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Hermite Polynomial and Least-Squares Technique for Solving Integro-differential Equations

Ahsan Fayez Shoushan

Thi-Qar Education Directorate, Iraq
e-mail: mathahsan9@gmail.com

Abstract

The goal of this project is to offer a new technique for solving integro-differential equations (IDEs) with mixed circumstances, which is based on the Hermite polynomial and the Least-Squares Technique (LST). Three examples will be given to demonstrate how the suggested technique works. The numerical results were utilized to demonstrate the correctness and efficiency of the existing method, and all calculations were carried out with the help of the MATLAB R2018b program.

1. Introduction

The authors look at the Hermit polynomial basic function and their squares, provide precise formulas for higher order derivatives, which can be viewed as ordinary (DE) or derivative polynomial, and derive explicit formulas and recurrence relations for the Hermit polynomial and their squares [1]. Using airfoil polynomials of the first kind, examines the numerical solution for a class of IDE with Cauchy kernel. Obtain a system of linear algebraic equations using this strategy [2]. With weakly singular kernels, a new collocation type approach for solving VIE of the second sort has been developed. For solving the VI, we employ the complex B-spline basics in collocation method. The findings of this technique are compared to the exact solution [3]. To obtain a trustworthy approximate solution to linear and nonlinear IEs and IDEs originating in ordinary life occurrences, a simple and effective method is used. The suggested method comprises solely of a series in which the undefined constants are found in the conventional manner. The results achieved using the method are in good agreement with the exact solution,

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demonstrating that it both effective and simple to use [4]. The Lyapunov's functional technique is used to conduct new research on the stability, asymptotically stability and instability of the zero solution, boundedness, integrability of solutions and integrability of derivatives of solutions of certain nonlinear (DIDEs). Three important Lyapunov functional are defined as main tools to achieve the goal of this research [5]. Several authors have worked semi analytical techniques such as the Taylor series expansion techniques [6], the Computational Methods [7], the Tau techniques [8, 9, 10], the Legendre wavelets method [11], the Legendre polynomials [12, 13], the Galerkin methods [14], the Legendre differential expressions [15], the Fourier analysis [16] for integral and integro-differential equations in recent.

$$z^{(k)}(s) = f(s) + \int_0^s k(s,y)z(y)dy, \qquad a \le s, y \le g, \tag{1.1}$$

under the mixed conditions

$$\sum_{i=0}^{N-1} (a_{ji} z^{(i)}(h) + g_{ji} z^{(i)}(g) = \beta_j, \qquad i = 0, 1, \dots, N-1, \text{ [17]}. \tag{1.2}$$

1.1. Hermite polynomial

The Hermite polynomials $H_i(S)$ are a collection of polynomials with coefficients in the range $[0, \infty]$, and the basic formula is [1].

$$e^{2sy-y^2} = \sum_{i=0}^{\infty} H_i(\mathrm{S}) \frac{y^i}{i!}. \tag{1.3}$$

The first three Hermite polynomials $H_i(\mathrm{S})$, for $0 \le i \le 5$

$$H_0(s) = 1, \quad H_1(s) = 2S, \quad H_2(s) = 4s^2 - 2.$$

2. Least-Square Method and Hermite Polynomial

To solve the equations (1.1) and (1.2), we will use a novel strategy based on the Hermite polynomial with LST as a foundation function.

