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q-Analogue of New Subclass of Salagean-type Harmonic Univalent Functions defined by Subordination

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Abstract

We introduce and investigate q-analogue of a new subclass of Salagean-type harmonic univalent functions defined by subordination. We first obtained a coefficient characterization of these functions. We give necessary and sufficient convolution conditions, distortion bounds, compactness and extreme points for this subclass of harmonic univalent functions with negative coefficients.

1 Introduction

Let $\mathbb{H}$ denote the class of continuous complex-valued harmonic functions which are harmonic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathbb{A}$ be the subclass of $\mathbb{H}$ consisting of functions which are analytic in $\mathbb{U}$. A function harmonic

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in $\mathbb{U}$ may be written as $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$, where $\mathfrak{h}$ and $\mathfrak{g}$ are analytic in $\mathbb{U}$. We call $\mathfrak{h}$ the analytic part and $\mathfrak{g}$ co-analytic part of $\mathfrak{f}$. A necessary and sufficient condition for $\mathfrak{f}$ to be locally univalent and sense-preserving in $\mathbb{U}$ is that $|\mathfrak{g}'(z)| < |\mathfrak{h}'(z)|$ (see [5]). To this end, without loss of generality, we may write

$$\mathfrak{h}(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } \mathfrak{g}(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1}$$

Let $\mathbb{HS}$ denote the class of functions $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ which are harmonic, univalent, and sense-preserving in $\mathbb{U}$ for which $\mathfrak{h}(0) = \mathfrak{h}'(0) - 1 = 0 = \mathfrak{g}(0)$. One shows easily that the sense-preserving property implies that $|b_1| < 1$.

Clunie and Sheil-Small ([5]) investigated the class $\mathbb{HS}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $\mathbb{HS}$ and its subclasses (see [20]).

We recollect here the q-difference operator that was used in geometric function theory and in several areas of science. We give basic definitions and properties about the q-difference operator that are used in this study (for details see [4] and [11]). For $0 < q < 1$, we defined the q-integer $[k]_q$ by

$$[k]_q = \frac{1-q^k}{1-q}, \quad (k = 1, 2, 3, ...).$$

Notice that if $q \to 1^-$, then $[k]_q \to k$.

In 1990, İsmail, Merkes and Styer ([10]) used q- calculus, in the theory of analytic univalent functions by defining a class of complex valued functions that are analytic on the open unit disk $\mathbb{U}$ with the normalizations $\mathfrak{f}(0) = 0$, $\mathfrak{f}'(0) = 1$, and $|\mathfrak{f}(qz)| \le |\mathfrak{f}(z)|$ on $\mathbb{U}$ for every q, $q \in (0, 1)$. Motivated by these authors, several researches used the theory of analytic univalent functions and q-calculus; for example see ([1] and [2]). The q-difference operator of analytic functions $\mathfrak{h}$ and $\mathfrak{g}$ given by (1) are by definition, given as follows (see [11])

$$\partial_q \mathfrak{h}(z) = \begin{cases} \frac{\mathfrak{h}(z)-\mathfrak{h}(qz)}{(1-q)z} & ; \ z \neq 0 \\ \mathfrak{h}'(0) & ; \ z = 0 \end{cases} \text{ and } \partial_q \mathfrak{g}(z) = \begin{cases} \frac{\mathfrak{g}(z)-\mathfrak{g}(qz)}{(1-q)z} & ; \ z \neq 0 \\ \mathfrak{g}'(0) & ; \ z = 0 \end{cases}.$$

Thus, for the function $\mathfrak{h}$ and $\mathfrak{g}$ of the form (1), we have

$$\partial_q \mathfrak{h}(z) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1} \text{ and } \partial_q \mathfrak{g}(z) = \sum_{k=1}^{\infty} [k]_q \, b_k z^{k-1}. \tag{2}$$

