



# Some Approximate Schemes for Solving Nonlinear Equations

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## Abstract

Some iterative algorithms for solving nonlinear equation  $f(x) = 0$  are suggested and investigated using Taylor series and homotopy perturbation technique. These algorithms can be viewed as extensions and generalization of well known methods such as Householder and Halley methods with cubic convergence. Convergence of the proposed methods has been discussed and analyzed. Several numerical examples are given to illustrate the efficiency of the suggested algorithms for solving nonlinear equations. Comparison with other iterative schemes is carried out to show the validity and performance of these algorithms.

## 1 Introduction

Solution of nonlinear equations are an important area of research in numerical analysis. In this branch of mathematics “Numerical analysis” usually deal with the continuous problem which come through daily life. These problems of real world phenomena occur in mathematics and all fields of sciences, such as, biology, economics, and physics.

We were familiar about the technique of hand calculation before the use of computers. Before the use of numerical algorithms analog devices were been used extensively in the science and amongst all the scientific Rule was common to the engineers which develop by Gunter in 1620.

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A nonlinear method in mathematics and science in which performance shift is not proportional to the input transition. Engineers, biologists, mathematicians and several other scientific scientists are involved in nonlinear problem. Newton method play central role in the development of technique for optimization most of current physical method for optimization can be viewed as variant form of the Newton method.

In many cases, it is not always possible to find the exact solution of nonlinear equations. Thus we use the approximate methods to find the solutions of nonlinear equations. For finding iterative approaches are usually applied for the solution of the nonlinear equation  $f(x) = 0$ . For instance the method of bisection, the Taylor series and the method of homotopy perturbation method. The homotopy perturbation method is used to solve a problem in pure and applied mathematics in different fields of science. This method is quiet efficient and allow us for choosing the auxiliary parameter arbitrary. The perturbation technique is used to solve the nonlinear problems. However the technique of perturbation based on the assumption of small parameters. Most of the nonlinear equation involving small parameter does not exist. These facts motivated the researchers to find approximate solutions of nonlinear equations  $f(x) = 0$ . These methods are as follows:

- (i) Bisection method
- (ii) Newton method
- (iii) Regula Falsi method
- (iv) Fixed point Iterative method
- (v) Secant method

In recent years, many modification have been established to above mention methods for computing nonlinear equations  $f(x) = 0$ . These methods have been constructed using different techniques such as Taylor series. Numerical methods have many application in pure and applied sciences. Most of the problems in applied and pure mathematics may lead to solve the nonlinear equations. For

more details, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and the reference therein. All the numerical methods which we used in this work to solve nonlinear equation have their own imitation and drawbacks. Thus its essential to discuss the efficiency of numerical methods but also to aware about its disadvantages. The working rule of numerical methods is to find the approximate solutions of any mathematical problems on a defined interval either in time or distance. After the hard work has been done by ancient Egyptian the Newton's and Leibniz developed calculus which led the foundation to solve the mathematical problems.

Inspired and motivated by the research activities in this dynamic and fascinating fields, we use the homotopy perturbation technique, which was introduced and studied by He [8, 9]. Noor [16, 17, 18, 19, 20, 21] and Noor et al.[22] modified the homotopy perturbation technique to suggest a wide class of iterative methods for solving nonlinear equations and related optimization problems. Some new iterative methods are proposed for solving nonlinear equations. To illustrate the efficiency of the methods, some numerical examples are given. Comparisons with the other known methods shows that the new methods perform better than the previous known methods.

## 2 Main Results

The homotopy perturbation method was introduced by He, [8, 9] it has been used in many branches of mathematics for finding solutions. Abbasbandy [1] applied the homotopy perturbation technique to derive various iterative methods for solving nonlinear equations. We again use this technique to suggest and propose some new iterative methods for solving the nonlinear equations. To be more precise, we consider the problem of solving the type of nonlinear equation of the type

$$f(x) = 0, \tag{1}$$

which can be decomposed as:

$$L(x) = N(x) + c, \quad (2)$$

where  $L$  is a linear operator,  $N$  is a non-linear operator and  $c$  is an arbitrary constant. Using the technique of [8], we can define homotopy  $H(v, p) : R \times [0, 1] \rightarrow R$  as:

$$\begin{aligned} H(v, p) &= (1 - p)[L(v) - L(v_0)] + p[L(v) - N(v) - c] \\ &= L(v) - pN(v) - c = 0, \quad p \in [0, 1]. \end{aligned} \quad (3)$$

Note that for  $p = 0, p = 1$  and from equation (3) we have,

$$\begin{aligned} H(v, 0) &= L(v) - L(v_0) = v - c = 0 \\ H(v, 1) &= [L(v) - N(v)] - c = 0. \end{aligned}$$

Here  $p \in [0, 1]$  is the embedding parameter which is increasing monotonically from 0 to 1. This implies that  $H(v, 0) = L(v) - c = v - c = 0$  is continuously deformed to basic problem as:  $H(v, 1) = L(v) - N(v) - c = v - N(v) - c = 0$ .

