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Some Geometric Properties for a Class of Analytic Functions Defined by Beta Negative Binomial Distribution Series

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Abstract

In the present paper, we introduce and study a subclass of analytic and univalent functions associated with Beta negative binomial distribution series which is defined in the open unit disk U. We discuss some important geometric properties of this subclass, like, coefficient estimates, extreme points and integral representation. Also, we obtain results about integral mean associated with fractional integral.

1. Introduction

Let A indicate the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C},\ |z| < 1\}$.

Let M denote the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0), \tag{1.2}$$

A function $f \in A$ is said to be univalent starlike of order α $(0 \leq \alpha < 1)$ if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U),$$

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and is said to be univalent convex of order α $(0 \leq \alpha < 1)$ if it satisfies the condition:

$$Re\left\{\frac{zf''(z)}{f'(z)}+1\right\} > \alpha \qquad (z \in U).$$

Denote by $S^*(\alpha)$ and $C(\alpha)$ the classes of univalent starlike and univalent convex functions of order α, respectively.

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial, the Borel have been partially studied in the Geometric Function Theory from a theoretical point of view (see for example [1,3,5,6,9]).

A discrete random variable x is said to have a beta negative binomial distribution if it takes the values $0,1,2,3,\ldots$ with the probabilities $\frac{\beta(\nu+\sigma,\rho)}{\beta(\nu,\rho)}, \mu\frac{\beta(\nu+\sigma,\rho+1)}{\beta(\nu,\rho)}, \frac{1}{2}\sigma(\sigma+1)\frac{\beta(\nu+\sigma,\rho+2)}{\beta(\nu,\rho)}, \ldots$ respectively, where ν, ρ, σ are named the parameters.

$$P(x=r) = \binom{\sigma+r-1}{r}\frac{\beta(\nu+\sigma,\rho+r)}{\beta(\nu,\rho)}, \quad r = 0,1,2,3,\ldots\ .$$

We can write the above probability as follows:

$$P(x=r) = \binom{\sigma+r-1}{r}\frac{\beta(\nu+\sigma,\rho+r)}{\beta(\nu,\rho)} = \frac{\Gamma(\sigma+r)}{r!\ \Gamma(\sigma)}\frac{\Gamma(\nu+\sigma)\Gamma(\rho+r)\Gamma(\nu+\rho)}{\Gamma(\nu+\sigma+\rho+r)\Gamma(\nu)\Gamma(\rho)}$$

$$= \frac{(\nu)_\sigma(\sigma)_r(\rho)_r}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_r\, r!},$$

where $(\kappa)_n$is the Pochhammer symbol defined by

$$(\kappa)_n = \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)} = \begin{cases} 1 & (n=0), \\ \kappa(\kappa+1)\ldots(\kappa+n-1) & (n \in \mathbb{N}). \end{cases}$$

Very recently, Wanas and Al-Ziadi [8] introduced the following power series whose coefficients are probabilities of the beta negative binomial distribution:

$$\wp_{\nu,\rho}^{\sigma}(z) = z - \sum_{n=2}^{\infty}\frac{(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!}z^n, \qquad (z \in U, \nu, \rho, \sigma > 0).$$

We note by the familiar Ratio Test that the radius of convergence of the above series is infinity.

The linear operator $\mathfrak{P}_{\nu,\rho}^{\sigma} : M \longrightarrow M$ is defined as follows (see [8])

$$\mathfrak{P}_{\nu,\rho}^{\sigma}(z) = \wp_{\nu,\rho}^{\sigma}(z) * f(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!} a_n z^n, \quad (a_n \geq 0),$$

where $*$ indicate the Hadamard product (or convolution) of two series.

For two functions f and g analytic in U, we say that f is subordinate to g in U, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a function ω which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$ such that $f(z) = g\big(\omega(z)\big)$, $(z \in U)$. In particular, if the function g is univalent in U, we have the following equivalence (see [7]):

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0), f(U) \subset g(U).$$

Definition 1.1 [2]**.** The fractional integral of order λ $(\lambda > 0)$ is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1.3}$$

where f is an analytic function in a simple connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Lemma 1.1 [4]**.** *If f and g are analytic in U with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \;\leq\; \int_0^{2\pi} |g(z)|^{\mu} \, d\theta \,.$$

Definition 1.2. A function $\mathfrak{P}_{\nu,\rho}^{\sigma} \in M$ is said to be in the class $\mathbb{B}(\gamma, \delta, \tau, \nu, \rho, \sigma)$ if it satisfies

$$\left| \frac{z\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)''}{\delta z\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)'' + (\delta+1)(1-\tau)\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)'} \right| < 1, \tag{1.4}$$

where $0 \leq \delta < 1,\ 0 \leq \tau < 1, \nu, \rho, \sigma > 0$ and $z \in U$.