For $\mathfrak{f} \in \mathbb{HS}$, $\nu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}$, $k \geq 1$, $0 < q < 1$, let:

$$\mathcal{D}_q^0 \mathfrak{f}(z) = \mathfrak{f}(z) = \mathfrak{h}(z) + \overline{\mathfrak{g}(z)},$$

$$\begin{aligned} \mathcal{D}_q^1 \mathfrak{f}(z) &= \mathcal{D}_q^1 \mathfrak{h}(z) - \overline{\mathcal{D}_q^1 \mathfrak{g}(z)} = z\partial_q \mathfrak{h}(z) - \overline{z\partial_q \mathfrak{g}(z)}, \\ &\vdots \\ \mathcal{D}_q^\nu \mathfrak{f}(z) &= \mathcal{D}_q^\nu \mathfrak{h}(z) + (-1)^\nu \overline{\mathcal{D}_q^\nu \mathfrak{g}(z)} = z\partial_q (\mathcal{D}_q^{\nu-1} \mathfrak{h}(z)) + (-1)^\nu \overline{z\partial_q (\mathcal{D}_q^{\nu-1} \mathfrak{g}(z))} \end{aligned} \tag{3}$$

where

$$\mathcal{D}_q^\nu \mathfrak{h}(z) = z + \sum_{k=2}^{\infty} [k]_q^\nu \, a_k z^k, \quad \mathcal{D}_q^\nu \mathfrak{g}(z) = \sum_{k=1}^{\infty} [k]_q^\nu \, b_k z^k.$$

We note that

(i) $\lim_{q \longrightarrow 1^-} \mathcal{D}_q^\nu \mathfrak{f}(z) = D^\nu \mathfrak{f}(z) = z + \sum\limits_{k=2}^{\infty} k^\nu a_k z^k$ for $\mathfrak{f} \in \mathbb{S}$ (see [16]),

(ii) $\lim_{q \longrightarrow 1^-} \mathcal{D}_q^\nu \mathfrak{f}(z) = D^\nu \mathfrak{f}(z) = D^\nu \mathfrak{h}(z) + (-1)^\nu \overline{D^\nu \mathfrak{g}(z)}$ for $\mathfrak{f} \in \mathbb{HS}$ where $D^\nu \mathfrak{h}(z) = z + \sum\limits_{k=2}^{\infty} k^\nu a_k z^k$ and $D^\nu \mathfrak{g}(z) = \sum\limits_{k=1}^{\infty} k^\nu b_k z^k$ (see [14]).

A harmonic function $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ defined by (1) is said to be $q-$harmonic, locally univalent and sense-preserving in $\mathbb{U}$ denoted by $\mathbb{HS}_q$, if and only if the second dilatation w_q satisfies the condition

$$|w_q(z)| = \left| \frac{\partial_q \mathfrak{g}(z)}{\partial_q \mathfrak{h}(z)} \right| < 1$$

where $0 < q < 1$ and $z \in \mathbb{U}$. Note that as $q \to 1^-$, $\mathbb{HS}_q$ reduces to the family $\mathbb{HS}$ ([1] and [13])

We also let the subclass $\mathbb{THS}_q$ consist of harmonic functions $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ in $\mathbb{HS}_q$ so that $\mathfrak{h}$ and $\mathfrak{g}$ are of the form

$$\mathfrak{h}(z) = z - \sum_{k=2}^{\infty} |a_k| \, z^k \text{ and } \mathfrak{g}(z) = (-1)^\nu \sum_{k=1}^{\infty} |b_k| \, z^k. \tag{4}$$

We say that an analytic function $\mathfrak{f}$ is subordinate to an analytic function F and write $\mathfrak{f} \prec F$, if there exists a complex valued function ϖ which maps $\mathbb{U}$ into oneself with $\varpi(0) = 0$, such that $\mathfrak{f}(z) = F(\varpi(z)) \quad (z \in \mathbb{U})$.

Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence:

$$\mathfrak{f}(z) \prec F(z) \Leftrightarrow \mathfrak{f}(0) = F(0) \text{ and } \mathfrak{f}(\mathbb{U}) \subset F(\mathbb{U}).$$

Denote by $\mathbb{HS}_q^{\nu}(\delta, A, B)$ the subclass of $\mathbb{HS}_q$ consisting of functions $\mathfrak{f}$ of the form (1) that satisfy the condition

$$\frac{\mathcal{D}_q^{\nu+1}\mathfrak{f}(z)}{\mathcal{D}_q^{\nu}\mathfrak{f}(z)} \prec (1-\delta)\frac{1+Az}{1+Bz} + \delta = \frac{1+[A+(B-A)\delta]z}{1+Bz}, \tag{5}$$

where $-B \leq A < B \leq 1$ and $0 \leq \delta < 1$.

Finally, we let $\mathbb{THS}_q^{\nu}(\delta, A, B) \equiv \mathbb{HS}_q^{\nu}(\delta, A, B) \cap \mathbb{THS}_q$. By suitably specializing the parameters, the classes $\mathbb{HS}_q^{\nu}(\delta, A, B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

$\mathbb{HS}_q^{\nu}(0, A, B) = S_{\mathcal{H}}(\nu, q, A, B)$ (see [17]),

$\mathbb{HS}_q^{0}(0, A, B) = S_{\mathcal{H}}^{*}(q, A, B)$ (see [19] and [17]),

$\mathbb{HS}_q^{1}(0, A, B) = \mathcal{CH}_q(A, B)$ (see [17]),

$\mathbb{HS}_q^{\nu}(0, (1+q)\alpha - 1, q) = H_q^{\nu}(\alpha)$ for $0 \leq \alpha < 1$ (see [13]),

$\mathbb{HS}_q^{0}(0, (1+q)\alpha - 1, q) = S_{H_q}^{*}(\alpha)$ for $0 \leq \alpha < 1$ (see [2]),

$\mathbb{HS}_q^{1}(0, (1+q)\alpha - 1, q) = S_{H_q}^{C}(\alpha)$ for $0 \leq \alpha < 1$ (see [1]),

$\mathbb{HS}_q^{\nu}(\delta, A, B) = SH(\nu, \delta, A, B)$ for $q \to 1^-$ (see [3]),

$\mathbb{HS}_q^{\nu}(0, A, B) = SH(\nu, A, B)$ for $q \to 1^-$ (see [8]),

$\mathbb{HS}_q^{0}(0, A, B) = S_{\mathcal{H}}^{*}(A, B)$ for $q \to 1^-$(see [6] and [7]),

$\mathbb{HS}_q^{1}(0, A, B) = \mathcal{CH}(A, B)$ for $q \to 1^-$(see [8]),

$\mathbb{HS}_q^{\nu}(0,(1+q)\alpha-1,q)=H^{\nu}(\alpha)$ for $0\leq\alpha<1$ and $q\to 1^{-}$(see [14]),

$\mathbb{HS}_q^{0}(0,(1+q)\alpha-1,q)=S_H^{*}(\alpha)$ for $0\leq\alpha<1$ and $q\to 1^{-}$ (see [12], [15]),

$\mathbb{HS}_q^{1}(0,(1+q)\alpha-1,q)=S_H^{C}(\alpha)$ for $0\leq\alpha<1$ and $q\to 1^{-}$ (see [12], [15]),

$\mathbb{HS}_q^{\nu}(0,-1,q)=H^{\nu}(0)$ for $q\to 1^{-}$ (see [14]),

$\mathbb{HS}_q^{0}(0,-1,q)=S_H^{*}$ for $q\to 1^{-}$ (see [18]),

$\mathbb{HS}_q^{1}(0,-1,q)=C_H$ for $q\to 1^{-}$ (see [18]).

Making use of the techniques and methodology used by Dziok (see [6] and [7]), Dziok et al. (see [8] and [9]), in this paper we find necessary and sufficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for the above defined class $\mathbb{THS}_q^{\nu}(\delta,A,B)$. In this paper we find necessary and sufficient conditions, distortion bounds, extreme points for the above defined class $\mathbb{THS}_q^{\nu}(\delta,A,B)$.

