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q-Power Quasi Binormal Operator

Alaa Hussein Mohammed

Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya, Iraq
e-mail: alaa.hussein@qu.edu.iq

Abstract

In this paper we introduce a new class of operators on Hilbert space called q-power quasi binormal operator. We study this operator and give some properties of it.

Introduction

Consider $B(H)$ be the algebra of all bounded linear operators on Hilbert space H. An operator S is called *normal* if $S^*S = SS^*$. Quasi normal operator was introduced by Brown in 1953 [1]. In [3] Campbell introduced the class binormal of operator which is defined as $S^*SSS^* = SS^*S^*S$.

In [5] Sid Ahmed generalize quasi normal operator to n-power quasi normal operator. In this paper we define a new class of operators on Hilbert space as $S^q(S^*SSS^*) = (SS^*S^*S)S^q$ called *q-power quasi binormal operator* and study some properties of it.

1. Main Results

Definition 1.1. Let S be bounded operator. Then S is called *q-power quasi binormal operator* if and only if $S^q(S^*SSS^*) = (SS^*S^*S)S^q$, where q is a nonnegative integer.

Proposition 1.2. *If S is a self adjoint and q-power quasi binormal operator, then S^* is a q-power quasi binormal operator.*

Proof. Since S is q-power quasi binormal operator, $S^q(S^*SSS^*) = (SS^*S^*S)S^q$.

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Let

$$(S^*)^q[((S^*)^* S^* S^*(S^*)^*)] = (S^*)^q(SS^*S^*S), \text{ since } S \text{ is a self adjoint}$$
$$= S^q(S^*SSS^*), \text{ since } S \text{ is } q\text{-power quasi binormal}$$
$$= (SS^*S^*S)S^q, \text{ since } S \text{ is a self adjoint}$$
$$= (S^*(S^*)^*(S^*)^*S^*)(S^*)^q.$$

Hence, S^* is q-power quasi binormal operator.

Proposition 1.3. *If S is a q-power quasi binormal operator, and if* S^{-1} *exist, then* S^{-1} *is a q-power quasi binormal operator.*

Proof. Since S is q-power quasi binormal operator, $S^q(S^*SSS^*) = (SS^*S^*S)S^q$.

Let

$$(S^{-1})^q[(S^{-1})^*S^{-1}S^{-1}(S^{-1})^*)] = (S^q)^{-1}[(S^*)^{-1}S^{-1}S^{-1}(S^*)^{-1})]$$
$$= (S^q)^{-1}[(SS^*)^{-1}(S^*S)^{-1})]$$
$$= (S^q)^{-1}[(S^*S)(SS^*)]^{-1}$$
$$= [[(S^*SSS^*)]S^q]^{-1}, \text{ since } S \text{ is binormal}$$
$$= [[(SS^*S^*S)]S^q]^{-1},$$

since S is a q-power quasi binormal,

$$= [S^q(S^*SSS^*)]^{-1}, \text{ since } S \text{ is binormal}$$
$$= [S^q(SS^*S^*S)]^{-1}$$
$$= [(SS^*S^*S)]^{-1}(S^q)^{-1}$$
$$= [(S^*S)^{-1}(SS^*)^{-1}](S^q)^{-1}$$
$$= [S^{-1}(S^*)^{-1}(S^*)^{-1}S^{-1}](S^q)^{-1}$$
$$= [S^{-1}(S^{-1})^*(S^{-1})^*S^{-1}](S^{-1})^q.$$

Hence, S^{-1} is q-power quasi binormal operator.

Definition 1.4 [4]**.** If A, B are bounded operator on Hilbert space H. Then A, B are *unitary equivalent* if there is an isomorphism $U: H \to H$ such that $B = UAU^*$.

Proposition 1.5. *If S is q-power quasi binormal operator and if $R \in B(H)$ is unitary equivalent to S, then R is q-power quasi binormal operator.*

Proof. Since R is unitary equivalent to S, $R = USU^*$, $(USU^*)^n = US^nU^*$ and since S is q-power quasi binormal operator, $S^q(S^*SSS^*) = (SS^*S^*S)S^q$.

Let

$$\begin{aligned} R^q(R^*RRR^*) &= (USU^*)^q[(USU^*)^*(USU^*)(USU^*)(USU^*)^*] \\ &= (US^qU^*) \;\; [(US^*U^*) \;\; (USU^*)(USU^*)(US^*U^*) \;\;] \\ &= U\,[S^q(S^*SSS^*)]U^*, \text{ since } S \text{ is } q\text{-power quasi binormal operator} \\ &= U[(SS^*S^*S)S^q]U^* \\ &= \left[(USU^*)(US^*U^*)(US^*U^*)(USU^*) \;\; \right](US^qU^*) = (RR^*R^*R)R^q. \end{aligned}$$

Hence R is q-power quasi binormal operator.