Let the solution of (1) can be expressed as

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (4)$$

The approximate solution of the nonlinear equation (1) is obtained as:

$$V = \lim_{p \rightarrow 1} v = v = v_0 + v_1 + v_2 + \dots \quad (5)$$

We can write equation (3) by using (5) and expanding Taylor series  $N(v)$  around  $v_0$  as

$$\begin{aligned} v_0 + pv_1 + p^2v_2 + \dots &= p^0c + p[N(v_0) + (v - v_0)N'(v_0) + \dots] \\ &= p^0c + p[N(v_0) + p(v_1 + pv_2 + \dots)N'(v_0) + \dots] \\ &= p^0c + pN(v_0) + p^2v_1N'(v_0). \end{aligned}$$

By comparing the coefficient of  $p$ , we have

$$p^0 : v_0 = c \tag{6}$$

$$p^1 : v_1 = N(v_0) \tag{7}$$

$$p^2 : v_2 = v_1 N'(v_0) \tag{8}$$

⋮

Consequently, we have

$$v_0 + v_1 + v_2 + \dots = c + N(v_0) + v_1 N'(v_0) + \dots \tag{9}$$

We use the equation (9) to obtain the main algorithm for the solution of nonlinear equation (1). First of all, we consider the equations

$$e^{\alpha x} f(x) = 0, \quad \alpha \in R. \tag{10}$$

Using the Taylor series, we obtain

$$\begin{aligned} 0 &= e^{\alpha x} f(x) = e^{\alpha \gamma} f(\gamma) + (x - \gamma)e^{\alpha \gamma} [f'(\gamma) + \alpha f(\gamma)] \\ &\quad + \frac{1}{2!}(x - \gamma)^2 e^{\alpha \gamma} [f''(\gamma) + 2\alpha(f'(\gamma)) + \alpha^2 f(\gamma)], \end{aligned} \tag{11}$$

which implies that

$$0 = f(\gamma) + (x - \gamma) [f'(\gamma) + \alpha f(\gamma)] + \frac{1}{2!}(x - \gamma)^2 [f''(\gamma) + 2\alpha(f'(\gamma) + \alpha^2 f(\gamma))] + \dots \tag{12}$$

Solving equation (12) for  $x$ , we have

$$\begin{aligned} x &= \gamma - \frac{f(\gamma)}{f'(\gamma) + \alpha f(\gamma)} - \frac{1}{2!}(x - \gamma)^2 \frac{[f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)]}{[f'(\gamma) + \alpha f(\gamma)]} \\ &= c + N(x), \end{aligned} \tag{13}$$

where

$$c = \gamma - \frac{f(\gamma)}{[f'(\gamma) + \alpha f(\gamma)]}, \tag{14}$$

$$N(x) = -\frac{1}{2!}(x - \gamma)^2 \frac{[f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)]}{[f'(\gamma) + \alpha f(\gamma)]}. \tag{15}$$

Also

$$N'(x) = -(x - \gamma) \frac{[f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)]}{[f'(\gamma) + \alpha f(\gamma)]}. \quad (16)$$

From (9) and (14), we have

$$x_0 = c = \gamma - \frac{f(\gamma)}{[f'(\gamma) + \alpha f(\gamma)]}, \quad (17)$$

$$x_1 = N(v_0) = -\frac{1}{2!}(x_0 - \gamma)^2 \frac{[f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)]}{[f'(\gamma) + \alpha f(\gamma)]} \quad (18)$$

$$x_2 = v_1 N'(v_0) = v_1 \left\{ -(x_0 - \gamma) \frac{[f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)]}{[f'(\gamma) + \alpha f(\gamma)]} \right\} \quad (19)$$

⋮

From (17), it follows that approximate  $x \approx x_0$  such that

$$x = c = \gamma - \frac{f(\gamma)}{[f'(\gamma) + \alpha f(\gamma)]} \quad (20)$$

which help us to suggest the following iterative scheme.

**Algorithm I:** For a given initial value  $x_0$ , compute the approx value  $x_n + 1$  by the iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{[f'(x_n) + \alpha f(x_n)]} \quad n = 0, 1, 2, 3, \dots,$$

which is well known modified Newton's method and has quadratic convergence.

