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General Variational Inclusions and Nonexpansive Mappings

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Abstract

In this paper, we introduce a new class of variational inclusions involving three operators. We suggest and analyze three-step iterations for finding the common element of the set of fixed points of a nonexpansive mappings and the set of the solutions of the variational inclusions using the resolvent operator technique. We also study the convergence criteria of three-step iterative method under some mild conditions. Inertial type methods are suggested and investigated for general variational inclusions. Our results include the previous results as special cases and may be considered as an improvement and refinement of the previously known results.

1 Introduction

Variational inclusions are useful and important extensions and generalizations of the variational inequalities, which were introduced by Stampacchia [37] in potential theory. It is amazing that variational inequalities have applications in industry, mathematical finance, economics, decision sciences, ecology,

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mathematical and engineering sciences. Using the projection operator, one can show that the variational inequalities are equivalent to the fixed point problem. This alternative formulation has been used to discuss the existence solution of the variational inequalities and to develop numerical methods. For the formulation, applications, numerical analysis, sensitivity analysis, dynamical systems and other aspects if variational inequalities, see [3–14, 16–24, 24–27, 30–33, 35, 37, 39, 40] and the references therein. Rockafellar [36] discussed the proximal point methods for solving variational inclusion. We would like to mention that the variational inequalities and variational inclusions are natural generalizations of the variational principles, which have played important and significant developments in various branches of pure and applied sciences.

It is well known that the projection method and its variant forms including the Wiener-Hopf equations can not be extended and modified for solving the variational inclusions. These facts and comments have motivated us to use the technique of the resolvent operators. In this technique, the given operator is decomposed into the sum of two(or more) monotone operators whose resolvent is easy to evaluate than the resolvent of the original operator. Such type of methods are called the operator splitting methods and have proved to be every effective and efficient in solving partial different equations, see Ames [2] and the references therein. This technique can lead to the development of very efficient and robust methods, since one can treat each part of the original operator independently. In the context of the variational inclusions, Noor [21–23] and Noor et al. [30,31] have used the resolvent technique to suggest and analyze some two-step and three-step methods. A useful feature of these two-step and three-step methods for solving variational inclusions is that the resolvent step involves the maximal monotone part only, while other parts facilitates the problem decomposition. Essentially using the resolvent technique, one can show that the variational inclusions are equivalent to the fixed point problems. This alternative equivalent formulation has played very crucial role in developing some very efficient methods for solving the variational inclusions and related optimization problems. Using the technique of updating the solution, Noor [24–26, 31, 32] suggested and analyzed several three-step iterative methods for solving different classes of variational inequalities. The main idea in this technique is to modify the resolvent method by performing an additional step forward and a resolvent at each iteration. It have been shown [4,5,13,24–26] that three-step schemes are numerically better than two-step and one-step methods.

Related to the variational inclusions, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to unify these two problems and find the common elements of the set of the solution of variational inclusions and the set of the fixed-points of the nonexpansive mappings. Noor [26] and Noor et al. [31] have considered some three-step iterative methods for finding the common element of the variational inequalities and nonexpansive mappings. Erturk et al. [9], Gupta [12] and Rathee at al. [35] have used the ideas and techniques of Noor [26] and Noor et al. [31] to used some new S-type iterative methods for solving some classes of variational inequalities and related optimization problems.

It is very important to develop some efficient iterative methods for solving the quasi variational inequalities. Alvarez et al. [1] used the inertial type projection methods for solving variational inequalities, the origin of which can be traced back to Polyak [34]. Noor [25] suggested and investigated inertial type projection methods for solving general variational inequalities. These inertial type methods have been modified in various directions for solving variational inequalities and related optimization problems. Noor et al [25, 30, 32] have analyzed some inertial projection methods for some classes of general variational inequalities.

Motivated and inspired by the ongoing research in these fields, we introduce a new class of variational inclusions involving three operators, which is called the general variational inclusion. We consider resolvent technique to suggest and analyze a three-step resolvent iterative method for finding the common element of the solution of the variational inclusions and the set of the fixed-points of the nonexpansive mappings. We also study the convergence criteria of the new iterative method under some mild conditions. The resolvent technique is used to suggest some inertial type methods for solving the general variational inclusions. Since the general variational inclusions include the mixed variational inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems.

2 Basic Results

Let K be a nonempty closed and convex set in a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let $T, A, g : H \longrightarrow H$ be three nonlinear operators and S be a nonexpansive operator.