2 Main Results

For functions $\mathfrak{f}_1$ and $\mathfrak{f}_2\in\mathbb{HS}_q$ of the form

$$\mathfrak{f}_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k+\sum_{k=1}^{\infty}\overline{b_{k,j}z^k},\qquad (z\in\mathbb{U},\ j=1,2),\tag{6}$$

we define the Hadamard product of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ by

$$(\mathfrak{f}_1*\mathfrak{f}_2)(z)=z+\sum_{k=2}^{\infty}a_{k,1}\ a_{k,2}z^k+\sum_{k=1}^{\infty}\overline{b_{k,1}\ b_{k,2}z^k}\qquad (z\in\mathbb{U}).$$

First we state and prove the necessary and sufficient conditions for harmonic functions in $\mathbb{HS}_q^{\nu}(\delta,A,B)$.

Theorem 1. *Let $\mathfrak{f}\in\mathbb{HS}_q$. Then $\mathfrak{f}\in\mathbb{HS}_q^{\nu}(\delta,A,B)$ if and only if*

$$\mathcal{D}_q^{\nu}\mathfrak{f}(z)*\Theta(z;\zeta)\neq 0,\qquad (\zeta\in\mathbb{C},\ |\zeta|=1,\ z\in\mathbb{U}\backslash\{0\}),$$

where

$$\Theta(z;\zeta) = \frac{(B-A)(1-\delta)\zeta z + (1+[A+(B-A)\delta]\zeta)\, qz^2}{(1-z)(1-qz)} - \frac{(2+[B+A+(B-A)\delta]\zeta)\,\overline{z} - (1+[A+(B-A)\delta]\zeta)\, q\overline{z}^2}{(1-\overline{z})(1-q\overline{z})}.$$

Proof. Let $\mathfrak{f} \in \mathbb{HS}_q$. Then $\mathfrak{f} \in \mathbb{HS}_q^{\nu}(\delta, A, B)$ if and only if (5) holds or equivalently

$$\frac{\mathcal{D}_q^{\nu+1}\mathfrak{f}(z)}{\mathcal{D}_q^{\nu}\mathfrak{f}(z)} \neq \frac{1+[A+(B-A)\delta]\zeta}{1+B\zeta} \quad (\zeta \in \mathbb{C},\ |\zeta| = 1,\ z \in \mathbb{U}\backslash\{0\}). \tag{7}$$

Now for

$$\mathcal{D}_q^{\nu}\mathfrak{f}(z) = \mathcal{D}_q^{\nu}\mathfrak{f}(z) * \left(\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}\right),$$

and

$$\mathcal{D}_q^{\nu+1}\mathfrak{f}(z) = \mathcal{D}_q^{\nu}\mathfrak{f}(z) * \left(\frac{z}{(1-z)(1-qz)} - \frac{\bar{z}}{(1-\bar{z})(1-q\bar{z})}\right)$$

the inequality (7) yields

$$\begin{aligned}
&(1+B\zeta)\mathcal{D}_q^{\nu+1}\mathfrak{f}(z) - \{1+[A+(B-A)\delta]\zeta\}\mathcal{D}_q^{\nu}\mathfrak{f}(z) \\
= \;& \mathcal{D}_q^{\nu}\mathfrak{h}(z) * \left[(1+B\zeta)\frac{z}{(1-z)(1-qz)} - \{1+[A+(B-A)\delta]\zeta\}\frac{z}{1-z}\right] \\
& -(-1)^{\nu}\overline{\mathcal{D}_q^{\nu}\mathfrak{g}(z)} * \left[(1+B\zeta)\frac{\bar{z}}{(1-\bar{z})(1-q\bar{z})} + \{1+[A+(B-A)\delta]\zeta\}\frac{\bar{z}}{1-\bar{z}}\right] \\
= \;& \mathcal{D}_q^{\nu}\mathfrak{f}(z) * \Theta(z;\zeta) \neq 0
\end{aligned}$$

□

Now we state and prove a sufficient coefficient bound for the class $\mathbb{HS}_q^{\nu}(\delta, A, B)$.