The following is a rough solution:

$$z_m(s) = \sum_{i=0}^{m} c_i H_i(s) \; a \le x \le g \tag{2.1}$$

where $H_i(s)$ are the Hermite polynomial and c_i are unknown constants of degrees (i). We obtain equation (2.2) by replacing equation (1.1) into equation (2.1).

$$\sum_{i=0}^{m} c_i H_i^{(k)}(s) = f(s) + \int_0^s k(s,y) \sum_{i=0}^{m} c_i H_i(y)\, dy. \tag{2.2}$$

The residual equation is as follows:

$$\varphi(s, c_i) = \varphi\big(s, z_m(c)\big) = \sum_{i=0}^{m} c_i H_i^{(k)} - \left\{ f(s) + \int_0^s k(s,y) \sum_{i=0}^{m} c_i H_i(y)\, dy \right\}. \tag{2.3}$$

Let

$$T(c_0, c_1, \dots, c_m) = \int_a^g [\varphi(s, c_i)]^2 w(s) ds, \tag{2.4}$$

where $w\,(s) = 1$, thus [18],

$$T(c_0, c_1, \dots, c_m) = \int_a^g \left[\sum_{i=0}^{m} c_i H_i^{(k)} - \left\{ f(s) + \int_0^s k(s,y) \sum_{i=0}^{m} c_i H_i(y)\, dy \right\} \right]^2 ds. \tag{2.5}$$

By decreasing the value of T, we may obtain the values of $c_i, i \geq 0$.

$$\frac{\partial T}{\partial c_i} = 0, i = 0,1, \dots, m. \tag{2.6}$$

By applying (2.6) we get:

$$\frac{\partial T}{\partial c_i} = \int_a^g \left[\sum_{i=0}^{m} c_i H_i^{(k)} - \left\{ f(s) + \int_0^s k(s,y) \sum_{i=0}^{m} c_i H_i(y)\, dy \right\} \right] ds$$

$$\times \int_a^g \left[H_i^{(k)} - \int_0^s k(s,y) H_i(s) ds \right] ds = 0. \tag{2.7}$$

As a result, (2.7) is formed as a $(m+1) * (m+1)$ algebraic system of equations in the undefined polynomial coefficient $c_i, i = 0, \cdots, m$ or as a matrix:

$$\Psi = \begin{pmatrix} \int_a^g \varphi(s,c_0)\Upsilon_0 ds & \int_a^g \varphi(s,c_1)\Upsilon_0 ds & \dots & \int_a^g \varphi(s,c_m)\Upsilon_0 ds \\ \int_a^g \varphi(s,c_0)\Upsilon_1 ds & \int_a^g \varphi(s,c_1)\Upsilon_1 ds & \dots & \int_a^g \varphi(s,c_m)\Upsilon_1 ds \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^g \varphi(s,c_0)\Upsilon_m ds & \int_a^g \varphi(s,c_1)\Upsilon_m ds & \dots & \int_a^g \varphi(s,c_m)\Upsilon_m ds \end{pmatrix} \tag{2.8}$$

$$\mathrm{E} = \begin{pmatrix} \int_a^g \{f(s)\}\Upsilon_0 ds \\ \int_a^g \{f(s)\}\Upsilon_1 ds \\ \vdots \\ \int_a^g \{f(s)\}\Upsilon_m ds \end{pmatrix} \tag{2.9}$$

where,

$$\Upsilon_i = H_i^{(k)}(s) - \int_0^s k(x,t)H_i(s)ds \tag{2.10}$$

$$\varphi(s,c_i) = \sum_{i=0}^{m} c_i H_i^{(k)} - \left\{ f(s) + \int_0^s k(s,y) \sum_{i=0}^{m} c_i H_i(y)\, dy \right\}, \tag{2.11}$$

$$\Psi C = \mathrm{E} \quad \text{or} \quad C = \ [\Psi : \mathrm{E}]. \tag{2.12}$$

Another variant of (2.12) can be described by using the conditions as follows:

$$[U_i : \beta_i], \qquad i = 0\,, \dots, N-1$$

where,

$$U_i = [u_{i0} u_{i1} u_{i2} \ \ \dots \ \ u_{iN}],\ i = 0\,, \dots, N-1. \tag{2.13}$$

Another solution can be obtained by substituting the matrices of the row (2.13) with the last rows (m) of the matrix form (2.12), [19].

