i}\binom{i}{j}$$
$$\times\binom{\beta(2j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{i+j+k+m}\,\bar{\alpha}^{i}(lnt)^{n}}{n!(n+m+1)}. \qquad (26)$$

since,

$$t^{x} = \sum_{n=0}^{\infty}\frac{(lnt)^{n}x^{n}}{n!}.$$

## 4.4 Renyi Entropy

An entropy is the degree of randomness or uncertainty associated with a random variable $X$. [11] defined an entropy of a random variable with pdf $f(x)$ as

$$\tau_R(\gamma) = \frac{1}{1-\gamma}log_e\left[\int f^{\gamma}(x)dx\right], \qquad \gamma > 0, \gamma \neq 1, \qquad (27)$$

From the definition, the Renyi entropy of the TMOETL distribution is thus obtain as

$$\tau_R(\gamma) = \frac{1}{1-\gamma}log_e\left[\int f^{\gamma}_{TMOETL}(x)dx\right], \qquad \gamma > 0, \gamma \neq 1, \qquad (28)$$

substituting Equation (10) into (28), we have

$$\tau_R(\gamma) = \frac{1}{1-\gamma}log_e\int_0^1\left[\frac{2\alpha\beta(1-x)\left(1-(1-x)^2\right)^{\beta-1}}{\left[1-\bar{\alpha}\left(1-(1-(1-x)^2)^{\beta}\right)\right]^2}\left[1+\lambda-2\lambda\left(\frac{\left(1-(1-x)^2\right)^{\beta}}{1-\bar{\alpha}\left(1-(1-(1-x)^2)^{\beta}\right)}\right)\right]\right]^{\gamma}dx, \quad (29)$$

using the generalized binomial expansion defined in equation (11), we have

$$\left[1+\lambda\left(1-2\left[\frac{\left(1-(1-x)^2\right)^{\beta}}{1-\bar{\alpha}\left(1-(1-(1-x)^2)^{\beta}\right)}\right]\right)\right]^{\gamma} = \sum_{i=0}^{\gamma}\binom{\gamma}{i}\lambda^{i}\left(1-2\left[\frac{\left(1-(1-x)^2\right)^{\beta}}{1-\bar{\alpha}\left(1-(1-(1-x)^2)^{\beta}\right)}\right]\right)^{i}, \quad (30)$$

$$\left(1-2\left[\frac{\left(1-(1-x)^2\right)^{\beta}}{1-\bar{\alpha}\left(1-(1-(1-x)^2)^{\beta}\right)}\right]\right)^{i} = \sum_{j=0}^{i}\binom{i}{j}(-1)^{j}2^{j}\left[\frac{\left(1-(1-x)^2\right)^{\beta}}{1-\bar{\alpha}\left(1-(1-(1-x)^2)^{\beta}\right)}\right]^{j}, \quad (31)$$

$$\left(1-\left(1-(1-x)^2\right)^{\beta}\right)^{2\gamma+j} = \sum_{k=0}^{2\gamma+j}\binom{2\gamma+j}{k}(-1)^{k}\bar{\alpha}^{k}\left[1-\left(1-(1-x)^2\right)^{\beta}\right]^{k}, \quad (32)$$

$$\left[1-\left(1-(1-x)^2\right)^{\beta}\right]^k = \sum_{m=0}^{k} \binom{k}{m} (-1)^m \left[1-(1-x)^2\right]^{\beta m}, \tag{33}$$

$$\left[1-(1-x)^2\right]^{\beta(m+\gamma+j)-\gamma} = \sum_{n=0}^{\beta(m+\gamma+j)-\gamma} \binom{\beta(m+\gamma+j)-\gamma}{n} (-1)^n (1-x)^{2n}, \tag{34}$$

$$(1-x)^{2n+\gamma} = \sum_{p=0}^{2n+\gamma} \binom{2n+\gamma}{p} (-1)^p x^p, \tag{35}$$

Thus, substituting Equations (30) - (35) into (29), we obtain the Renyi entropy of the TMOETL distribution as

$$\tau_R(\gamma) = \frac{1}{1-\gamma} log_e \sum_{i=0}^{\gamma} \sum_{j=0}^{i} \sum_{k=0}^{2\gamma+j} \sum_{m=0}^{k} \sum_{n=0}^{\beta(m+\gamma+j)-\gamma} \sum_{p=0}^{2n+\gamma} \binom{\gamma}{i} \binom{i}{j} \binom{2\gamma+j}{k} \times \binom{k}{m} \binom{\beta(m+\gamma+j)-\gamma}{n} \binom{2n+\gamma}{p} \frac{(-1)^{j+k+m+n+p}(2\alpha\beta)^{\gamma}\lambda^i 2^j \bar{\alpha}^k}{p+1}. \tag{36}$$