2. Coefficient Estimates

The first theorem gives a necessary and sufficient condition for a function $\mathfrak{P}_{\nu,\rho}^{\sigma}$ to be in the class $\mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$.

Theorem 2.1. *A function* $\mathfrak{P}_{\nu,\rho}^{\sigma} \in M$ *is in the class* $\mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{n(n-\tau)(\delta+1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!} a_n \leq (\delta+1)(1-\tau), \tag{2.1}$$

where $0 \leq \delta < 1, \quad 0 \leq \tau < 1, \nu, \rho, \sigma > 0$ *and* $z \in U$.

The result is sharp for the function $\mathfrak{P}_{\nu,\rho}^{\sigma}$ *given by*

$$\mathfrak{P}_{\nu,\rho}^{\sigma}(z) = z - \frac{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}} z^n, \quad (n \geq 2). \tag{2.2}$$

Proof. Assume that inequality (2.1) holds true and $|z| = 1$. Then we have

$$\left|z\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)''\right| - \left|\delta z\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)'' + (\delta+1)(1-\tau)\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)'\right|$$

$$= \left|-\sum_{n=2}^{\infty} \frac{n(n-1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!} a_n z^{n-1}\right|$$

$$-\left|(\delta+1)(1-\tau) - \sum_{n=2}^{\infty} \frac{n(\delta(n-\tau)+1-\tau)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!} a_n z^{n-1}\right|$$

$$\leq \sum_{n=2}^{\infty} \frac{n(n-1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!} a_n |z|^{n-1} - (\delta+1)(1-\tau)$$

$$+\sum_{n=2}^{\infty} \frac{n(\delta(n-\tau)+1-\tau)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!} a_n |z|^{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{n(n-\tau)(\delta+1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!} a_n - (\delta+1)(1-\tau) \leq 0,$$

by hypothesis. Hence by maximum modulus principle, we obtain $\mathfrak{P}_{\nu,\rho}^{\sigma} \in \mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$.

To show the converse, let $\mathfrak{P}_{\nu,\rho}^{\sigma} \in \mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$. Then

$$\left|\frac{z\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)''}{\delta z\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)''+(\delta+1)(1-\tau)\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)'}\right|$$

$$=\left|\frac{-\sum_{n=2}^{\infty}\frac{n(n-1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!}a_n z^{n-1}}{(\delta+1)(1-\tau)-\sum_{n=2}^{\infty}\frac{n(\delta(n-\tau)+1-\tau)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!}a_n z^{n-1}}\right|<1$$

Since $Re(z) \leq |z|$ for all z, then we get

$$Re\left(\frac{-\sum_{n=2}^{\infty}\frac{n(n-1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!}a_n z^{n-1}}{(\delta+1)(1-\tau)-\sum_{n=2}^{\infty}\frac{n(\delta(n-\tau)+1-\tau)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!}a_n z^{n-1}}\right)<1, \qquad (2.3)$$

Now taking the value of z on the real axis so that $\frac{z\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)''}{\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)'}$ is real. Upon clearing the denominator of (2.3) and letting $z \longrightarrow 1^{-}$ through real values, we obtain the inequality (2.1).

Corollary 2.1. *If* $\mathfrak{P}_{\nu,\rho}^{\sigma} \in \mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$, *then*

$$a_n \leq \frac{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}, \qquad (n \geq 2), \qquad (2.4)$$

3. Extreme Points

Theorem 3.1. *Let* $\mathfrak{P}_{\nu,\rho_1}^{\sigma}(z)=z$ *and*

$$\mathfrak{P}_{\nu,\rho_n}^{\sigma}(z)=z-\frac{(\nu+\rho)_{\sigma}(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_{\sigma}(\sigma)_{n-1}(\rho)_{n-1}}z^n, \qquad (n \geq 2).$$

Then $\mathfrak{P}_{\nu,\rho}^{\sigma} \in \mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$ *if and only if it can be expressed in the form*

$$\mathfrak{P}_{\nu,\rho}^{\sigma}(z)=\sum_{n=1}^{\infty}\lambda_n \mathfrak{P}_{\nu,\rho_n}^{\sigma}(z), \qquad (3.1)$$

where $\lambda_n \geq 0,\ n \geq 1$ *and* $\sum_{n=1}^{\infty}\lambda_n = 1$.