$$
[\tilde{\Psi}:\tilde{\mathrm{E}}] = \begin{pmatrix}
\int_a^g \varphi(\mathrm{s},\mathrm{c}_0)\Upsilon_0 \mathrm{ds} & \int_a^g \varphi(\mathrm{s},\mathrm{c}_1)\Upsilon_0 \mathrm{ds} & \dots & \int_a^g \varphi(\mathrm{s},\mathrm{c}_m)\Upsilon_0 \mathrm{ds} & ; & \mathrm{E}_0 \\
\int_a^g \varphi(\mathrm{s},\mathrm{c}_0)\Upsilon_1 \mathrm{ds} & \int_a^g \varphi(\mathrm{s},\mathrm{c}_1)\Upsilon_1 \mathrm{ds} & \dots & \int_a^g \varphi(\mathrm{s},\mathrm{c}_m)\Upsilon_1 \mathrm{ds} & ; & \mathrm{E}_1 \\
\vdots & \vdots & \ddots & \vdots & & \vdots \\
\int_a^g \varphi(\mathrm{s},\mathrm{c}_0)\Upsilon_{m_{N_0}} \mathrm{ds} & \int_a^g \varphi(\mathrm{s},\mathrm{c}_1)\Upsilon_{m_{N_1}} \mathrm{ds} & \dots & \int_c^g \varphi(\mathrm{s},\mathrm{c}_m)\Upsilon_{m_{N_m}} \mathrm{ds} & ; & \mathrm{E}_{m-N} \\
 & \mathrm{u}_{00}\mathrm{u}_{01}\mathrm{u}_{0N} & & ; & \beta_0 & \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
\mathrm{u}_{(m-1)0}\mathrm{u}_{(m-1)1} & \dots & \mathrm{u}_{(m-1)N} & ; & \beta_{N-1} &
\end{pmatrix}
$$

$$C = \tilde{\Psi}^{-1}\tilde{\mathrm{E}}.$$

As a result, the matrix C can only be determined once. In addition, there is only one solution to the equation (1.1) given conditions (1.2).

3. Analyze Convergence

Now, using the numerical methods established in (2), we will review an estimate of the error above and show that as m, the approximate solution $z_m(s)$ will converge to the precise solution $z(s)$ of (1).

Theorem [1]. *For $m \geq 0$, the Hermite polynomial $H_m(s)$ and their derivatives $H'_m(s)$ satisfy*

$$H'_0(s) = 0, H'_m(s) = 2mH_{m-1}(s). \tag{3.1}$$

4. Illustrative Problems

In this paragraph, we have investigated the combination of LST for solving linear IDEs with Hermite polynomial as the basis function. The problems are solved to explain them precisely, and time of accomplishment of the method. The absolute error has been defined

$$\text{Error}= |z(s) - z_m(s)|, \qquad a \leq s \leq g,\ m = 1,2,\dots$$

where $z_m(s)$ and $z(s)$ are the approximate solutions andaccurate solution.

Problem 4.1 [17]**.** Consider the following integro-differential equations,

$$z''(s) = -15s + \int_0^s \int_0^1 syz(y)dyds$$

with the initial condition, $z(0) = 1, z'(0) = 0$, the exact solution of this problem $z(s) = 1 - \frac{5}{2}s^3$.

We can see that $z_5(s)$ corresponds to the exact solution.