## 4.5 Parameter Estimation

Let $x = (x_1, x_2, \cdots, x_n)$ be random samples of size $n$ associated with the Transmuted Marshall-Olkin Topp-Leone distribution with pdf defined in equation (10). Then the log-likelihood function is defined by

$$\begin{aligned}
\ell &= \sum_{i=1}^{n} log \left\{ \frac{2\alpha\beta(1-x_i)\left(1-(1-x_i)^2\right)^{\beta-1}}{\left[\alpha+\bar{\alpha}\left(1-(1-x_i)^2\right)^{\beta}\right]^2} \left[1+\lambda-2\lambda\left(\frac{\left(1-(1-x_i)^2\right)^{\beta}}{\alpha+\bar{\alpha}\left(1-(1-x_i)^2\right)^{\beta}}\right)\right] \right\}, \\
&= nlog(2\alpha\beta) + \sum_{i=1}^{n} log(1-x_i) + (\beta-1)\sum_{i=1}^{n} log\left(1-(1-x_i)^2\right) \\
&\quad - 2\sum_{i=1}^{n} log\left[\alpha+\bar{\alpha}(1-(1-x_i)^2)^{\beta}\right] + \sum_{i=1}^{n} log\left[1+\lambda-2\lambda\left(\frac{\left(1-(1-x_i)^2\right)^{\beta}}{\alpha+\bar{\alpha}\left(1-(1-x_i)^2\right)^{\beta}}\right)\right]. \quad (37)
\end{aligned}$$

Differentiating the log-likelihood function defined in Equation (37) with respect to the parameters $\Phi = (\alpha, \beta, \lambda)$, we obtain the score function as

$$U(\Phi) = \left(\frac{\partial\ell}{\partial\alpha} \quad \frac{\partial\ell}{\partial\beta} \quad \frac{\partial\ell}{\partial\lambda}\right)^T,$$

where

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \frac{\alpha + (\bar{\alpha}-2)(1-(1-x_i)^2)^{\beta}}{\alpha(1+\lambda) + \left[(\bar{\alpha}+(\lambda-1))\left(1-(1-x_i)^2\right)^{\beta}\right]},$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{2\lambda(1-(1-x_i)^2)^{\beta}\left[(1-(1-x_i)^2)^{\beta}-1\right]}{\left[\alpha(1+\lambda)+(\bar{\alpha}+(\lambda-1))\left(1-(1-x_i)^2\right)^{\beta}\right]\left[\alpha+\bar{\alpha}(1-(1-x_i)^2)^{\beta}\right]} - 2\sum_{i=1}^{n} \frac{1-(1-(1-x_i)^2)^{\beta}}{\alpha+\bar{\alpha}(1-(1-x_i)^2)^{\beta}},$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} = {} & \frac{n}{\beta} + \sum_{i=1}^{n} log(1-(1-x_i)^2) - 2\sum_{i=1}^{n}\left[\frac{\bar{\alpha} log(1-(1-x_i)^2)\left(1-(1-x_i)^2\right)^{\beta}}{\alpha+\bar{\alpha}(1-(1-x_i)^2)^{\beta}}\right] \\ & - 2\lambda\sum_{i=1}^{n}\left[\frac{\left[\alpha log(1-(1-x_i)^2) - log(1-(1-x_i)^2)^{\beta}\right]\left(1-(1-x_i)^2\right)^{2\beta}}{\left[\alpha(1+\lambda)+(\bar{\alpha}+(\lambda-1))\left(1-(1-x_i)^2\right)^{\beta}\right]\left[\alpha+\bar{\alpha}(1-(1-x_i)^2)^{\beta}\right]}\right] \\ & - 2\lambda\sum_{i=1}^{n}\left[\frac{log(1-(1-x_i)^2)\left(1-(1-x_i)^2\right)^{\beta}}{\left[\alpha(1+\lambda)+(\bar{\alpha}+(\lambda-1))\left(1-(1-x_i)^2\right)^{\beta}\right]}\right].\end{aligned}$$

The derivatives of the log-likelihood function of the TMOETL distribution were obtained using the WOLFRAM MATHEMATICA 11.1 software. The unknown parameter estimates say $\hat{\Phi} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, can be obtained by solving the system of non-linear equation $\frac{\delta \ell}{\delta \Phi} = 0$. This is achieved by employing a standard numeric optimization algorithm such as the Newton-Raphson Iterative Scheme given by

$$\hat{\Phi} = \Phi_k - H^{-1}(\Phi_k)U(\Phi_k), \tag{38}$$

where $U(\Phi_k)$ is the score function and $H(\Phi_k)$ is the Hessian matrix, which is the second derivative of the log-likelihood function. The Hessian matrix is defined by

$$H(\Phi_k) = H(\Phi_k) = \begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} & \frac{\partial^2 \ell}{\partial \lambda^2} \end{pmatrix}.$$

The elements of the hessian matrix of the TMOETL distribution were omitted due to the complexity of the derivatives. The unknown parameter estimates of the TMOETL distribution were obtained using the *fitdistrplus* package in R statistical software.

## 5 Data Analysis

In this section, we illustrate the usefulness of the TMOETL Distribution along side with some well-known bounded distributions in analyzing a real data set. The distributions include the Marshall-Olkin Extended Kumaraswamy (MOEKwD), Transmuted Kumaraswamy (TKwD), Kumaraswamy (Kumar) and Beta distributions, etc. The data set represents 20 observations of the maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania and is reported in [7]. The data set is positively skewed with skewness value ($S_k = 0.9940$) and leptokurtic with kurtosis value ($K_s = 3.3054$). The data set include; 0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74.