Theorem 2. *Let* $\mathfrak{f}$ *be of the form (1). If* $-B \leq A < B \leq 1$, $0 \leq \delta < 1$ *and*

$$\sum_{k=1}^{\infty} \left(\Phi_k |a_k| + \Psi_k |b_k|\right) \leq 2(B-A)(1-\delta), \tag{8}$$

where

$$\Phi_k = [k]_q^{\nu} \left\{ (B+1) [k]_q - (A+1) - (B-A)\delta \right\} \tag{9}$$

and

$$\Psi_k = [k]_q^{\nu} \left\{ (B+1) [k]_q + (A+1) + (B-A)\delta \right\} \tag{10}$$

then $\mathfrak{f}$ *is harmonic, sense-preserving, locally univalent in* $\mathbb{U}$, *and* $\mathfrak{f} \in \mathbb{HS}_q^{\nu}(\delta, A, B)$.

Proof. Since

$$\begin{aligned}
|\mathcal{D}_q \mathfrak{h}(z)| &\geq 1 - \sum_{k=2}^{\infty} [k]_q |a_k| |z|^{k-1} \\
&> 1 - \sum_{k=2}^{\infty} \frac{[k]_q^{\nu} \left\{ (B+1) [k]_q - (A+1) - (B-A)\delta \right\}}{(B-A)(1-\delta)} |a_k| \\
&\geq \sum_{k=1}^{\infty} \frac{[k]_q^{\nu} \left\{ (B+1) [k]_q + (A+1) + (B-A)\delta \right\}}{(B-A)(1-\delta)} |b_k| \\
&> \sum_{k=1}^{\infty} \frac{[k]_q^{\nu} \left\{ (B+1) [k]_q + (A+1) + (B-A)\delta \right\}}{(B-A)(1-\delta)} |b_k| |z|^{k-1} \\
&\geq \sum_{k=1}^{\infty} [k]_q |b_k| |z|^{k-1} \geq |\mathcal{D}_q \mathfrak{g}(z)|,
\end{aligned}$$

it follows that $\mathfrak{f} \in \mathbb{HS}_q$. On the other hand, $\mathfrak{f} \in \mathbb{HS}_q^{\nu}(\delta, A, B)$ if and only if there exists a complex valued function ϖ; $\varpi(0) = 0$, $|\varpi(z)| < 1$ $(z \in \mathbb{U})$ such that

$$\frac{\mathcal{D}_q^{\nu+1} \mathfrak{f}(z)}{\mathcal{D}_q^{\nu} \mathfrak{f}(z)} = \frac{1 + [A + (B-A)\delta] \varpi(z)}{1 + B\varpi(z)}$$

or equivalently

$$\left| \frac{\mathcal{D}_q^{\nu+1} \mathfrak{f}(z) - \mathcal{D}_q^{\nu} \mathfrak{f}(z)}{B\mathcal{D}_q^{\nu+1} \mathfrak{f}(z) - [A + (B-A)\delta] \mathcal{D}_q^{\nu} \mathfrak{f}(z)} \right| < 1. \tag{11}$$

Substituting for $\mathcal{D}_q\mathfrak{h}(z)$ and $\mathcal{D}_q\mathfrak{g}(z)$ in (11), we obtain