Theorem 1.6. *The set of all q-power quasi binormal operators on H is a closed subset of $B(H)$ under scalar multiplication.*

Proof. Let

$$M(H) = \{S \in B(H): S \text{ is } q\text{-power quasi binormal operator on } H \text{ for some nonnegative integer } q\}$$

Let $S \in W(H)$, then we have S is q-power quasi binormal operator and thus $S^q(S^*SSS^*) = (SS^*S^*S)S^q$.

Let θ be a scalar, hence

$$\begin{aligned} (\theta S)^q[(\theta S)^*(\theta S)(\theta S)(\theta S)^*] &= \theta^q S^q[\bar{\theta}S^*(\theta S)(\theta S))\bar{\theta}S^*] \\ &= \theta^q\bar{\theta}\theta\theta\bar{\theta}[S^q(S^*SSS^*)] \\ &= \theta^q\bar{\theta}\theta\theta\bar{\theta}[\;\;(SS^*S^*S)S^q] \\ &= [(\theta S)(\theta S)^*(\theta S)^*(\theta S)](\theta S)^q. \end{aligned}$$

Thus $\theta S \in M(H)$.

Let S_k be a sequence in $M(H)$ and converge to S, then we can get that

$$\|S^q(S^*SSS^*) - (SS^*S^*S)S^q\|$$

$$= \left\|S^q(S^*SSS^*) - S_k^q(S_k^*S_kS_kS_k^*) + (S_kS_k^*S_k^*S_k)S_k^q - (SS^*S^*S)S^q\right\|$$

$$\leq \left\|S^q(S^*SSS^*) - S_k^q(S_k^*S_kS_kS_k^*)\right\| + \left\|(S_kS_k^*S_k^*S_k)S_k^q - (SS^*S^*S)S^q\right\| \to 0 \text{ as } k \to \infty.$$

Hence, $S^q(S^*SSS^*) = (SS^*S^*S)S^q$.

Therefore $S \in M(H)$. Then, $M(H)$ is closed subset.

Theorem 1.7. *If T and S are normal, q-power quasi binormal operators on H, and let S commute with T, then (ST) is q-power quasi binormal operator on H.*

Proof. Since T and S are q-power quasi binormal operators $S^q(S^*SSS^*) = (SS^*S^*S)S^q$ and $T^q(T^*TTT^*) = (TT^*T^*T)T^q$,

$$\begin{aligned}
(ST)^q((ST)^*(ST)(ST)(ST)^*) &= (T^qS^q)[(T^*S^*)(ST)(ST)(T^*S^*)] \\
&= (T^qS^q)[T^*S^*ST\ ST\ T^*S^*] \\
&= (T^qS^q)[T^*SS^*T\ S\ T^*TS^*] \\
&= (T^qS^q)[ST^*S^*T\ S\ T^*S^*T] \\
&= (T^qS^q)[ST^*TS^*\ T^*SS^*T] \\
&= (T^qS^q)[STT^*S^*\ T^*S^*ST] \\
&= T^qSS^qTT^*S^*\ T^*S^*ST \\
&= ST^qS^qTT^*S^*\ T^*S^*ST \\
&= ST^qTS^qT^*S^*\ T^*S^*ST \\
&= STT^qS^qT^*S^*\ T^*S^*ST \\
&\ \vdots \\
&= [(ST)(T^*S^*)(T^*S^*)(ST)](T^qS^q) \\
&= ((ST)(ST)^*(ST)^*(ST))(ST)^q.
\end{aligned}$$

Then (ST) is q-power quasi binormal operator.

Theorem 1.8. *Let $T_1, T_2, \dots, T_k$ are q-power quasi binormal operators on H. Then the direct sum $(T_1 \oplus T_2 \oplus \dots \oplus T_k)$ is q-power quasi binormal operator on H.*