We consider the problem of finding $x \in H$ such that

$$0 \in \rho Tx + x - g(x) + \rho A(x), \tag{2.1}$$

where $\rho > 0$ is a constant. Inclusion of the type (2.1) is called the variational inclusion. Problem (2.1) is also known as finding the zero of the sum of three (or more) monotone operators. Variational inclusions and related problems are being studied extensively by many authors and have important applications in operations research, optimization, mathematical finance, decision sciences and other several branches of pure and applied sciences, see [8, 13, 14, 16, 36, 38] and the references therein.

If $A(.) \equiv \partial\varphi(.)$, where $\partial\varphi(.)$ is the subdifferential of a proper, convex and lower-semicontinuous function $\varphi : H \longrightarrow R \cup \{+\infty\}$, then the problem (2.1) reduces to finding $x \in H$ such that

$$0 \in Tx + x - g(x) + \partial\varphi(x), \tag{2.2}$$

or equivalently, finding $x \in H$ such that

$$\langle Tx + x - g(x), y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in H. \tag{2.3}$$

The inequality (2.3) is called the mixed general variational inequality or the general variational inequality of the second kind, which was introduced and studied by Ullah and Noor [39]. It has been shown that a wide class of linear and nonlinear problems arising in various branches of pure and applied sciences can be studied in the unified framework of mixed variational inequalities.

We note that, if φ is the indicator function of a closed convex set $K \subseteq H$, that is,

$$\varphi(x) \equiv I_K(x) = \begin{cases} 0, & \text{if} \quad x \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then the mixed general variational inequality (2.3) is equivalent to finding $x \in K$ such that

$$\langle Tx + x - g(x), y - x \rangle \geq 0, \quad \forall y \in K, \tag{2.4}$$

which is called the general variational inequality and appears to be a new one. We would like to point out that general variational inequality (2.4) is quite and distinctly different from other general variational inequalities, which were introduced and studied by Noor [19, 20, 24–27].

If $g(x) = x$, then problem (2.4) reduces to finding $x \in K$, such that

$$\langle Tx, y - x \rangle \geq 0, \quad \forall y \in K, \tag{2.5}$$

which is called the classical variational inequality, introduced and studied by Stampacchia [37] in 1964. For the recent trends and developments in variational inclusions and inequalities, see [3–14, 16, 17, 19–24, 24–28, 30–33, 35, 37, 39, 40] and the references therein.

We also need the following well known concepts and results.

Definition 2.1. *[6]* If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(x) = (I + \rho A)^{-1}(x), \quad \forall x \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal monotone, if and only if, its resolvent operator is defined everywhere. In addition, the resolvent operator is a single-valued and nonexpansive, that is,

$$\|J_A(x) - J_A(y)\| \leq \|x - y\|. \quad \forall x, y \in H.$$

Remark 2.1. It is well known that the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ is a maximal monotone operator, we denote by

$$J_\varphi(x) = (I + \rho\partial\varphi)^{-1}(x), \quad \forall x \in H,$$

the resolvent operator associated with $\partial\varphi$, which is defined everywhere on H. In particular, the resolvent operator J_φ has the following interesting characterization.

Lemma 2.1. *[6] For a given $z \in H$, $x \in H$ satisfies the inequality*

$$\langle x - z, y - x\rangle + \rho\varphi(y) - \rho\varphi(x) \geq 0, \quad \forall y \in H,$$

if and only if

$$x = J_\varphi z,$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator.

This property of the resolvent operator J_φ plays an important part in developing the numerical methods for solving the mixed variational inequalities.

3 Iterative Methods and Convergence Analysis

In this section, we show that the general variational inclusions are equivalent to the fixed point problem. This alternative equivalent formulation is used to suggest and analyze several iterative method.

Lemma 3.1. *The element $x \in H$ is a solution of the general variational inclusion (2.1), if and only if, $x \in H$ satisfies the relation*

$$x = J_A[g(x) - \rho Tx], \tag{3.1}$$

where $\rho > 0$ is a constant and $J_A = (I+\rho A)^{-1}$ is the resolvent operator associated with the maximal monotone operator.