From (17) and (18), we obtain

$$x_1 = -\frac{1}{2!} \left[ \frac{f(\gamma)}{[f'(\gamma) + \alpha f(\gamma)]} \right]^2 \frac{[f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)]}{[f'(\gamma) + \alpha f(\gamma)]}. \quad (21)$$

One can find the approximate  $x$  as

$$\begin{aligned} x &\approx x_0 + x_1 = x_0 + N(x_0) \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma) + \alpha f(\gamma)} - \frac{1}{2!} \left[ \frac{f(\gamma)}{[f'(\gamma) + \alpha f(\gamma)]} \right]^2 \frac{[f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)]}{[f'(\gamma) + \alpha f(\gamma)]}. \end{aligned} \quad (22)$$

This enable us to suggest the following one step method.

**Algorithm II:** For a given  $x_0$ , compute the approx sol  $x_{n+1}$  by iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{[f'(x_n) + \alpha f(x_n)]} - \frac{1}{2!} \left[ \frac{f(x_n)}{[f'(x_n) + \alpha f(x_n)]} \right]^2 \left[ \frac{f''(x_n) + 2\alpha f'(x_n) + \alpha^2 f(x_n)}{f'(x_n) + \alpha f(x_n)} \right], \quad n = 0, 1, 2, \dots$$

From (18), (19) and (21), we have,

$$\begin{aligned} x_2 &= x_1 N'(x_0) = N(x_0) N'(x_0) \\ &= \frac{1}{2!} (x_0 - \gamma)^3 \left( \frac{f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)}{f'(\gamma) + \alpha f(\gamma)} \right)^2. \end{aligned}$$

Consequently, we have

$$\begin{aligned} x &\approx x_0 + x_1 + x_2 = x_0 + N(x_0) + x_1 N'(0) \\ &= \gamma - \frac{f(\gamma)}{[f'(\gamma) + \alpha f(\gamma)]} - \frac{1}{2!} (x_0 - \gamma)^2 \frac{[f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)]}{[f'(\gamma) + \alpha f(\gamma)]} + \\ &\quad \frac{1}{2!} (x_0 - \gamma)^3 \left[ \frac{f''(\gamma) + 2\alpha f'(\gamma) + \alpha^2 f(\gamma)}{f'(\gamma) + \alpha f(\gamma)} \right]^2. \end{aligned}$$

This formulation help us to construct a following algorithms.

**Algorithm III:** The following iterative method is used to obtain  $x_{n+1}$  as  $x_0$  is provided

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{[f'(x_n) + \alpha f(x_n)]} - \frac{1}{2!} (x_{n+1} - \gamma)^2 \frac{[f''(x_n) + 2\alpha f'(x_n) + \alpha^2 f(x_n)]}{[f'(x_n) + \alpha f(x_n)]} \\ &\quad + \frac{1}{2!} (x_{n+1} - \gamma)^3 \left[ \frac{f''(x_n) + 2\alpha f'(x_n) + \alpha^2 f(x_n)}{f'(x_n) + \alpha f(x_n)} \right]^2, \end{aligned} \tag{23}$$

which is an implicit method.

To implement the implicit method, we use the predictor-corrector technique. Using modified Newton method as predictor and Algorithm III as corrector method, we can suggest the following two-step method.

**Algorithm IV:** For a given initial value  $x_0$ , compute the approximate value  $x_{n+1}$  by the iterative scheme

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n) + \alpha f(x_n)} \\ x_{n+1} &= y_n - \frac{1}{2!}(x - \gamma)^2 \left( \frac{f''(x_n) + 2\alpha f'(x_n) + \alpha^2 f(x_n)}{f'(x_n) + \alpha f(x_n)} \right) \\ &\quad + \frac{1}{2!}(y_n - \gamma)^3 \left( \frac{f''(x_n) + 2\alpha f'(x_n) + \alpha^2 f(x_n)}{f'(x_n) + \alpha f(x_n)} \right)^2, \end{aligned}$$

which is equivalent to the following method.

**Algorithm V:** For a given initial value  $x_0$ , compute the approximate solution  $x_{n+1}$  by the iterative scheme:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n) + \alpha f(x_n)} \\ &\quad - \frac{1}{2!} \left( \frac{f(x_n)}{f'(x_n) + \alpha f(x_n)} \right)^2 \left( \frac{f''(x_n) + 2\alpha f'(x_n) + \alpha^2 f(x_n)}{f'(x_n) + \alpha f(x_n)} \right) \\ &\quad - \frac{1}{2!} \left( \frac{f(x_n)}{f'(x_n) + \alpha f(x_n)} \right)^3 \left( \frac{f''(x_n) + 2\alpha f'(x_n) + \alpha^2 f(x_n)}{f'(x_n) + \alpha f(x_n)} \right)^2, \end{aligned}$$

which is an explicit method and appears to be new one.