Proof. Suppose that $\mathfrak{P}_{\nu,\rho}^{\sigma}$ is expressed in the form (3.1). Then

$$\mathfrak{P}_{\nu,\rho}^{\sigma}(z)=\sum_{n=1}^{\infty}\lambda_n\mathfrak{P}_{\nu,\rho_n}^{\sigma}(z)$$

$$=\lambda_1\mathfrak{P}_{\nu,\rho_1}^{\sigma}(z)+\sum_{n=2}^{\infty}\lambda_n\mathfrak{P}_{\nu,\rho_n}^{\sigma}(z)$$

$$=\left(1-\sum_{n=2}^{\infty}\lambda_n\right)z+\sum_{n=2}^{\infty}\lambda_n\left(z-\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}z^n\right)$$

$$=z-\sum_{n=2}^{\infty}\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}\lambda_n z^n.$$

Now

$$\sum_{n=2}^{\infty}\frac{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}$$

$$\times\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}\lambda_n$$

$$=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\leq 1.$$

This shows that $\mathfrak{P}_{\nu,\rho}^{\sigma}\in\mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$.

Conversely, assume that $\mathfrak{P}_{\nu,\rho}^{\sigma}\in\mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$. Then by Corollary 2.1, we find that

$$a_n\leq\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}},\qquad (n\geq 2).$$

We can set

$$\lambda_n=\frac{n(n-\tau)(\delta+1)((\nu)_\sigma)^2((\sigma)_{n-1})^2((\rho)_{n-1})^2}{((\nu+\rho)_\sigma)^2((\sigma+\nu+\rho)_{n-1})^2\left((n-1)!\right)^2(\delta+1)(1-\tau)}a_n\quad (n\geq 2),$$

where $\lambda_1=1-\sum_{n=2}^{\infty}\lambda_n$. Thus

$$\mathfrak{P}_{\nu,\rho}^{\sigma}(z)=z-\sum_{n=2}^{\infty}\frac{(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!}a_n z^n$$

$$= z - \sum_{n=2}^{\infty} \frac{(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!}$$

$$\times \frac{((\nu+\rho)_\sigma)^2((\sigma+\nu+\rho)_{n-1})^2\big((n-1)!\big)^2(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)((\nu)_\sigma)^2((\sigma)_{n-1})^2((\rho)_{n-1})^2}\lambda_n z^n$$

$$= z - \sum_{n=2}^{\infty} \frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}\lambda_n z^n$$

$$= z - \sum_{n=2}^{\infty}\left(z - \mathfrak{P}^{\sigma}_{\nu,\rho_n}(z)\right)\lambda_n$$

$$= \left(1 - \sum_{n=2}^{\infty}\lambda_n\right) z + \sum_{n=2}^{\infty}\lambda_n \mathfrak{P}^{\sigma}_{\nu,\rho_n}(z)$$

$$= \lambda_1 z + \sum_{n=2}^{\infty}\lambda_n \mathfrak{P}^{\sigma}_{\nu,\rho_n}(z) = \sum_{n=1}^{\infty}\lambda_n \mathfrak{P}^{\sigma}_{\nu,\rho_n}(z).$$

That is the required representation.

4. Integral Representation

Theorem 4.1. *Let* $\mathfrak{P}^{\sigma}_{\nu,\rho} \in \mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$. *Then*

$$\mathfrak{P}^{\sigma}_{\nu,\rho}(z) = \int_0^z \int_0^z exp\left[\int_0^z \frac{(\delta-\tau)\psi(t_1)+p-2}{t_1\big(1-\gamma\psi(t_1)\big)}dt_1\right] dt_2 dt_3\ ,$$

where $|\psi(z)| < 1,\ z \in U$.

Proof. By setting $\frac{z\left(\mathfrak{P}^{\sigma}_{\nu,\rho}(z)\right)''}{\left(\mathfrak{P}^{\sigma}_{\nu,\rho}(z)\right)'} = N(z)$ in (1.4), we obtain

$$\left|\frac{N(z)}{\delta N(z)+(\delta+1)(1-\tau)}\right| < 1,$$

or equivalently

$$\frac{N(z)}{\delta N(z)+(\delta+1)(1-\tau)} = \psi(z), \quad (|\psi(z)| < 1,\ \ z \in U).$$

So

$$\frac{\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)''}{\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)'}=\frac{(\delta+1)(1-\tau)\psi(z)}{z\big(1-\delta\psi(z)\big)}.$$

After integration, we get

$$\log\left(\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)''\right)=\int_0^z\frac{(\delta+1)(1-\tau)\psi(t_1)}{t_1\big(1-\delta\psi(t_1)\big)}\,dt_1.$$