Proof. Let $x \in H$ be a solution of (2.1). Then

$$\begin{aligned} 0 &\in \rho Tx + x - g(x) + \rho A(x), \quad \rho > 0 \\ &\Longleftrightarrow (g(x) - \rho Tx) + (I + \rho A)(x) \\ &\Longleftrightarrow x = (I + \rho A)^{-1}[g(x) - \rho Tx] = J_A[g(x) - \rho Tx], \end{aligned}$$

the required result. □

It is clear from Lemma 3.1 that variational inclusion (2.1) and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Let S be a nonexpansive mapping. We denote the set of the fixed points of S by $F(S)$ and the set of the solutions of the variational inequalities (2.1) by $VI(H,T)$. We can characterize the problem. If $x^* \in F(S) \cap VI(H,T)$, then

$x^* \in F(S)$ and $x^* \in VI(H,T)$. Thus, from Lemma 3.1, it follows that

$$x^* = Sx^* = J_A[g(x^*) - \rho Tx^*] = SJ_A[g(x^*) - \rho Tx^*], \tag{3.2}$$

where $\rho > 0$ is a constant.

Using the technique of updating the solution, we can rewrite the equation (3.1) in the following form as:

$$\begin{aligned} z &= SJ_A[g(x) - \rho Tx] \\ y &= SJ_A[g(z) - \rho Tz] \\ x &= SJ_A[g(y) - \rho Ty]. \end{aligned}$$

This fixed point formulation is used to suggest the following three-step iterative methods for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings S and the variational inclusion (2.1).

Algorithm 3.1. *For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes*

$$z_n = (1 - c_n)x_n + c_n SJ_A[g(x_n) - \rho Tx_n], \tag{3.3}$$

$$y_n = (1 - b_n)x_n + b_n SJ_A[g(z_n) - \rho Tz_n], \tag{3.4}$$

$$x_{n+1} = (1 - a_n)x_n + a_n SJ_A[g(y_n) - \rho Ty_n], \tag{3.5}$$

where $a_n, b_n, c_n \in [0,1]$ for all $n \geq 0$ and S is the nonexpansive operator.

Algorithm 3.1 is a three-step predictor-corrector method.

Note that for $c_n \equiv 0$, Algorithm 3.1 reduces to:

Algorithm 3.2. *For an arbitrarily chosen initial point $x_0 \in H$, compute the sequence the approximate solution$\{x_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1 - b_n)x_n + b_n SJ_A[g(x_n) - \rho Tx_n], \\ x_{n+1} &= (1 - a_n)x_n + a_n SJ_A[g(y_n) - \rho Ty_n], \end{aligned}$$

where $a_n, b_n \in [0,1]$ for all $n \geq 0$ and S is the nonexpansive operator.

Algorithm 3.2 is also called the two-step (Ishikawa iterations) iterative method.

For $b_n = 1, \quad a_n = 1$, Algorithm 3.2 reduces to:

Algorithm 3.3. *For an arbitrarily chosen initial point $x_0 \in H$, compute the sequence $\{x_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= SJ_A[g(x_n) - \rho T x_n], \\ x_{n+1} &= SJ_A[g(y_n) - \rho T y_n]. \end{aligned}$$

Remark 3.1. It should be remarked that our Algorithm 3.3 is a two-step method, which may be regarded as a predictor-corrector method. Moreover, Algorithm 3.2 covers the case in Algorithm 3.3 whenever $a_n \equiv 1$ for all $n \geq 0$.

Algorithm 3.3 can be written as

$$x_{n+1} = SJ_A[g(SJ_A[g(x_n) - \rho T x_n]) - \rho T SJ_A[g(x_n) - \rho T x_n]],$$

which is called extraresolvent Algorithm.

For $b_n \equiv 0, c_n \equiv 0$ and $g = I$, Algorithm 3.1 collapses to the following iterative method, which is known as the Mann iteration or one-step method for solving the variational inclusion (2.1).

Algorithm 3.4. *For a given $x_0 \in H$, compute the approximate solution x_{n+1} by the iterative schemes:*

$$x_{n+1} = (1 - a_n)x_n + a_n SJ_A[x_n - \rho T x_n]. \tag{3.6}$$

We now discuss some special cases of Algorithm 3.1 for solving the mixed variational inequalities (2.3) and the classical variational inequalities (2.5).

I. If $A(.) \equiv \varphi(.)$, the subdifferential of a proper lower-semicontinuous and convex function φ, then $J_A = J_\varphi$, and consequently Algorithm 3.1 collapses to:

Algorithm 3.5. *For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes*

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n S J_\varphi[g(x_n) - \rho T x_n], \\ y_n &= (1-b_n)x_n + b_n S J_\varphi[g(z_n) - \rho T z_n], \\ x_{n+1} &= (1-a_n)x_n + a_n S J_\varphi[g(y_n) - \rho T y_n], \end{aligned}$$

where $a_n, b_n, c_n \in [0,1]$ for all $n \geq 0$ and S is the the nonexpansive mapping.

