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Construction of Lyapunov Functions for the Stability of Fifth Order Nonlinear Ordinary Differential Equations

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Abstract

This study employed Lyapunov function method to examine the stability of nonlinear ordinary differential equations. Using direct Lyapunov method, we constructed Lyapunov function to investigate the stability of fifth order nonlinear ordinary differential equations. $V(x)$, a quadratic form and positive definite and $U(x)$ which is also positive definite was chosen such that the derivative of $V(x)$ with respect to time would be equal to the negative value of $U(x)$. We adopted the pre-multiplication of the given equation by $\ddddot{x}$ and obtained a Lyapunov function which established local and global stability of a fifth order differential equation.

1. Introduction

Lyapunov functions are useful tools in determining stability, asymptotic stability, uniform stability, global stability or out-right instability of differential system and boundedness of solution of a real scalar fourth-order differential equation [1-4]. Asymptotic stability is intimately linked to the existence of a Lyapunov's function, that is, a proper, non-negative function varnishing only on an invariant set and decreasing along those curved paths of the system not evolving in the invariant set. Lyapunov

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theorem allows stability of linear and nonlinear system to be verified without differential equations solution being required. The presence of Lyapunov function implies asymptotic stability for linear time-invariant systems [9].

The concept of stability in problems arising from theory and application of differential equations is very important and an effective approach is the second approach of Lyapunov [7]. The method of Lyapunov functions was introduced by Aleksandra M. Lyapunov, a Russian Mathematician. The fundamental of his proof was centred on the established fact that the sum of energy in a system is decreasing or constant as it approaches state of equilibrium. Lyapunov functions have been constructed for linear equations on the platform that given any V that is definite positive, we have another definite positive function$U$ such that $-U = V^*$ and for the nonlinear case, a correlation is taken between the constant coefficient equations of linear and nonlinear equations which leads to the appropriate Lyapunov functions for the nonlinear case [1], [5], [6], [8].

$$x^* = Ex, \tag{1.1}$$

$x \in \Re^n$, E is a constant matrix and that E possessed eigen values whose real part is negative. Accordingly, given any quadratic positive definite form $U(x)$, we have another quadratic definite positive form $V(x)$ such that

$$-U = V^* \tag{1.2}$$

along the solution paths of (1.1). This result in (1.2) has been extended to hold for positive semi definite quadratic $U(x)$ as well. As a matter of fact, our basis for construction of Lyapunov function in this work would ultimately satisfy equation (1.2). We consider V to be definite positive function, then V must eventually decreases, and approaches zero. That is, for a system that is stable, all curved paths must move so that the values of V are diminishing.

2. Statement of Problems, Preliminaries and Definitions

Consider the fifth-order differential equation (2.1)

$$\overset{\cdot\cdot\cdot\cdot\cdot}{x} + a\overset{\cdot\cdot\cdot\cdot}{x} + b\dddot{x} + c\ddot{x} + d\dot{x} + e = 0 \tag{2.1}$$

where a, b, c, d and e are constants with

$$a > 0,\ b > 0,\ c > 0,\ d > 0 \text{ and } e > 0. \tag{2.2}$$

The equation (2.1) is equivalent to the following five systems.

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w \\ \dot{w} &= r \\ \dot{r} &= -ar - bw - cz - d - ex \end{aligned}\right\}. \tag{2.3}$$

The system (2.3) has negative real parts if and only if $a > 0$, $b > 0$, $c > 0$, $d > 0$ and $e > 0$. There is therefore need to have a positive definite continuous quadratic function V and another positive quadratic form U such that

$$\dot{V} = -U \tag{2.4}$$

along the solution paths of (2.1) or (2.3). Before now, the result in equation (2.4) has been extended and is established to hold for positive semi definite quadratic $U(x)$ as well. It is our interest therefore to construct a Lyapunov function that would ultimately satisfy equation (2.4).

Definition.

(i) A continuous function $V(x,\ t) = V(x_1,\ x_2,\ \ldots,\ x_n,\ t)$ is *positive definite* if $\lim_{|x|\to 0} V(x,t) = 0$ and there exists $\phi(\|x\|)$ such that

$$V(x,\ t) > \phi(\|x\|). \tag{2.5}$$

(ii) A continuous function $V(x,\ t) = V(x_1,\ x_2,\ \ldots,\ x_n,\ t)$ is *positive semi-definite* if $\lim_{|x|\to 0} V(x,t) = 0$ and there exists $\phi(\|x\|)$ such that