Thus

$$\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)''=\exp\left[\int_0^z\frac{(\delta+1)(1-\tau)\psi(t_1)}{t_1\big(1-\delta\psi(t_1)\big)}\,dt_1\right].$$

By integration once again, we have

$$\left(\mathfrak{P}_{\nu,\rho}^{\sigma}(z)\right)'=\int_0^z\exp\left[\int_0^z\frac{(\delta+1)(1-\tau)\psi(t_1)}{t_1\big(1-\delta\psi(t_1)\big)}\,dt_1\right]dt_2.$$

Also, after integration, we conclude that

$$\mathfrak{P}_{\nu,\rho}^{\sigma}(z)=\int_0^z\int_0^z\exp\left[\int_0^z\frac{(\delta+1)(1-\tau)\psi(t_1)}{t_1\big(1-\delta\psi(t_1)\big)}\,dt_1\right]dt_2\,dt_3.$$

This gives the required result.

5. Integral Mean Inequalities for the Fractional Integral

Theorem 5.1. *If* $\mathfrak{P}_{\nu,\rho}^{\sigma}\in\mathbb{B}(\delta,\tau,\nu,\rho,\sigma)$ *and suppose that* ${\mathfrak{P}_{\nu,\rho}^{\sigma}}_n$ *is defined by*

$${\mathfrak{P}_{\nu,\rho}^{\sigma}}_n(z)=z-\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}}z^n,\quad(n\geq 2).\quad(5.1)$$

Also let

$$\sum_{j=2}^{\infty}(j-\eta)_{\eta+1}a_j$$

$$\leq\frac{(\sigma+\nu+\rho)((\nu+\rho)_\sigma)^2(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)\Gamma(\lambda+\eta+3)}{n\sigma\rho(n-\tau)(\delta+1)((\nu)_\sigma)^2(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)\Gamma(2-\eta)},$$

for $0\leq\eta\leq j,\lambda>0,$ *where* $(j-\eta)_{\eta+1}$ *denotes the Pochhammer symbol defined by* $(j-\eta)_{\eta+1}=\frac{\Gamma(j+1)}{\Gamma(j-\eta)}$.

If there exists an analytic function ω defined by

$$(\omega(z))^n = \frac{n(n-\tau)(\delta+1)((\nu)_\sigma)^2(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)}{((\nu+\rho)_\sigma)^2(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)}$$

$$\times \sum_{j=2}^{\infty} \frac{(\sigma)_{j-1}(\rho)_{j-1}}{(\sigma+\nu+\rho)_{j-1}(j-1)!}(j-\eta)_{\eta+1}\, T(j) a_j z^{j-1},$$

where $n \geq \eta$ *and*

$$T(j) = \frac{\Gamma(j-\eta)}{\Gamma(j+\lambda+\eta+1)}, \quad (\lambda > 0\,, \; j \geq 2),$$

then for $z = re^{i\theta}$ *and* $0 < r < 1$

$$\int_0^{2\pi} \left| D_z^{-\lambda-\eta} \mathfrak{P}_{\nu,\rho}^{\sigma}(z) \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| D_z^{-\lambda-\eta} \mathfrak{P}_{\nu,\rho_n}^{\sigma}(z) \right|^{\mu} d\theta, \quad (\lambda > 0, \mu > 0).$$

Proof. Assume that

$$\mathfrak{P}_{\nu,\rho}^{\sigma}(z) = z - \sum_{j=2}^{\infty} \frac{(\nu)_\sigma(\sigma)_{j-1}(\rho)_{j-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{j-1}(j-1)!} a_j z^j.$$

For $\eta \geq 0$ and Definition 1.1, we have

$$D_z^{-\lambda-\eta} \mathfrak{P}_{\nu,\rho}^{\sigma}(z)$$

$$= \frac{z^{\lambda+\eta+1}}{\Gamma(\lambda+\eta+2)} \left(1 - \sum_{j=2}^{\infty} \frac{\Gamma(j+1)\Gamma(\lambda+\eta+2)(\nu)_\sigma(\sigma)_{j-1}(\rho)_{j-1}}{\Gamma(j+\lambda+\eta+1)(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{j-1}(j-1)!} a_j z^{j-1} \right)$$

$$= \frac{z^{\lambda+\eta+1}}{\Gamma(\lambda+\eta+2)} \left(1 - \sum_{j=2}^{\infty} \frac{\Gamma(\lambda+\eta+2)(\nu)_\sigma(\sigma)_{j-1}(\rho)_{j-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{j-1}(j-1)!} (j-\eta)_{\eta+1}\, T(j) a_j z^{j-1} \right),$$