Algorithm 3.5 is also a three-step method for solving the mixed variational inequalities (2.3).

II. If φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto the closed convex set K. In this case Algorithm 3.5 reduces to the following method.

Algorithm 3.6. *For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes*

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n S P_K[g(x_n) - \rho T x_n], \\ y_n &= (1-b_n)x_n + b_n S P_K[g(z_n) - \rho T z_n], \\ x_{n+1} &= (1-a_n)x_n + a_n S P_K[g(y_n) - \rho T y_n], \end{aligned}$$

where $a_n, b_n, c_n \in [0,1]$ for all $n \geq 0$ and S is the the nonexpansive mappings.

Algorithm 3.6 is a three-step method for solving the classical variational inequalities (2.3). Noor [26] and Noor and Huang [31] have studied the convergence analysis of Algorithm 3.6 and its various special cases.

From the above discussion, it is clear that Algorithm 3.1 is quite general and it includes several new and previously known algorithms for solving variational inequalities and related optimization problems.

Definition 3.1. A mapping $T : H \to H$ is called μ-Lipschitzian if for all $x, y \in H$, there exists a constant $\mu > 0$, such that

$$||Tx - Ty|| \leq \mu ||x - y||.$$

Definition 3.2. A mapping $T : H \to H$ is called α-inverse strongly monotonic if for all $x, y \in H$, there exists a constant $\alpha > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha ||Tx - Ty||^2.$$

Definition 3.3. A mapping $T : H \to H$ is called r-strongly monotonic if for all $x, y \in K$, there exists a constant $r > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq r ||x - y||^2.$$

Definition 3.4. A mapping $T : H \to H$ is called relaxed (γ, r)-cocoercive if for all $x, y \in K$, there exists constants $\gamma > 0$ and $r > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq -\gamma ||Tx - Ty||^2 + r ||x - y||^2.$$

Remark 3.2. Clearly a r-strongly monotonic mapping must be a relaxed (γ, r)-cocoercive mapping, or a γ-inverse strongly monotonic mapping must be a relaxed (γ, r)-cocoercive mapping whenever $r = 0$, but the converse is not true. Therefore the class of the relaxed (γ, r)-cocoercive mappings is the most general class, and hence definition 3.4 includes both the definition 3.2 and the definition 3.3 as special cases.

Lemma 3.2. *Suppose $\{\delta_k\}_{k=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{k+1} \leq (1 - \lambda_k)\delta_k + \sigma_k, \ \ k \geq 0$$

with $\lambda_k \in [0, 1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$, and $\sigma_k = o(\lambda_k)$. Then $\lim_{k \to \infty} \delta_k = 0$.

We now investigate the strong convergence of Algorithm 3.1 and Algorithm 3.4 in finding the common element of two sets of solutions of the variational inclusions (2.1) and $F(S)$ and this is the main motivation of this paper.

Theorem 3.1. *Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of H into H. Let g be a relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian mapping. Let S be a nonexpansive mapping of H into H such that $F(S) \cap VI(H,T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by Algorithm 3.1, for any initial point $x_0 \in H$, with conditions*

$$\left\| \rho - \frac{r - \gamma\mu^2}{\mu^2} \right\| < \frac{\sqrt{(r-\gamma\mu)^2 - \mu^2 k(2-k)}}{\mu^2}, \quad r > \gamma\mu^2 + \mu\sqrt{k(2-k)}, \quad k < 1 \tag{3.7}$$

$a_n, b_n, c_n \in [0,1]$ *and* $\sum_{n=0}^{\infty} a_n = \infty$, *and* $k = \sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2}$.

Then x_n obtained from Algorithm 3.1 converges strongly to $x^* \in F(S) \cap VI(H,T)$.