$$V(x,\ t) \geq \phi(\|x\|). \tag{2.6}$$

(iii) A continuous function $V(x,\ t) = V(x_1,\ x_2,\ \ldots,\ x_n,\ t)$ is *negative definite* if $\lim_{|x|\to 0} V(x,t) = 0$ and there exists $\phi(\|x\|)$ such that

$$V(x,\ t) < -\phi(\|x\|). \tag{2.7}$$

(iv) A continuous function $V(x,\ t) = V(x_1,\ x_2,\ \ldots,\ x_n,\ t)$ is *negative semi-definite* if $\lim_{|x|\to 0} V(x,\ t) = 0$ and there exists $\phi(\|x\|)$ such that

$$V(x,\ t) \leq -\phi(\|x\|). \tag{2.8}$$

(v) A continuous function $V(x,\ t) = V(x_1,\ x_2,\ \ldots,\ x_n,\ t)$ is *indefinite* if it assumes both positive and negative values in any arbitrary neighbourhood of the origin in a domain D.

At a glance we have

$V(0) = 0,\ V(x) = 0$, for $x \neq 0$ (Positive semi-definite)

$V(0) = 0,\ V(x) > 0$, for $x \neq 0$ (Positive definite)

$V(0) = 0,\ V(x) \leq 0$, for $x \neq 0$ (Negative semi-definite)

$V(0) = 0,\ V(x) < 0$, for $x \neq 0$ (Negativedefinite)

$\|x\| \to \infty,\ V(x) \to \infty$ (Radially unbounded)

Given a set of nonlinear first order differential equations

$$\dot{x}_i = f_i(x_1,\ x_2,\ \ldots,\ x_n) \text{ for } i = 1,\ 2,\ \ldots,\ n, \tag{2.9}$$

where $x_i = x_i(t)$ for some t and $\dot{x}_i$ stands for the time derivative of x_i for $i = 1,\ 2,\ \ldots,\ n$. Whereas f_i are analytic functions such that $f_i(0,\ \ldots,\ 0) = 0$ for $i = 1,\ 2,\ \ldots,\ n$ so that the origin $x = 0$ is an equilibrum point.

Lyapunov Test Function: For a function, $V(x)$, where $x = (x_1,\ x_2,\ \ldots,\ x_n)$, if the following conditions are satisfied:

(i) $V(x)$ and $\frac{\partial V}{\partial x_i}$ are continuous, for all $x \in \Re^n$ and $i = 1, 2, ..., n,$ not necessarily at the origin.

(ii) $V(0) = 0$.

Then we say that $V(x)$ is a possible Lyapunov test function for system (2.9).

Theorem 2.1 (Lyapunov's Direct Method). *Using an appropriate Lyapunov test function, it may be possible to investigate the stability of an equilibrium point of the system of nonlinear differential equation* (2.9), *as explained by Parks* [10] *by examining the rate of change with respect to time of* $V(x)$ *calculated as the Lyapunov derivative*:

$$V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x). \tag{2.10}$$

We can then interpret stability from V as follows:

Stable: If $V(x) > 0$ for $x \neq 0$ and $\dot{V}(x) \leq 0$, then we say that $V(x)$ is positive definite and the origin $(0, 0)$ of the system of ordinary differential equations (2.9) is stable.

Asymptotically stable: If $V(x) > 0$ for $x \neq 0$ and $\dot{V}(x) < 0$, then we say that $V(x)$ is positive definite and the origin $(0,\ 0)$ of the system of ordinary differential equations (2.9) is asymptotically stable.

Unstable: If $\dot{V}(x) > 0$, then we say that $\dot{V}(x)$ is positive and the origin $(0,\ 0)$ of the system of ordinary differential equations (2.9) is unstable.

Theorem 2.2 (Lassale's Invariant Principle). *Assume that $V(x)$ is a Lyapunov function of* (2.9) *on a subset $G \subset \Re^n,\ n \geq 1$. Define $S = \{x \in \bar{G}: V(x) = 0\}$, where $\bar{G}$ is the closure of G. Let M be the maximal invariant subset S. Then for $t \geq 0$, every bounded trajectory of* (2.9) *that remains in G approaches the set M as $t \to +\infty$.*

3. Methodology and Discussions

The system under investigation is

$$\overset{\cdot\cdot\cdot\cdot\cdot}{x} + a\overset{\cdot\cdot\cdot\cdot}{x} + b\dddot{x} + c\ddot{x} + d\dot{x} + e = 0$$

$$\dot{X} = AX = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -e & -d & -c & -b & -a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \\ r \end{pmatrix}, \tag{3.1}$$

where $a, b, c, d, e\ >\ 0$ for the system to have a negative real path. Consider

$$2V = k_1x^2 + k_2y^2 + k_3z^2 + k_4w^2 + k_5r^2 + 2k_6xy + 2k_7xz + 2k_8xw$$
$$+2k_9xr + 2k_{10}yz + 2k_{11}yw + 2k_{12}yr + 2k_{13}zw + 2k_{14}zr + 2k_{15}wr. \tag{3.2}$$

