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Applications of Fractional Calculus on a Certain Class of Univalent Functions Associated with Wanas Operator

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Abstract

The purpose of this work is to use fractional integral and Wanas operator to define a certain class of analytic and univalent functions defined in the open unit disk U. Also, we obtain some results for this class such as integral representation, inclusion relationship and argument estimate.

1. Introduction

Denote by $\mathcal{A}$ the class of all functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions f and g which are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g\big(w(z)\big), (z \in U)$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1.1 [16]**.** For $f \in \mathcal{A}$. The Wanas differential operator $W_{\alpha,\beta}^{k,\eta}$ is defined by

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$$W_{\alpha,\beta}^{k,\eta} f(z) = z + \sum_{n=2}^{\infty} \left[\sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha^{\mu} + n\beta^{\mu}}{\alpha^{\mu} + \beta^{\mu}} \right) \right]^{\eta} a_n z^n ,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $k \in \mathbb{N}$, $\mu, \eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Special cases of the Wanas operator can be found in [1,2,4,5,6,9,11,13,14,15]. For more details see [18].

Definition 1.2 [3]**.** The fractional integral of order λ $(\lambda > 0)$ is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt ,$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

From Definition 1.1 and Definition 1.2, we conclude that

$$D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\lambda)} \left[\sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha^{\mu} + n\beta^{\mu}}{\alpha^{\mu} + \beta^{\mu}} \right) \right]^{\eta} a_n z^{n+\lambda} . \quad (1.2)$$

From [17] we need this result

$$z \left(D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z) \right)' = \left[1 + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta} \right)^{\mu} \right] D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z) + \left[\lambda - \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta} \right)^{\mu} \right] D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z). \quad (1.3)$$

Let T stand for the class of mappings h of the form:

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n,$$

which are analytic and convex univalent in U and satisfy the condition:

$$Re\{h(z)\} > 0, \quad (z \in U).$$

In order to prove our main results, we need the following lemmas:

Lemma 1.1 [8]. *Let $u, v \in \mathbb{C}$ and suppose that ψ is convex and univalent in U with $\psi(0) = 1$ and $Re\{u\psi(z) + v\} > 0, (z \in U)$. If q is analytic in U with $q(0) = 1$, then the subordination*

$$q(z) + \frac{zq'(z)}{uq(z) + v} \prec \psi(z)$$

implies that $q(z) \prec \psi(z)$.

Lemma 1.2 [10]. *Let h be convex univalent in U and $\mathcal{T}$ be analytic in U with $Re\{\mathcal{T}(z)\} \geq 0$, $(z \in U)$. If q is analytic in U and $q(0) = h(0)$, then the subordination*

$$q(z) + \mathcal{T}(z)zq'(z) \prec h(z)$$

implies that $q(z) \prec h(z)$.

Lemma 1.3 [7]. *Let q be analytic in U with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$. If there exists two points $z_1, z_2 \in U$ such that*

$$-\frac{\pi}{2}b_1 = arg\big(q(z_1)\big) < arg\big(q(z)\big) < arg\big(q(z_2)\big) = \frac{\pi}{2}b_2\,,$$

for some b_1 and $b_2 (b_1 > 0, b_2 > 0)$ and for all $z(|z| < |z_1| = |z_2|)$, then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{b_1 + b_2}{2}\right)m \qquad and \qquad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{b_1 + b_2}{2}\right)m\,,$$

where

$$m \geq \frac{1 - |\varepsilon|}{1 + |\varepsilon|} \quad and \quad \varepsilon = i\,tan\frac{\pi}{4}\left(\frac{b_2 - b_1}{b_1 + b_2}\right).$$

2. Main Results

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}(k, \eta, \alpha, \beta, \lambda, \delta; h)$ if it satisfies the following differential subordination condition:

$$\frac{1}{1-\delta}\left(\frac{z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)} - \delta\right) \prec h(z), \tag{2.1}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ with $\alpha + \beta > 0$, $k \in \mathbb{N}, \eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leq \delta < 1$ and $h \in T$.