Proof. Since every operator of $T_1, T_2, \dots, T_k$ is q-power quasi binormal,

$$T_i^q(T_i^*T_iT_iS^*)$$
$$= (T_iT_i^*T_i^*T_i)T_i^q \text{ for all } i = 1,2,\dots,k$$
$$= (T_1 \oplus T_2 \oplus \dots \oplus T_k)^q[(T_1 \oplus T_2 \oplus \dots \oplus T_k)^*(T_1 \oplus T_2 \oplus \dots \oplus T_k)$$
$$\times (T_1 \oplus T_2 \oplus \dots \oplus T_k)(T_1 \oplus T_2 \oplus \dots \oplus T_k)^*]$$
$$= (T_1{}^q \oplus T_2{}^q \oplus \dots \oplus T_k{}^q)[(T_1{}^* \oplus T_2{}^* \oplus \dots \oplus T_k{}^*)(T_1 \oplus T_2 \oplus \dots \oplus T_k)$$
$$\times (T_1 \oplus T_2 \oplus \dots \oplus T_k)(T_1{}^* \oplus T_2{}^* \oplus \dots \oplus T_k{}^*)]$$
$$= T_1{}^q(T_1{}^*T_1T_1T_1{}^*) \oplus T_2{}^q(T_2{}^*T_2T_2T_2{}^*) \oplus \dots \oplus T_k{}^q(T_k{}^*T_kT_kT_k{}^*)$$
$$= (T_1T_1{}^*T_1{}^*T_1)T_1{}^q \oplus (T_2T_2{}^*T_2{}^*T_2)T_2{}^q \oplus \dots \oplus (T_kT_k{}^*T_k{}^*T_k)T_k{}^q$$
$$= [(T_1 \oplus T_2 \oplus \dots \oplus T_k)(T_1 \oplus T_2 \oplus \dots \oplus T_k)^*(T_1 \oplus T_2 \oplus \dots \oplus T_k)^*(T_1 \oplus T_2 \oplus \dots \oplus T_k) \quad]$$
$$\times (T_1 \oplus T_2 \oplus \dots \oplus T_k)^q$$

Thus, $(T_1 \oplus T_2 \oplus \dots \oplus T_k)$ is q-power quasi binormal operator on H.

Theorem 1.9. *Let $T_1, T_2, \dots, T_k$ are q-power quasi binormal operators on H. Then the tenser product ($T_1 \otimes T_2 \otimes \dots \otimes T_k$) is q-power quasi binormal operator on H.*

Proof. Since every operator of $S_1, S_2, \dots, S_k$ is a q-power quasi binormal,

$$T_i^q(T_i^*T_iT_iS^*)$$
$$= (T_iT_i^*T_i^*T_i)T_i^q \text{ for all } i = 1,2,\dots,k$$
$$= (T_1 \otimes T_2 \otimes \dots \otimes T_k)^q[(T_1 \otimes T_2 \otimes \dots \otimes T_k)^*(T_1 \otimes T_2 \otimes \dots \otimes T_k)$$
$$\times (T_1 \otimes T_2 \otimes \dots \otimes T_k)(T_1 \otimes T_2 \otimes \dots \otimes T_k)^*](\boldsymbol{x_1} \otimes \boldsymbol{x_2} \otimes \dots \otimes \boldsymbol{x}_{\mathbf{k}})$$
$$= (T_1{}^q \otimes T_2{}^q \otimes \dots \otimes T_k{}^q)[(T_1{}^* \otimes T_2{}^* \otimes \dots \otimes T_k{}^*)(T_1 \otimes T_2 \otimes \dots \otimes T_k)$$
$$\times (T_1 \otimes T_2 \otimes \dots \otimes T_k)(T_1{}^* \otimes T_2{}^* \otimes \dots \otimes T_k{}^*)](\boldsymbol{x_1} \otimes \boldsymbol{x_2} \otimes \dots \otimes \boldsymbol{x}_{\mathbf{k}})$$
$$= T_1{}^q(T_1{}^*T_1T_1T_1{}^*)\boldsymbol{x_1} \otimes T_2{}^q(T_2{}^*T_2T_2T_2{}^*)\boldsymbol{x_2} \otimes \dots \otimes \; T_k{}^q(T_k{}^*T_kT_kT_k{}^*)\boldsymbol{x}_{\mathbf{k}}$$
$$= [T_1{}^q(T_1{}^*T_1T_1T_1{}^*)]\boldsymbol{x_1} \otimes [(T_2T_2{}^*T_2{}^*T_2)T_2{}^q]\boldsymbol{x_2} \otimes \dots \otimes \; [(T_kT_k{}^*T_k{}^*T_k)T_k{}^q]\boldsymbol{x}_{\mathbf{k}}$$
$$= [(T_1 \otimes T_2 \otimes \dots \otimes T_k)(T_1 \otimes T_2 \otimes \dots \otimes T_k)^*(T_1 \otimes T_2 \otimes \dots \otimes T_k)^*(T_1 \otimes T_2 \otimes \dots \otimes T_k) \quad]$$
$$\times (T_1 \otimes T_2 \otimes \dots \otimes T_k)^q(\boldsymbol{x_1} \otimes \boldsymbol{x_2} \otimes \dots \otimes \boldsymbol{x}_{\mathbf{k}}).$$

Thus $(T_1 \otimes T_2 \otimes \dots \otimes T_k)$ is q-power quasi binormal operator.

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