There is need to obtain the derivative of (3.2) with respect to t

$$2\dot{V} = 2k_1x\dot{x} + 2k_2y\dot{y} + 2k_3z\dot{z} + 2k_4w\dot{w} + 2k_5r\dot{r} + 2k_6(y\dot{x} + x\dot{y})$$
$$+2k_7(z\dot{x} + x\dot{z}) + 2k_8(w\dot{x} + x\dot{w}) + 2k_9(r\dot{x} + x\dot{r}) + 2k_{10}(z\dot{x} + y\dot{z})$$
$$+2k_{11}(w\dot{y} + y\dot{w}) + 2k_{12}(r\dot{y} + y\dot{r}) + 2k_{13}(w\dot{z} + z\dot{w}) + 2k_{14}(r\dot{z} + z\dot{r})$$
$$+2k_{15}(r\dot{w} + w\dot{r}), \tag{3.3}$$

$$\dot{V} = k_1x\dot{x} + k_2y\dot{y} + k_3z\dot{z} + k_4w\dot{w} + k_5r\dot{r} + 2k_6(y\dot{x} + x\dot{y})$$
$$+k_7(z\dot{x} + x\dot{z}) + k_8(w\dot{x} + x\dot{w}) + k_9(r\dot{x} + x\dot{r}) + k_{10}(z\dot{y} + y\dot{z})$$
$$+k_{11}(w\dot{y} + y\dot{w}) + k_{12}(r\dot{y} + y\dot{r}) + k_{13}(w\dot{z} + z\dot{w}) + k_{14}(r\dot{z} + z\dot{r})$$
$$+k_{15}(r\dot{w} + w\dot{r}). \tag{3.4}$$

But from equation (2.3)

$$\begin{aligned}\dot{V} &= k_1xy + k_2yz + k_3zw + k_4wr + k_5r(-ar - bw - cz - dy - ex) + k_6(yy + xz) \\ &+k_7(zy + xw) + k_8(wy + xr) + k_9\big(ry + x(-ar - bw - cz - dy - ex)\big) \\ &+k_{10}(zz + yw) + k_{11}(wz + yr) + k_{12}\big(rz + y(-ar - bw - cz - dy - ex)\big) \\ &+k_{13}(ww + zr) + k_{14}\big(rw + z(-ar - bw - cz - dy - ex)\big) \\ &+k_{15}\big(rr + w(-ar - bw - cz - dy - ex)\big). \end{aligned} \tag{3.5}$$

Expanding (3.5), we have

$$\begin{aligned}\dot{V} &= k_1xy + k_2yz + k_3zw + k_4wr - k_5ar^2 - k_5brw - k_5crz - k_5dry - k_5erx \\ &+y^2k_6 + xzk_6 + xyk_7 + xwk_7 + wyk_8 + xrk_8 + ryk_9 \\ &-arxk_9 - bwxk_9 - czxk_9 - dxyk_9 - ex^2k_9 + z^2k_{10} + ywk_{10} \\ &+wzk_{11} + yrk_{11} + rzk_{12} - aryk_{12} - bwyk_{12} - cyzk_{12} - dy^2k_{12} \\ &-exyk_{12} + w^2k_{13} + zrk_{13} + rwk_{14} - arzk_{14} - bwzk_{14} - cz^2k_{14} \\ &-dyzk_{14} - exzk_{14} + r^2k_{15} - arwk_{15} - bw^2k_{15} - cwzk_{15} - dwyk_{15} \\ &-ewxk_{15} \end{aligned} \tag{3.6}$$

bringing the terms and respective coefficients of (3.6)

Terms	Coefficients
x^2	$-ek_9$
y^2	$k_6 = dk_{12}$
z^2	$k_{10} - ck_{14}$
w^2	$k_{13} - bk_{15}$
r^2	$k_{15} - ak_5$
xy	$k_1 - dk_9 - ek_{12}$
yz	$k_2 + k_7 - ck_{12} - dk_{14}$
zw	$k_3 + k_{11} - bk_{14} - ck_{15}$
wr	$k_4 - bk_5 + k_{14} - ak_{15}$
zr	$k_{12} - ck_5 + k_{13} - ak_{14}$
yr	$k_9 - dk_5 + k_{11} - ak_{12}$
xr	$k_8 - ek_5 - ak_9$
xz	$k - ck_9 - ek_{14}$
xw	$k_7 - bk_9 - ek_{15}$
yw	$k_8 + k_{10} - bk_{12} - dk_{15}$