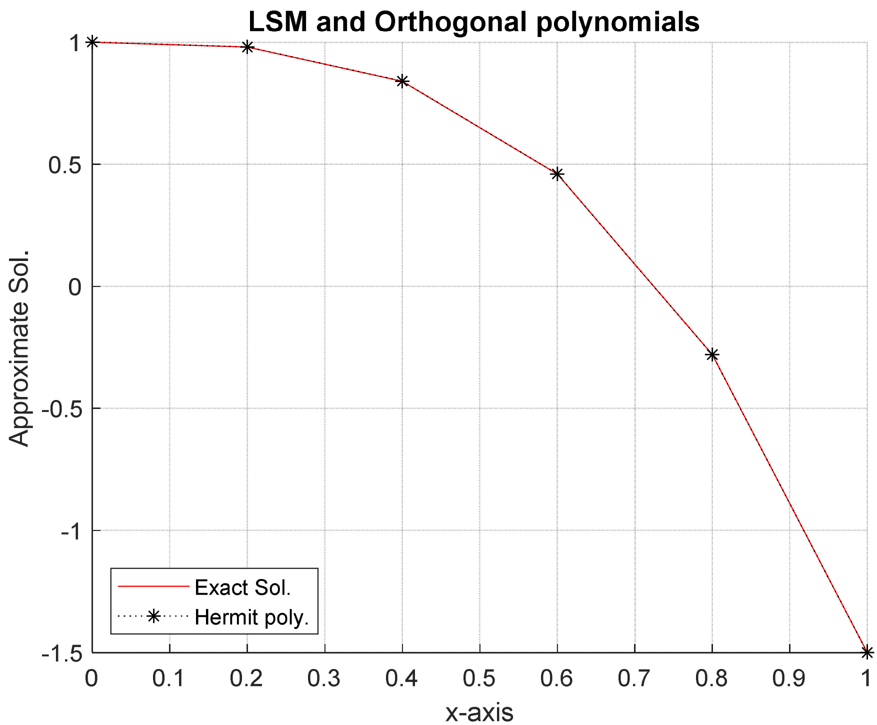

Figure 4.1. Approximate values and Exact, Problem 4.1.

Problem 4.2 [20]**.** Consider the following integro-differential equations

$$z'(s) = 8s + \frac{5}{4}s^2 + \int_0^s \int_0^1 (1 - sy)z(y)dyds,$$

with the initial condition, $z(0) = 2$, the exact solution of this problem $z(s) = 2 + 6s^2$.

We can see that $z_5(s)$ corresponds to the exact solution.

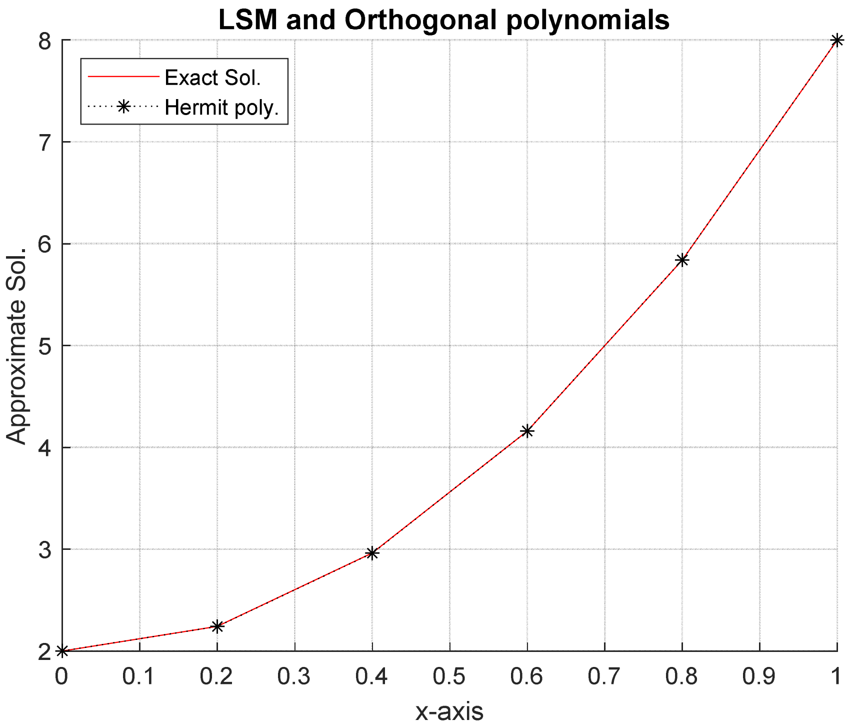

Figure 4.2. Approximate values and Exact, Problem 4.2.