$$
\begin{aligned}
&\left|\mathcal{D}_q^{\nu+1}\mathfrak{f}(z)-\mathcal{D}_q^{\nu}\mathfrak{f}(z)\right|-\left|B\mathcal{D}_q^{\nu+1}\mathfrak{f}(z)-[A+(B-A)\delta]\,\mathcal{D}_q^{\nu}\mathfrak{f}(z)\right| \\
= &\left|\sum_{k=2}^{\infty}[k]_q^{\nu}\left([k]_q-1\right)a_k z^k-(-1)^{\nu}\sum_{k=1}^{\infty}[k]_q^{\nu}\left([k]_q+1\right)\overline{b_k z^k}\right| \\
&-\Bigg|(B-A)(1-\delta)z+\sum_{k=2}^{\infty}[k]_q^{\nu}\left(B[k]_q-A-(B-A)\delta\right)a_k z^k \\
&-(-1)^{\nu}\sum_{k=1}^{\infty}[k]_q^{\nu}\left(B[k]_q+A+(B-A)\delta\right)\overline{b_k z^k}\Bigg| \\
\le &\sum_{k=2}^{\infty}[k]_q^{\nu}\left([k]_q-1\right)|a_k|\,|z|^k+\sum_{k=1}^{\infty}[k]_q^{\nu}\left([k]_q+1\right)|b_k|\,|z|^k \\
&-(B-A)(1-\delta)|z|+\sum_{k=2}^{\infty}[k]_q^{\nu}\left(B[k]_q-A-(B-A)\delta\right)|a_k|\,|z|^k \\
&+\sum_{k=1}^{\infty}[k]_q^{\nu}\left(B[k]_q+A+(B-A)\delta\right)|b_k|\,|z|^k \\
\le &|z|\left\{\sum_{k=2}^{\infty}\Phi_k|a_k|\,|z|^{k-1}+\sum_{k=1}^{\infty}\Psi_k|b_k|\,|z|^{k-1}-(B-A)(1-\delta)\right\}<0,
\end{aligned}
$$

The harmonic function

$$
\mathfrak{f}(z)=z+\sum_{k=2}^{\infty}\tfrac{(B-A)(1-\delta)x_k}{\Phi_k}z^k+\sum_{k=1}^{\infty}\tfrac{(B-A)(1-\delta)y_k}{\Psi_k}\overline{z^k} \tag{12}
$$

where

$$
\sum_{k=1}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1
$$

shows that the coefficient bound given by (8) is sharp. The functions of the form (12) are in $\mathbb{HS}_q^{\nu}(\delta,A,B)$ because

$$
\sum_{k=1}^{\infty}\left(\tfrac{\Phi_k}{2(B-A)(1-\delta)}|a_k|+\tfrac{\Psi_k}{2(B-A)(1-\delta)}|b_k|\right)=\sum_{k=1}^{\infty}\left(|x_k|+|y_k|\right)=1.
$$

by (8). □

Next we show that the bound (8) is also necessary for $\mathbb{THS}_q^{\nu}(\delta, A, B)$.

Theorem 3. *Let $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ with $\mathfrak{h}$ and $\mathfrak{g}$ of the form* (4). *Then $\mathfrak{f} \in \mathbb{THS}_q^{\nu}(\delta, A, B)$ if and only if the condition* (8) *holds.*

Proof. In view of Theorem 2, we only need to show that $\mathfrak{f} \notin \mathbb{THS}_q^{\nu}(\delta, A, B)$ if condition (8) does not hold. We note that a necessary and sufficient condition for $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ given by (4) to be in $\mathbb{THS}_q^{\nu}(\delta, A, B)$ is that the coefficient condition (8) to be satisfied. Equivalently, we must have

$$\left| \frac{\sum_{k=2}^{\infty} [k]_q^{\nu}([k]_q - 1)|a_k| z^k + \sum_{k=1}^{\infty} [k]_q^{\nu}([k]_q + 1)|b_k| \bar{z}^k}{(B-A)(1-\delta)z - \sum_{k=2}^{\infty} [k]_q^{\nu}\big(B[k]_q - A - (B-A)\delta\big)|a_k| z^k - \sum_{k=1}^{\infty} [k]_q^{\nu}\big(B[k]_q + A + (B-A)\delta\big)|b_k| \bar{z}^k} \right| < 1.$$