(3.7)

Next is to determine $\dot{V}$ such that one of the following conditions hold:

$$\left.\begin{array}{ll} (i) & \dot{V} \leq -\alpha x^2 \\ (ii) & \dot{V} \leq -\alpha y^2 \\ (iii) & \dot{V} \leq -\alpha z^2 \\ (iv) & \dot{V} \leq -\alpha w^2 \\ (v) & \dot{V} \leq -\alpha r^2 \\ (vi) & \dot{V} \leq -\alpha(x^2 + y^2 + z^2 + w^2 + r^2) \end{array}\right\}. \tag{3.8}$$

For the realization of any of these cases, one is required to impose conditions by realization of any of these terms as follows:

$$k_9 = 0, \tag{3.9a}$$

$$ak_5 - k_{15} = 0, \tag{3.9b}$$

$$k_{13} - bk_{15} = 0, \tag{3.9c}$$

$$k_{10} - ck_{14} = 0, \tag{3.9d}$$

$$k_6 - dk_{12} > 0, \tag{3.9e}$$

$$k_{15} = ak_5, \quad \text{from equation (3.9b)}$$

$$k_{13} = bk_{15} \Rightarrow k_{13} = abk, \tag{3.10}$$

Since $k_9 = 0$ and from equation (3.7)

$$k_1 = ek_{12}, \tag{3.11a}$$

$$k_6 = ek_{14}, \tag{3.11b}$$

$$k_7 = ek_{15}, \tag{3.11c}$$

$$k_8 = ek_5, \tag{3.11d}$$

$$k_2 = dk_{14} - k_7 + c, \tag{3.11e}$$

$$k_{10} = dk_{15} - k_8 + b, \tag{3.11f}$$

$$k_{11} = dk_5 + a, \tag{3.11g}$$

$$k_{13} = ak_{14} + ck_5 - k_{12}, \tag{3.11h}$$

$$k_3 = ck_{15} + bk_{14} - k_{11}, \tag{3.11m}$$

$$k_4 = ak_{15} + bk_5 - k_{14}. \tag{3.11n}$$

Recall that one interest is on equation (3.9e)

$$k_6 - dk_{12} > 0.$$

In (3.11b), $k_6 = ek_{14}$, thus

$$ek_{14} - dk_{12} > 0. \tag{3.12}$$

In (3.11g),

$$k_{11} = dk_5 + ak_{12}.$$

In (3.11h),

$$k_{13} = ak_{14} + ck_5 - k_{12}.$$

But $k_{13} = abk_5$, thus

$$abk_5 = ak_{14} + ck_5 - k_{12},$$

$$k_{12} = ak_{14} + ck_1 - abk_5. \tag{3.13}$$

From (3.11f),

$$k_{10} = dk_{15} - k_8 + bk_{12}. \tag{3.14}$$

From (3.11d),

$$k_8 = ek_5.$$

From (3.9d),

$$k_{10} - ck_{14} = 0 \Rightarrow k_{10} = ck_{14},$$

$$k_{14} = \frac{k_{10}}{c}. \tag{3.15}$$

Substituting (3.11d) into (3.14), we have

$$k_{10} = dk_{15} - ek_5 + bk_{12}. \tag{3.16}$$

From (3.9b),

$$k_{15} = ak_5.$$

Substituting (3.9b) into (3.16), we have

$$k_{10} = adk_5 - ek_5 + bk_{12},$$

$$k_{14} = \frac{adk_5 - ek_5 + bk_{12}}{c}, \tag{3.17}$$

$$k_{12} = a\left(\frac{adk_5 - ek_5 + bk_{12}}{c}\right) + ck_5 - abk_5,$$

$$k_{12} = \frac{a^2dk_5 - aek_5 + abk_{12} + c^2k_5 - abck_5}{c},$$

$$k_{12} = \frac{[a^2a - ae + c^2 - abc]}{c - ab}k_5, \tag{3.18}$$

Let $\pi = \frac{[a^2a - ae + c^2 - abc]}{c - ab}$,

$$k_{12} = \pi k_5. \tag{3.19}$$

From (3.12)

$$ek_{14} - dk_{12} > 0.$$

From (3.1) and (3.9b)