Proof. Let $x^* \in H$ be the solution of $F(S) \cap VI(H,T)$. Then

$$\begin{aligned} x^* &= (1-c_n)x^* + c_n SJ_A[g(x^*) - \rho Tx^*] &(3.8)\\ &= (1-b_n)x^* + b_n SJ_A[g(x^*) - \rho Tx^*] &(3.9)\\ &= (1-a_n)x^* + a_n SJ_A[g(x^*) - \rho Tx^*], &(3.10)\end{aligned}$$

where $a_n, b_n, c_n \in [0,1]$ are some constants. From (3.3), (3.8), and the nonexpansive property of the resolvent J_A and the nonexpansive mapping S, we have

$$\begin{aligned} &||x_{n+1} - x^*|| \\ = \ &||(1-a_n)x_n + a_n SJ_A[g(y_n) - \rho Ty_n] - (1-a_n)x^* - a_n SJ_A[g(x^*) - \rho Tx^*]|| \\ \leq \ &(1-a_n)||x_n - x^*|| + a_n||SJ_A[g(y_n) - \rho Ty_n] - SJ_A[g(x^*) - \rho Tx^*]|| \\ \leq \ &(1-a_n)||x_n - x^*|| + a_n||y_n - x^* - \rho(Ty_n - Tx^*)|| \\ &+\|y_n - x^* - (g(y_n) - g(x^*))\|. \qquad (3.11)\end{aligned}$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on T,

$$\begin{aligned}
& ||y_n - x^* - \rho(Ty_n - Tx^*)||^2 \\
= & ||y_n - x^*||^2 - 2\rho\langle Ty_n - Tx^*, y_n - x^*\rangle + \rho^2||Ty_n - Tx^*||^2 \\
\leq & ||y_n - x^*||^2 - 2\rho[-\gamma||Ty_n - Tx^*||^2 + r||y_n - x^*||^2] \\
& +\rho^2||Ty_n - Tx^*||^2 \\
\leq & ||y_n - y^*||^2 + 2\rho\gamma\mu^2||y_n - x^*||^2 - 2\rho r||y_n - x^*||^2 + \rho^2\mu^2||y_n - x^*||^2 \\
= & [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]||y_n - x^*||^2 \\
= & \theta_1^2\|y_n - x^*\|^2, \qquad (3.12)
\end{aligned}$$

where

$$\theta_1^2 = \sqrt{1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2}. \qquad (3.13)$$

In a similar way, using the relaxed (γ_1, r_1)-cocoercivity and μ_1-Lipschitzian of the operator g, we have

$$\|y_n - x^* - (g(y_n) - g(x^*))\| \leq k\|y_n - x^*\|, \qquad (3.14)$$

where k is defined as:

$$k = \sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2}. \qquad (3.15)$$

Combining (3.11), (3.12), (3.13), (3.14) and (3.15), we have

$$\|x_{n+1} - x^*\| \leq (1 - a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\|. \qquad (3.16)$$

In a similar, from (3.4) and (3.9), we have

$$||y_n - x^*|| \leq (1 - b_n)||x_n - x^*|| + b_n\theta||z_n - x^*||. \qquad (3.17)$$

and from (3.5) and (3.10), it follows that

$$\begin{aligned}
\|z_n - x^*\| & \leq (1 - c_n)\|x_n - x^*\| + c_n\theta\|x_n - x^*\|, \\
& = \{(1 - c_n(1 - \theta))\}\|x_n - x^*\| \\
& \leq \|x_n - x^*\|. \qquad (3.18)
\end{aligned}$$

From (3.16), (3.17) and (3.18), we obtain that

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq (1 - a_n)||x_n - x^*|| + a_n\theta||y_n - x^*|| \\ &\leq (1 - a_n)||x_n - x^*|| + a_n\theta||z_n - x^*|| \\ &\leq (1 - a_n)\|x_n - x^*\| + a_n\theta\|x_n - x^*\| \\ &= [1 - a_n(1 - \theta)]||x_n - x^*||, \end{aligned} \tag{3.19}$$

and hence by Lemma 3.2, $\lim_{n\to\infty} ||x_n - x^*|| = 0$, completing the proof. □

Next we will provide and prove the strong convergence theorem of Algorithm 3.4 under the α-inverse strongly monotonicity. With the following result, we can obtain the result of [31] as a special case.

Theorem 3.2. *Let $\alpha > 0$. Let T be an α-inverse strongly monotonic mapping of H into H, and S be a nonexpansive mapping of such that $F(S) \cap VI(H, T) \neq \emptyset$. Let $\{x_n\}$ be the approximate solution obtained from Algorithm 3.4 for any initial point $x_0 \in H$, where $\rho \in [a, b] \subset (0, 2\alpha)$ and $a_n \in [c, d]$ for some constants $c, d \in (0, 1)$. Then the sequence $\{x_n\}$ converges strongly to $x^* \in F(S) \cap VI(H, T)$.*