For $z = r < 1$ we obtain

$$\frac{\sum_{k=2}^{\infty} [k]_q^{\nu}([k]_q - 1)|a_k| r^{k-1} + \sum_{k=1}^{\infty} [k]_q^{\nu}([k]_q + 1)|b_k| r^{k-1}}{(B-A)(1-\delta) - \sum_{k=2}^{\infty} [k]_q^{\nu}\big(B[k]_q - A - (B-A)\delta\big)|a_k| r^{k-1} - \sum_{k=1}^{\infty} [k]_q^{\nu}\big(B[k]_q + A + (B-A)\delta\big)|b_k| r^{k-1}} < 1. \tag{13}$$

If condition (8) does not hold, then condition (13) does not hold for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient (13) is greater than 1. This contradicts the required condition for $\mathfrak{f} \in \mathbb{THS}_q^{\nu}(\delta, A, B)$ and so the proof is complete. □

Theorem 4. *Let $\mathfrak{f} \in \mathbb{THS}_q^{\nu}(\delta, A, B)$. Then for $|z| = r < 1$, we have*

$$|\mathfrak{f}(z)| \leq (1 + |b_1|)\, r + \frac{(B-A)(1-\delta) - [2 + A + B + (B-A)\delta]\, |b_1|}{[2]_q^{\nu}\, [(B+1)q + (B-A)(1-\delta)]} r^2,$$

and

$$|\mathfrak{f}(z)| \geq (1 - |b_1|)\, r - \frac{(B-A)(1-\delta) - [2 + A + B + (B-A)\delta]\, |b_1|}{[2]_q^{\nu}\, [(B+1)q + (B-A)(1-\delta)]} r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $\mathfrak{f} \in \mathbb{THS}_q^\nu(\delta, A, B)$. Taking the absolute value of $\mathfrak{f}$ we have

$$\begin{aligned}
|\mathfrak{f}(z)| &\leq (1+|b_1|)\, r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)\, r^k \\
&\leq (1+|b_1|)\, r + \tfrac{r^2}{[2]_q^\nu [(B+1)q+(B-A)(1-\delta)]} \sum_{k=2}^{\infty} (\Phi_k\, |a_k| + \Psi_k\, |b_k|) \\
&\leq (1+|b_1|)\, r + \frac{(B-A)(1-\delta) - [2+A+B+(B-A)\delta]\, |b_1|}{[2]_q^\nu \left[(B+1)q + (B-A)(1-\delta)\right]} r^2.
\end{aligned}$$

□

Theorem 5. *Set*

$$\mathfrak{h}_1(z) = z,\ \ \mathfrak{h}_k(z) = z - \frac{(B-A)(1-\delta)}{\Phi_k} z^k,\ \ (k = 2, 3, ...),$$

and

$$\mathfrak{g}_k(z) = z + (-1)^\nu \frac{(B-A)(1-\delta)}{\Psi_k} \bar{z}^k,\ \ (k = 1, 2, ...).$$

Then $\mathfrak{f} \in \mathbb{THS}_q^\nu(\delta, A, B)$ if and only if it can be expressed as

$$\mathfrak{f}(z) = \sum_{k=1}^{\infty} (x_k \mathfrak{h}_k(z) + y_k \mathfrak{g}_k(z))$$

where $x_k \geq 0$, $y_k \geq 0$ and $\sum_{k=1}^{\infty} (x_k + y_k) = 1$. In particular, the extreme points of $\mathbb{THS}_q^\nu(\delta, A, B)$ are $\{\mathfrak{h}_k\}$ and $\{\mathfrak{g}_k\}$.