In the following theorem, we find integral representation of the class $\mathcal{N}(k,\eta,\alpha,\beta,\lambda,\delta;h)$.

Theorem 2.1. *Let* $f \in \mathcal{N}(k,\eta,\alpha,\beta,\lambda,\delta;h)$. *Then*

$$D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z) = z.\exp\left[(1-\delta)\int_0^z \frac{h(w(s))-1}{s}ds\right],$$

where w *is analytic in* U *with* $w(0)=0$ *and* $|w(z)|<1, (z \in U)$.

Proof. Assume that $f \in \mathcal{N}(k,\eta,\alpha,\beta,\lambda,\delta;h)$. It is easy to see that subordination condition (2.1) can be written as follows

$$\frac{z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)} = (1-\delta)h(w(z))+\delta, \tag{2.2}$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1$, $(z \in U)$.

From (2.2), we find that

$$\frac{\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)} - \frac{1}{z} = (1-\delta)\frac{h(w(z))-1}{z}, \tag{2.3}$$

After integrating both sides of (2.3), we have

$$\log\left(\frac{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)}{z}\right) = (1-\delta)\int_0^z \frac{h(w(s))-1}{s}ds. \tag{2.4}$$

Therefore, from (2.4), we obtain the required result.

Theorem 2.2. *Let* $Re\left\{(1-\delta)h(z)+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right\}>0$. *Then*

$$\mathcal{N}(k,\eta+1,\alpha,\beta,\lambda,\delta;h) \subset \mathcal{N}(k,\eta,\alpha,\beta,\lambda,\delta;h).$$

Proof. Let $f \in \mathcal{N}(k,\eta+1,\alpha,\beta,\lambda,\delta;h)$ and put

$$q(z) = \frac{1}{1-\delta}\left(\frac{z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)} - \delta\right). \tag{2.5}$$

Then q is analytic in U with $q(0) = 1$. In view of the identity (1.3), we find from (2.5) that

$$\left[1 + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\left(\frac{\alpha}{\beta}\right)^{\mu}\right)\right] \frac{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z)}{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z)}$$

$$= (1-\delta)q(z) + \delta - \lambda + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta}\right)^{\mu}. \tag{2.6}$$

Differentiating both sides of (2.6) with respect to z and multiplying by z, we have

$$q(z) + \frac{zq'(z)}{(1-\delta)q(z) + \delta - \lambda + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta}\right)^{\mu}}$$

$$= \frac{1}{1-\delta} \left(\frac{z\left(D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z)\right)'}{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z)} - \delta \right) \prec h(z). \tag{2.7}$$

Since $Re\left\{(1-\delta)h(z) + \delta - \lambda + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta}\right)^{\mu}\right\} > 0$, applying Lemma 1.1 to the subordination (2.7), yields $q(z) \prec h(z)$, which implies $f \in \mathcal{N}(k, \eta, \alpha, \beta, \lambda, \delta; h)$.

Theorem 2.3. *Let* $f \in \mathcal{A}$, $0 < a_1, a_2 \leq 1$ *and* $0 \leq \delta < 1$. *If*

$$-\frac{\pi}{2} a_1 < arg \left(\frac{z\left(D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z)\right)'}{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} g(z)} - \delta \right) < \frac{\pi}{2} a_2,$$

for some $g \in \mathcal{N}\left(k, \eta+1, \alpha, \beta, \lambda, \delta; \frac{1+AZ}{1+Bz}\right), (-1 \leq B < A \leq 1)$, *then*

$$-\frac{\pi}{2} b_1 < arg \left(\frac{z\left(D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z)\right)'}{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} g(z)} - \delta \right) < \frac{\pi}{2} b_2,$$

where b_1 *and* $b_2 (0 < b_1, b_2 \leq 1)$ *are the solutions of the equations*:

$$a_1 = \begin{cases} b_1 + \frac{2}{\pi}\tan^{-1}\left(\dfrac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{\begin{array}{c}2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B}+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)\\ +(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t\end{array}}\right), & B \neq -1 \\ b_1, & B = -1 \end{cases} \tag{2.8}$$

and

$$a_2 = \begin{cases} b_2 + \frac{2}{\pi}\tan^{-1}\left(\dfrac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{\begin{array}{c}2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B}+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)\\ +(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t\end{array}}\right), & B \neq -1 \\ b_2, & B = -1 \end{cases} \tag{2.9}$$

with $\varepsilon = i\tan\frac{\pi}{2}\left(\frac{b_2-b_1}{b_1+b_2}\right)$ *and*

$$t = \frac{2}{\pi}\sin^{-1}\left(\frac{(A-B)(1-\delta)}{\left(\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)(1-B^2)+(1-\delta)(1-AB)}\right). \tag{2.10}$$