Proof. Let T is α-inverse strongly monotonic with the constant $\alpha > 0$. Then T is $\frac{1}{\alpha}$–Lipschitzian continuous. Consider

$$\begin{aligned} &\|x_n - x^* - \rho[Tx_n - Tx^*]\|^2 \\ = &\ \|x_n - x^*||^2 + \rho^2||Tx_n - Tx^*\|^2 - 2\rho\langle Tx_n - Tx^*, x_n - x^*\rangle \\ \leq &\ \|x_n - x^*\|^2 + \rho^2\|Tx_n - Tx^*\|^2 - 2\rho\alpha||Tx_n - Tx^*||^2 \\ = &\ \|x_n - x^*\|^2 + (\rho^2 - 2\rho\alpha)\|Tx_n - Tx^*\|^2 \\ \leq &\ \|x_n - x^*\|^2 + (\rho^2 - 2\rho\alpha) \cdot \frac{1}{\alpha^2}\|x_n - x^*\|^2 \\ = &\ (1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2})\|x_n - x^*\|^2. \end{aligned} \tag{3.20}$$

Set $\theta_2 = (1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2})^{1/2}$.

Then from the condition $\rho \in [a, b] \subset (0, 2\alpha)$, it follows that $\theta_2 \in (0, 1)$.

From (3.4), (3.8), (3.20) and the nonexpansive property of the operators of S and J_A, we have

$$\begin{aligned} & ||x_{n+1} - x^*|| \\ \leq\ & (1-a_n)||x_n - x^*|| + a_n||SJ_A[x_n - \rho Tx_n] - SJ_A[x^* - \rho Tx^*]|| \\ \leq\ & (1-a_n)||x_n - x^*|| + a_n||x_n - x^* - \rho(Tx_n - Tx^*)|| \\ \leq\ & (1-a_n)||x_n - x^*|| + a_n\theta_2||x_n - x^*|| \\ =\ & [1 - a_n(1-\theta_2)]||x_n - x^*||. \end{aligned}$$

Therefore, it follows $\lim_{n\to\infty} \|x_n - x^*\| = 0$, from Lemma 3.2. □

From the fixed point formulation (3.2), we have

$$x = Sx = SJ_A[g(x) - \rho Tx],$$

which can be written as

$$x = SJ_A[g((1-\gamma)x + \gamma x)) - \rho T((1-\gamma)x + \gamma x)], \tag{3.21}$$

where γ is a constant.

The fixed point formulation is used to suggest the following iterative method for solving the general variational inclusion (2.1).

Algorithm 3.7. *For given initial values x_0, x_1, compute the approximate solution x_{n+1} be the iterative scheme*

$$x_{n+1} = SJ_A[g((1-\gamma)x_n + \gamma x_{n-1})) - \rho T((1-\gamma)x_n + \gamma x_{n-1})], \quad n = 0, 1, 1, 2, ..., \tag{3.22}$$

which is called the inertial proximal point method.

Algorithm 3.7 is equivalent to the following two-step inertial iterative method.

Algorithm 3.8. *For given initial values x_0, x_1, compute the approximate solution x_{n+1} be the iterative scheme*

$$\begin{aligned} y_n &= (1-\gamma_n)x_n + \gamma_n x_{n-1} \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n SJ_A[g(y_n) - \rho T(y_n)], \quad n = 0, 1, 1, 2, ..., \end{aligned}$$

where $\alpha_n > 0$ and γ_n are constants.

In a similar way, we can suggest the four-step inertial methods for solving the general variational inequalities (2.1).

Algorithm 3.9. *For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)x_n + \gamma_n x_{n-1} \\ z_n &= (1-c_n)y_n + c_n S J_A[g(y_n) - \rho T y_n], \\ w_n &= (1-b_n)z_n + b_n S J_A[g(z_n) - \rho T z_n], \\ x_{n+1} &= (1-a_n)w_n + a_n S J_A[g(w_n) - \rho T w_n], \end{aligned}$$

where $a_n, b_n, c_n, \gamma_n \in [0,1], \quad \forall n \geq 0$ and S is the nonexpansive operator.

For different choices of the parameters $a_n, b_n, c_n, \lambda_n \in [0,1]$, and the operators, one can contain Mann iteration, Ishikawa iterations and Noor iterations as special cases of Algorithm 3.9. Using the techniques of Jabeen et al. [15] and Noor et al. [30].

Remark 3.3. Using the above techniques and ideas of Noor et al. [29], we can suggest and analyze several inertial type methods for solving the general variational inclusion and its variant forms. Comparison and implementation of these methods need further research efforts.

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