Proof. Suppose

$$\begin{aligned}
\mathfrak{f}(z) &= \sum_{k=1}^{\infty} (x_k \mathfrak{h}_k(z) + y_k \mathfrak{g}_k(z)) \\
&= \sum_{k=1}^{\infty} (x_k + y_k) z - \sum_{k=2}^{\infty} \frac{(B-A)(1-\delta)}{\Phi_k} x_k z^k \\
&\quad + (-1)^\nu \sum_{k=1}^{\infty} \frac{(B-A)(1-\delta)}{\Psi_k} y_k \bar{z}^k.
\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=2}^{\infty}\Phi_k\,|a_k|+\sum_{k=1}^{\infty}\Psi_k\,|b_k| &= (B-A)\,(1-\delta)\sum_{k=2}^{\infty}x_k+(B-A)\,(1-\delta)\sum_{k=1}^{\infty}y_k\\ &= (B-A)\,(1-\delta)(1-x_1)\le B-A\end{aligned}$$

and so $\mathfrak{f}\in\mathbb{THS}_q^{\nu}(\delta,A,B)$. Conversely, if $\mathfrak{f}\in\mathbb{THS}_q^{\nu}(\delta,A,B)$, then

$$|a_k|\le\frac{(B-A)\,(1-\delta)}{\Phi_k}\ \text{and}\ |b_k|\le\frac{(B-A)\,(1-\delta)}{\Psi_k}.$$

Set

$$x_k=\frac{\Phi_k}{(B-A)\,(1-\delta)}|a_k|\ (k=2,3,...),$$

and

$$y_k=\frac{\Psi_k}{(B-A)\,(1-\delta)}|b_k|\ (k=1,2,...).$$

Then note by Theorem 3, $0\le x_k\ \le 1\ (k=2,3,...)$ and $0\le y_k\ \le 1\ (k=1,2,...)$. We define

$$x_1=1-\sum_{k=2}^{\infty}x_k-\sum_{k=1}^{\infty}y_k$$

and note that by Theorem 3, $x_1\ge 0$. Consequently, we obtain

$$\mathfrak{f}(z)=\sum_{k=1}^{\infty}(x_k\mathfrak{h}_k(z)+y_k\mathfrak{g}_k(z))$$

as required. □

Now we show that $\mathbb{THS}_q^{\nu}(\delta,A,B)$ is closed under convex combinations of its members.

Theorem 6. *The class $\mathbb{THS}_q^{\nu}(\delta,A,B)$ is closed under convex combination.*

Proof. For $j=1,2,3,...$ let $\mathfrak{f}_j\in\mathbb{THS}_q^{\nu}(\delta,A,B)$, where $\mathfrak{f}_j$ is given by

$$\mathfrak{f}_j(z)=z-\sum_{k=2}^{\infty}|a_{k_j}|z^k+(-1)^{\nu}\ \sum_{k=1}^{\infty}|b_{k_j}|\bar{z}^k.$$

Then by (8),

$$\sum_{k=1}^{\infty}\left(\Phi_k|a_{k_j}|+\Psi_k|b_{k_j}|\right)\leq 2\,(B-A)\,(1-\delta).$$

For $\sum_{j=1}^{\infty}\lambda_j=1,\ 0\leq\lambda_j\leq 1$, the convex combination of $\mathfrak{f}_j$ may be written as

$$\sum_{j=1}^{\infty}\lambda_j\mathfrak{f}_j(z)=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{\infty}\lambda_j|a_{k_j}|\right)z^k+(-1)^{\nu}\ \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\lambda_j|b_{k_j}|\right)\bar{z}^k.$$

Then by (8),

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\Phi_k\sum_{j=1}^{\infty}\lambda_j|a_{k_j}|+\Psi_k\sum_{j=1}^{\infty}\lambda_j|b_{k_j}|\right) &= \sum_{j=1}^{\infty}\lambda_j\left(\sum_{k=1}^{\infty}\left[\Phi_k|a_{k_j}|+\Psi_k|b_{k_j}|\right]\right)\\ &\leq 2\,(B-A)\,(1-\delta)\sum_{j=1}^{\infty}\lambda_j\\ &= 2\,(B-A)\,(1-\delta).\end{aligned}$$

This is the condition required by (8) and so $\sum_{j=1}^{\infty}\lambda_j\mathfrak{f}_j(z)\in\mathbb{THS}_q^{\nu}(\delta,A,B)$. □

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