### 3 Numerical Illustration

In this section, we check some numerical examples to illustrate the efficiency of our suggested new developed algorithm for solving nonlinear equation. Numerical result are compared with other well known method like Newton method (NM), Householder method (HM) and Halley method. The stopping criteria followed by  $|x_{n+1} - x_n| < 10^{-15}$ .

**Example 1:** Consider the nonlinear equation  $f(x) = x^2 - e^x - 3x + 2 = 0$ ,  $x_0 = 0.5$ .



Table 1: Comparison of NM, HM with Algorithm II and Algorithm III.

Method			N	$vx_n$	$f(x_n)$
NM			5	0.257530285439861	0
HM			4	0.257530285439861	0
HHM			4	0.257530285439861	0
4*Algorithm II	4* $\alpha$	-1	5	0.257530285439861	0
		-0.5	4	0.257530285439861	0
		-0.25	4	0.257530285439861	0
		0	4	0.257530285439861	0
4*Algorithm III	4* $\alpha$	-1	4	0.257530285439861	0
		-0.5	4	0.257530285439861	$4.440892098500626e^{-016}$
		-0.25	4	0.257530285439861	0
		0	4	0.257530285439861	0

**Example 2:** Consider  $f(x) = e^{x^2+7x-30} - 1$ ,  $x_0 = 3.5$ .

Table 2: Comparison of NM, HH, with algorithm III and Algorithm IV.

Method			Iteration	$f_n$	$f(x_n)$
NM			13	3	$3.289812866569264e-012$
HM			7	3	$7.105427357601002e-015$
HHM			9	3	0
4*Algorithm III	4* $\alpha$	-1	8	3	$1.421085471520200e-014$
		-0.5	8	3	$8.935074902183260e-012$
		-0.25	8	3	$1.444497854663496e-010$
		0	9	3	0
4*Algorithm IV	4* $\alpha$	-1	7	3	0
		-0.5	7	3	$4.440892098500626e^{-016}$
		-0.25	7	3	0
		0	7	3	$7.105427357601002e-015$

**Example 3:** Consider  $f(x) = \sin^2 x - x^2 + 1$ ,  $x_0 = 1$ .

Table 3: Comparison of NM, HM, with new Alorithm III.

Method			Iteration	$x_n$	$f(x_n)$
NM			6	1.404491648215341	-7.593925488436071e-013
HM			4	1.404491648215341	6.667999485898690e-013
HHM			6	1.404491648215341	-4.440892098500626e-016
4*Algorithm III	4* $\alpha$	-1	4	1.404491648215341	-4.440892098500626e-016
		-0.5	4	1.404491648215341	2.764455331316640e-014
		-0.25	5	1.404491648215341	3.330669073875470e-016
		0	6	1.404491648215341	-4.440892098500626e-016
4*Algorithm IV	4* $\alpha$	-1	4	1.404491648215341	-4.440892098500626e-016
		-0.5	4	1.404491648215341	3.330669073875470e-016
		-0.25	4	1.404491648215341	1.221245327087672e-015
		0	4	1.404491648215341	6.667999485898690e-013

Table 3 shows that new Algorithms are more efficient than Newton method.

**Example 4:** Consider  $f(x) = x^3 - 2x^2 + 2$ ,  $x_0 = 0$ .

Table 4: Comparison of NM, HM with Algorithms III.

Method			Iteration	$x_n$	$f(x_n)$
NM			-	fails	-
HM			-	fails	-
HHM			-	fails	-
5*Algorithm III	5* $\alpha$	1	5	-0.83928675521416	0
		0	fail	-	-
		-0.25	22	-0.839286755214161	0
		-0.5	10	-0.839286755214161	2.22044604925031e-16
		-1	18	-0.839286755214161	2.22044604925031e-16
5*Algorithm II	5* $\alpha$	1	5	-0.839286755214161	0
		0.5	5	-0.839286755214161	0
		0.25	6	-0.839286755214161	0
		0	fail	-	-
		-0.5	-0.5	-0.839286755214161	2.220446049250313e-16

These numerical results indicate that the suggested algorithms perform better than the Newton method and have the same defines as Householder and Halley method. In the last example shows that the newly developed algorithms give approximate solution of given nonlinear equation, where Newton, Householder and Halley method fail to find approximate.

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