In Figure 4.1-4.2, the exact solution is compared to the approximate solution derived via the suggested method for $m = 5$, the numerical and precise answers are quite consistent. When $m = 5$ is used, an approximate solution is obtained that is valid for all Hermite polynomials.

Problem 4.3 [21]**.** Consider the following integro-differential equations

$$z'(s) = -\frac{2}{3}e^{-s} - \frac{2}{3}s + se^{-2} + e^{-2} + \int_0^s (s-y)z(y)dy + \int_0^1 (y+ys)z(y)dy,$$

with the initial condition, $z(0) = \frac{1}{3}$, the exact solution of this problem $z(s) = \frac{1}{3}e^{-s}$.

Table 4.1. Exact, approximate solutions and the absolute errors, Problem 4.3.

x	**Exact solution**	Hermite Poly.		CLSCM [10]
		Approximate	Error	Error
0	0.3333	0.3333	0	0
0.1	0.3016	0.3059	0.0043	0.012
0.2	0.2729	0.2799	0.007	0.0228
0.3	0.2469	0.2553	0.0084	0.0078
0.4	0.2234	0.2321	0.0087	0.0059
0.5	0.2022	0.2103	0.0081	0.0118
0.6	0.1829	0.1899	0.007	0.0112
0.7	0.1655	0.1710	0.0055	0.0023
0.8	0.1498	0.1534	0.0036	0.0107
0.9	0.1355	0.1373	0.0018	0.0235
1	0.1226	0.1226	0	0.0061

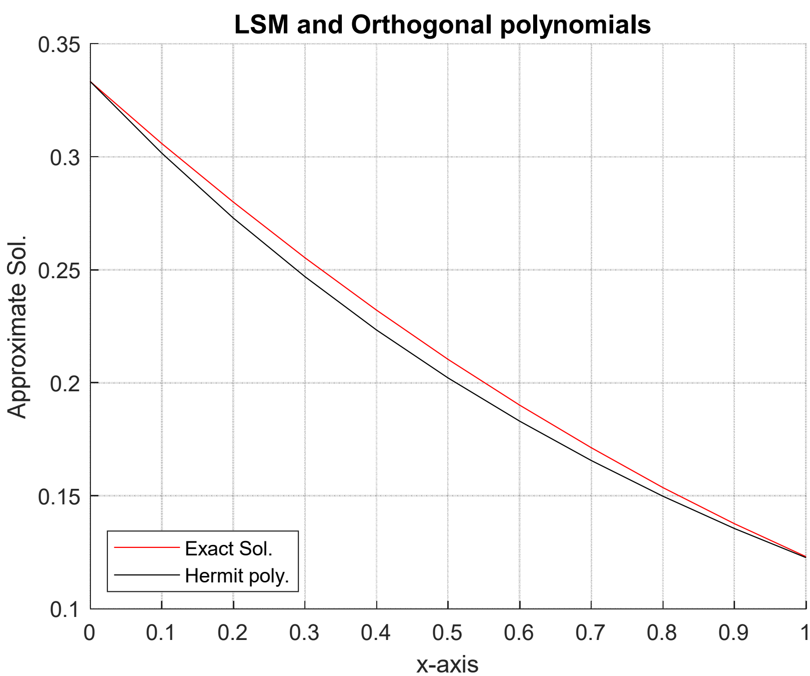

Figure 4.3. Approximate values and Exact, Problem 4.3.

Figure 4.3, the exact solution is compared to the approximate solution derived via the suggested method for $m = 2$, shows the exactness and correctness of the proposed method. When $m = 2$ is used, an approximate solution is obtained that is valid for all Hermite polynomial.

5. Conclusions

In this research, suggested a technique for solving integro-differential equations using the Least-Squares Technique (LST) and Hermite polynomial. Three problems were solved using the proposed technique. The accuracy and efficiency of the technique used were obtained by solving the problems.

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