$$e\left(\frac{adk_5 - ek_5 + b\pi k_5}{c}\right) - d\pi k_5 > 0,$$

$$adek_5 - e^2k_5 + be\pi k_5 - cd\pi k_5 > 0,$$

$$(ade - e^2 + be\pi - cd\pi)k_5 > 0,$$

$$\left.\begin{array}{l}
k_1 = ek_{12} = e\pi k_5 \\
k_2 = \frac{ad^2k_5}{c} - \frac{edk_5}{c} + \frac{bd\pi k_5}{c} - aek_5 + c\pi k_5 \\
k_3 = ack_5 + \frac{abdk_5}{c} - \frac{bek_5}{c} + \frac{b^2\pi k_5}{c} - dk_5 + a\pi k_5 \\
k_4 = a^2k_5 + bk_5 - \frac{adk_5}{c} + \frac{ek_5}{c} - \frac{b\pi k_5}{c} \\
k_7 = ek_{15} = aek_5 \\
k_8 = ek_5 \\
k_9 = 0 \\
k_{10} = adk_5 - ek_5 + b\pi k_5 \\
k_{11} = dk_5 + a\pi k_5 \\
k_{12} = \pi k_5 \\
k_{13} = abk_5 \\
k_{14} = \frac{1}{c}(adk_5 - ek_5 + b\pi)k_5 \\
k_{15} = ak_5
\end{array}\right\}. \tag{3.20}$$

Ploughing (3.20) back into equation (3.12) gives

$$2V = k_5 e\pi x^2 + k_5\left(\frac{ad}{c} - \frac{ed}{c} + \frac{bd\pi}{c} - ae + c\pi\right)y^2$$

$$+k_5\left(ac+\frac{abd}{c}-\frac{be}{c}+\frac{b^2\pi}{c}-d+a\pi\right)z^2$$

$$+k_5\left(a^2+b-\frac{ad}{c}+\frac{e}{c}-\frac{b\pi}{c}\right)w^2+r^2$$

$$+2k_5(ade-e^2+be\pi)\frac{1}{c}xy+2k_5aexz+2k_5exw$$

$$+2k_5(as-e+b\pi)yz+2k_5(d+a\pi)yw+2k_5\pi yr+2k_5abzw$$

$$+2k_5(ad-e+b\pi)\frac{1}{c}zr+2k_5awr. \quad (3.21)$$

By setting $k_5 = 1$ in (3.21) and dividing both sides by 2, we have

$$V=\frac{e}{2}\pi x^2+\frac{1}{2}\left(\frac{ad}{c}-\frac{ed}{c}+\frac{bd\pi}{c}-ae+c\pi\right)y^2$$

$$+\frac{1}{2}\left(ac+\frac{abd}{c}-\frac{be}{c}+\frac{b^2\pi}{c}-d+a\pi\right)z^2$$

$$+\frac{1}{2}\left(a^2+b-\frac{ad}{c}+\frac{e}{c}-\frac{b\pi}{c}\right)w^2+\frac{1}{2}r^2$$

$$+(ade-e^2+be\pi)\frac{1}{c}xy+aexz+exw$$

$$+(ad-e+b\pi)yz+(d+a\pi)yw+\pi yr+2k_5abzw$$

$$+(ad-e+b\pi)\frac{1}{c}zr+awr. \quad (3.22)$$

This is positive definite provided that $ad > e$ and $c > ab$ and if we so choose that $a = b = c = d = e > 1$. This is an indication of positive definiteness and the corresponding time derivative.

$$\dot{V}=-(ade-e^2+be\pi-cd\pi)y^2. \quad (3.23)$$

Since equation (3.23) satisfies V defined by equation (3.22) satisfies equation (2.4) if U is replaced by $(ade-e^2+be\pi-cd\pi)y^2$, then V define by (3.22) is a Lyapunov function for the fifth order system (2.1). The existence of Lyapunov function guarantee the stability of nonlinear ordinary differential equations and by Lasalle's theorem on stability of a system in (Theorem 2.2), the system is locally and globally asymptotically stable.

4. Conclusion

The study applied Lyapunov direct method to construct a Lyapunov function to investigate the stability of fifth order nonlinear differential equations. An appropriate quadratic form and positive definite $V(x)$ and also positive definite $U(x)$ was chosen such that the derivative of $V(x)$ with respect to time along the solution paths of the five scales system is equal to the negative $U(x)$, that is, $V = -U$. The existence of Lyapunov function for the fifth order nonlinear system guarantee local and global asymptotic stability of the system as corroborated by Lassale's Invariant theorem

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