Proof. Define the function G by

$$G(z) = \frac{1}{1-\tau}\left(\frac{z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}g(z)} - \tau\right), \tag{2.11}$$

where $g \in \mathcal{N}\left(k,\eta+1,\alpha,\beta,\lambda,\delta;\frac{1+AZ}{1+Bz}\right)$, $(-1 \leq B < A \leq 1)$ and $0 \leq \tau < 1$.

Then G is analytic in U with $G(0)=1$. By making use of (1.3) and (2.11), we observe that

$$\left((1-\tau)G(z)+\tau\right)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}g(z)$$

$$= \left(1+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z)$$

$$+\left(\lambda-\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z).$$

Differentiating above relation with respect to z and multiplying by z, we get

$$\big((1-\tau)G(z)+\tau\big)z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}g(z)\right)'+(1-\tau)zG'(z)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}g(z)$$

$$=\left(1+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z)\right)'$$

$$+\left(\lambda-\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z)\right)'. \tag{2.12}$$

Suppose that

$$H(z)=\frac{1}{1-\delta}\left(\frac{z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}g(z)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}g(z)}-\delta\right).$$

Using (1.3) again, we have

$$\left(1+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)\frac{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}g(z)}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}g(z)}$$

$$=(1-\delta)H(z)+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}. \tag{2.13}$$

From (2.12) and (2.13), we easily get

$$G(z)+\frac{zG'(z)}{(1-\delta)H(z)+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}}$$

$$=\frac{1}{1-\tau}\left(\frac{z\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}g(z)}-\tau\right). \tag{2.14}$$

Notice that from Theorem 2.2, $g \in \mathcal{N}\left(k, \eta+1, \alpha, \beta, \lambda, \delta; \frac{1+AZ}{1+Bz}\right)$ implies $g \in \mathcal{N}\left(k, \eta, \alpha, \beta, \lambda, \delta; \frac{1+AZ}{1+Bz}\right)$. Thus,

$$H(z) \prec \frac{1+AZ}{1+Bz} \qquad (-1 \leq B < A \leq 1).$$

By applying the result of Silverman and Silvia [12], we have

$$\left|H(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (B \neq -1, z \in U) \tag{2.15}$$

and

$$Re\{H(z)\} > \frac{1-A}{2} \quad (B = -1, z \in U). \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\left|(1-\delta)H(z) + \delta - \lambda + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta}\right)^{\mu} - \frac{\left(\delta - \lambda + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta}\right)^{\mu}\right)(1-B^2) + (1-\delta)(1-AB)}{1-B^2}\right|$$

$$< \frac{(A-B)(1-\delta)}{1-B^2}, \quad (B \neq -1,\ z \in U)$$

and

$$Re\left\{(1-\delta)H(z) + \delta - \lambda + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta}\right)^{\mu}\right\}$$

$$> \frac{(1-A)(1-\delta)}{2} + \delta - \lambda + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta}\right)^{\mu}, \quad (B = -1,\ z \in U).$$

Putting

$$(1-\delta)H(z) + \delta - \lambda + \sum_{\mu=1}^{k} \binom{k}{\mu} (-1)^{\mu+1} \left(\frac{\alpha}{\beta}\right)^{\mu} = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$-\frac{(A-B)(1-\delta)}{\left(\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)(1-B^2)+(1-\delta)(1-AB)}<\phi$$

$$<\frac{(A-B)(1-\delta)}{\left(\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)(1-B^2)+(1-\delta)(1-AB)}, \quad (B\neq -1)$$

and $-1<\phi<1,\ (B=-1)$, then

$$\frac{(1-A)(1-\delta)}{1-B}+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}$$

$$<\rho<\frac{(1+A)(1-\delta)}{1+B}+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}, \ (B\neq -1)$$

and

$$\frac{(1-A)(1-\delta)}{1-B}+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}<\rho<\infty, \quad (B=-1).$$

An application of Lemma 1.2 with $\mathcal{T}(z)=\frac{1}{(1-\delta)H(z)+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}}$, yields $G(z)\prec h(z)$.