The parameter estimates, maximized log-likelihood ($\ell$), Akiake Information criterion [$AIC = 2k - 2\log(L)$], Komolgorov-Smirnov ($K - S$) test statistic, Cramer-von Mises ($W^*$) test statistic and the Anderson Darling ($A^*$) test statistic with the corresponding $p$-value is employed to evaluate the fit of each distribution.

Table 3: Summary Statistics for the Maximum Flood Level Data Set.

| *Models* | *Parameter Estimates* | $\ell$ | *AIC* | $(K-S)$ $(p-value)$ | $W^*$ $(p-value)$ | $A^*$ $(p-value)$ |
|---|---|---|---|---|---|---|
| TMOETLD | $\alpha = 0.0216$ | | | | | |
| | $\beta = 8.6519$ | 16.1644 | -26.3289 | 0.1214 | 0.0387 | 0.2641 |
| | $\lambda = 0.0024$ | | | (0.9299) | (0.9440) | (0.9622) |
| MOEKwD | $\alpha = 0.0153$ | | | | | |
| | $\beta = 6.4530$ | 15.9235 | -25.8471 | 0.1297 | 0.0411 | 0.3133 |
| | $\lambda = 5.4128$ | | | (0.8899) | (0.9320) | (0.9271) |
| TKwD | $\alpha = 3.7431$ | | | | | |
| | $\beta = 11.1474$ | 13.6401 | -21.2802 | 0.2001 | 0.1262 | 0.7702 |
| | $\lambda = 0.6207$ | | | (0.3997) | (0.4746) | (0.5012) |
| Kumar | $\alpha = 3.3773$ | 12.9733 | -21.9465 | 0.2176 | 0.1654 | 0.9368 |
| | $\beta = 12.0005$ | | | (0.3002) | (0.3479) | (0.3909) |
| Beta | $\alpha = 6.8307$ | 14.1836 | -24.3671 | 0.2062 | 0.1242 | 0.7301 |
| | $\beta = 9.2364$ | | | (0.3627) | (0.4823) | (0.5323) |

Table 3 presents the parameter estimates, maximized log-likelihood, Akaike Information Criterion $[AIC = 2k - 2\log(L)]$, Komolgorov-Smirnov $(K-S)$ test statistic, Cramer-von Mises $(W^*)$ test statistic and the Anderson Darling $(A^*)$ test statistic with the corresponding $p$-values of the distributions for the maximum flood level data set. We clearly observe that the TMOETL distribution having the maximized log-likelihood and the least value of the $AIC, (K-S), A^*$ and $W^*$ demonstrates superiority over the Marshall-Olkin Extended Kumaraswamy (MOEKwD), Transmuted Kumaraswamy (TKwD), Kumaraswamy (Kumar) and Beta distributions in analyzing the real data set under study. We further validated this claim by inspecting the plot of the density and cumulative distribution fit of the distributions for the data set presented in Figure 3.

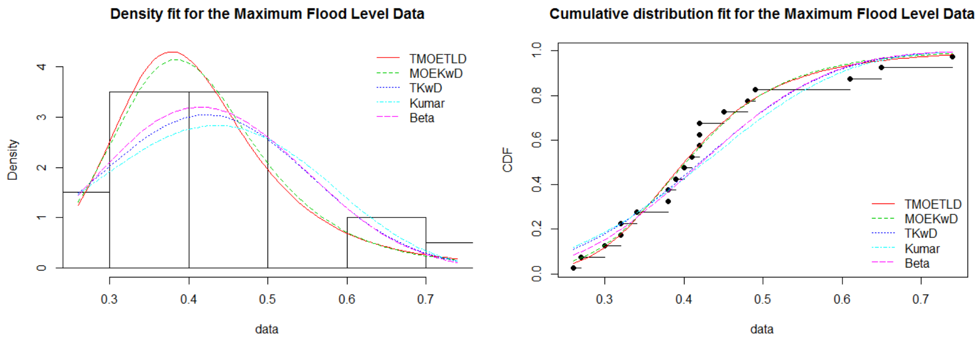


Figure 3: Density and the cumulative distribution fit for the Maximum Flood Level Data.

# 6 Conclusion

In this paper, we have introduced an extension of the Marshall-Olkin Extended Topp-Leone distribution, called the Transmuted Marshall-Olkin Extended Topp-Leone (TMOETL) distribution. Some statistical properties of the TMOETL distribution were derived and the maximum likelihood estimation method was employed to estimate the unknown parameters of the proposed distribution. A real data set was used to illustrate the usefulness of the TMOETL distribution and results obtained reveal the superiority of the proposed distribution over the Marshall-Olkin Extended Kumaraswamy (MOEKwD), Transmuted Kumaraswamy (TKwD), Kumaraswamy (Kumar) and Beta distributions.

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