where

$$T(j) = \frac{\Gamma(j-\eta)}{\Gamma(j+\lambda+\eta+1)}, \quad (\lambda > 0, \; j \geq 2).$$

since T is a decreasing function of j, then we get

$$0 < T(j) \leq T(2) = \frac{\Gamma(2-\eta)}{\Gamma(\lambda+\eta+3)}.$$

similarly, from (5.1) and Definition 1.1, we find that

$$D_z^{-\lambda-\eta}\mathfrak{P}_{\nu,\rho_n}^{\sigma}(z)=\frac{z^{\lambda+\eta+}}{\Gamma(\lambda+\eta+2)}$$

$$\times\left(1-\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)\,\Gamma(\lambda+\eta+2)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)}z^{n-1}\right).$$

For $\mu>0$ and $z=re^{i\theta}$ $(0<r<1)$, we must show that

$$\int_0^{2\pi}\left|1-\sum_{j=2}^{\infty}\frac{\Gamma(\lambda+\eta+2)(\nu)_\sigma(\sigma)_{j-1}(\rho)_{j-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{j-1}(j-1)!}(j-\eta)_{\eta+1}\,T(j)a_jz^{j-1}\right|^{\mu}d\theta$$

$$\leq\int_0^{2\pi}\left|1-\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)\,\Gamma(\lambda+\eta+2)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)}z^{n-1}\right|^{\mu}d\theta.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1-\sum_{j=2}^{\infty}\frac{\Gamma(\lambda+\eta+2)(\nu)_\sigma(\sigma)_{j-1}(\rho)_{j-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{j-1}(j-1)!}(j-\eta)_{\eta+1}\,T(j)a_jz^{j-1}$$

$$\prec 1-\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)\,\Gamma(\lambda+\eta+2)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)}z^{n-1}.$$

By setting

$$1-\sum_{j=2}^{\infty}\frac{\Gamma(\lambda+\eta+2)(\nu)_\sigma(\sigma)_{j-1}(\rho)_{j-1}}{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{j-1}(j-1)!}(j-\eta)_{\eta+1}\,T(j)a_jz^{j-1}$$

$$=1-\frac{(\nu+\rho)_\sigma(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)\,\Gamma(\lambda+\eta+2)}{n(n-\tau)(\delta+1)(\nu)_\sigma(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)}\big(\omega(z)\big)^{n-1},$$

it follows that

$$\big(\omega(z)\big)^{n}=\frac{n(n-\tau)(\delta+1)((\nu)_\sigma)^2(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)}{((\nu+\rho)_\sigma)^2(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)}$$

$$\times\sum_{j=2}^{\infty}\frac{(\sigma)_{j-1}(\rho)_{j-1}}{(\sigma+\nu+\rho)_{j-1}(j-1)!}(j-\eta)_{\eta+1}\,T(j)a_jz^{j-1},$$

which readily yields $\omega(0)=0$. For such a function ω, we obtain

$$|\omega(z)|^n \leq \frac{n(n-\tau)(\delta+1)((\nu)_\sigma)^2(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)}{((\nu+\rho)_\sigma)^2(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)}$$

$$\times \sum_{j=2}^{\infty} \frac{(\sigma)_{j-1}(\rho)_{j-1}}{(\sigma+\nu+\rho)_{j-1}(j-1)!}(j-\eta)_{\eta+1}\, T(j)a_j|z|^{j-1}$$

$$\leq \frac{n\sigma\rho(n-\tau)(\delta+1)((\nu)_\sigma)^2(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)}{(\sigma+\nu+\rho)((\nu+\rho)_\sigma)^2(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)} T(2)|z|$$

$$\times \sum_{j=2}^{\infty} (j-\eta)_{\eta+1} a_j$$

$$= |z| \frac{n\sigma\rho(n-\tau)(\delta+1)((\nu)_\sigma)^2(\sigma)_{n-1}(\rho)_{n-1}\Gamma(n+\lambda+\eta+1)\Gamma(2-\eta)}{(\sigma+\nu+\rho)((\nu+\rho)_\sigma)^2(\sigma+\nu+\rho)_{n-1}(n-1)!\,(\delta+1)(1-\tau)\Gamma(n+1)\Gamma(\lambda+\eta+3)}$$

$$\times \sum_{j=2}^{\infty} (j-\eta)_{\eta+1} a_j$$

$$\leq |z| < 1.$$

This completes the proof of the theorem.

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