If there exist two points $z_1, z_2\in U$ such that

$$-\frac{\pi}{2}b_1=arg\big(G(z_1)\big)<arg\big(G(z)\big)<arg\big(G(z_2)\big)=\frac{\pi}{2}b_2\,,$$

then by Lemma 1.3, we get

$$\frac{z_1G'(z_1)}{G(z_1)}=-\frac{mi}{2}(b_1+b_2) \qquad \text{and} \qquad \frac{z_2G'(z_2)}{G(z_2)}=\frac{mi}{2}(b_1+b_2)\,,$$

where

$$m\geq\frac{1-|\varepsilon|}{1+|\varepsilon|} \quad \text{and} \quad \varepsilon=i\tan\frac{\pi}{4}\left(\frac{b_2-b_1}{b_1+b_2}\right).$$

Now, for the case $B\neq -1$, we obtain

$$arg\left(\frac{1}{1-\tau}\left(\frac{z_1\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z_1)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}g(z_1)}-\tau\right)\right)$$

$$=arg\left(G(z_1)+\frac{z_1G'(z_1)}{(1-\delta)H(z_1)+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}}\right)$$

$$=arg\big(G(z_1)\big)+arg\left(1+\frac{z_1G'(z_1)}{\left[(1-\delta)H(z_1)+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right]G(z_1)}\right)$$

$$=-\frac{\pi}{2}b_1+arg\left(1-\frac{mi}{2\rho}(b_1+b_2)e^{-i\frac{\pi}{2}\phi}\right)$$

$$=-\frac{\pi}{2}b_1+arg\left(1-\frac{m}{2\rho}(b_1+b_2)\cos\frac{\pi}{2}(1-\phi)+\frac{mi}{2\rho}(b_1+b_2)\sin\frac{\pi}{2}(1-\phi)\right)$$

$$\leq-\frac{\pi}{2}b_1-\tan^{-1}\left(\frac{m(b_1+b_2)\sin\frac{\pi}{2}(1-\phi)}{2\rho+m(b_1+b_2)\cos\frac{\pi}{2}(1-\phi)}\right)$$

$$\leq-\frac{\pi}{2}b_1-\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B}+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right)$$

$$=-\frac{\pi}{2}a_1,$$

where a_1 and t are given by (2.8) and (2.10), respectively.

Also,

$$arg\left(\frac{1}{1-\tau}\left(\frac{z_2\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z_2)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}g(z_2)}-\tau\right)\right)$$

$$\geq \frac{\pi}{2} b_2 + \tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{\begin{array}{c}2(1+|\varepsilon|)\left(\frac{(1+A)(1-\delta)}{1+B}+\delta-\lambda+\sum_{\mu=1}^{k}\binom{k}{\mu}(-1)^{\mu+1}\left(\frac{\alpha}{\beta}\right)^{\mu}\right)\\ +(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t\end{array}}\right)$$

$$= \frac{\pi}{2} a_2,$$

where a_2 and t are given by (2.9) and (2.10), respectively.

Similarly, for the case $B = -1$, we have

$$arg\left(\frac{1}{1-\tau}\left(\frac{z_1\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z_1)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}g(z_1)}-\tau\right)\right) \leq -\frac{\pi}{2} b_1$$

and

$$arg\left(\frac{1}{1-\tau}\left(\frac{z_2\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z_2)\right)'}{D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}g(z_2)}-\tau\right)\right) \geq \frac{\pi}{2} b_2.$$

The above two cases disagree with the assumptions. Therefore, the proof of the theorem is complete.

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