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Semi-analytic Solution of the Nonlinear Advection Problem using Homotopy Perturbation Method

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Abstract

This paper focuses on finding the solution of some nonlinear partial differential equations with initial and boundary conditions. This is achieved using the homotopy perturbation method. The solutions obtained are said to be analytic approximate in nature. The applications basically are on inhomogeneous partial differential equations.

1. Introduction

Nonlinear physical processes play a pivotal role in field of science such as Physics, Mathematics and Engineering. Most life occurrences are nonlinear in nature. Linear problems are solved analytically but unfortunately it is quite difficult to solve nonlinear problems analytically. The analytic methods are fast developing, but still have some deficiencies [15]. Homotopy perturbation method was first introduced by He [13-14]. The method has been applied by many authors to solve linear and nonlinear problems [8-12]. The homotopy perturbation method is a combination of classical perturbation technique and the homotopy map used in topology [5]. This technique was used by [17] to obtain solution of system of Volterra integral equations. [20] investigated the solution of fourth order nonlinear parabolic equation using Homotopy Perturbation Method. The HPM is an efficient method for solving ordinary, partial and coupled differential equation. The HPM yields a very rapid convergence of the solution series in most cases, usually only in a few [21]. The aim of this paper is to extend the application of the He's HPM to advection equation in one or two dimensions. The advection equation governs

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the motion of conserved scalar field as it is transported by a notable velocity vector field. We shall illustrate the HPM introduction by [14].

2. Homotopy Perturbation Method

Consider a nonlinear differential equation

$$D(u) - f(r) = 0, \quad r \in \Omega \tag{2.1}$$

with boundary conditions

$$Bu, \frac{\partial u}{\partial n} = 0; \quad r \in \Gamma \tag{2.2}$$

where D is a differential operator, B is boundary operator, $f(r)$ is known analytic function, Γ is the boundary of the domain Ω. The operator D can be divided into linear part L and a nonlinear part N; Therefore Equation (2.2) is written as:

$$L(u) + N(u) - f(r) = 0. \tag{2.3}$$

He [2] constructs a homotopy of Equation (2.1), $v(r,\ p) = \Omega \times [0,\ 1] \to R$ which satisfies

$$H(v,\ p) = (1-p)[L(v) - L(v_0)] + p[D(v) - f(r)] = 0 \quad p \in [0,\ 1],\ r \in \Omega \tag{2.4}$$

which is equivalent to

$$H(v,\ p) = L(v) - L(v_0) + p[N(v) - f(r)] = 0, \tag{2.5}$$

where $p \in [0, 1]$ is an embedded parameter and v_0 is and initial guess approximation of Equation (2.1) which satisfies the boundary conditions. It follows from Equation (2.4) and (2.5) that

$$H(v,\ 0) = L(v) - L(v_0) = 0 \tag{2.6}$$

$$H(v,\ 1) = A(v) - f(r) = 0. \tag{2.7}$$

Hence, the changing process of p from zero to unity is just that of $v(r,\ p)$ from $v_0(r)$ to $v(r)$ in topology, this is called deformation and $L(v) - L(v_0)$ and $D(v) - f(r)$ are said to be homotopic. Assuming that the solution of Equation (2.3) can be written as a power series in p

$$v = v_0 + pv_1 + p^2 v_2 + \cdots$$

setting $p = 1$, the approximate solution of Equation (2.1) is obtained readily

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{2.8}$$

3. Applications

Example 1. Consider the nonhomogeneous advection problem in one-dimension

$$u_t - uu_x = u^2 + e^{t-x} \tag{3.1}$$

$$u(x,\ 0) = e^{-x}. \tag{3.2}$$

We construct a homotopy for the PDE as

$$H(u,\ p) = u_t - (u_0)_t - p[uu_x + u^2 + e^{t-x}]$$

substituting the value of the initial condition, we have

$$u_t - (e^{-x})_t = p[uu_x + u^2 + e^{t-x}]. \tag{3.3}$$

Then,

$$u_t = p[uu_x + u^2 + e^{t-x}] \tag{3.4}$$

let,

$$u = (p^0u_0 + p^1u_1 + p^2u_2 + \cdots) \tag{3.5}$$

it follows that,

$$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)$$

$$= p[(u_0 + u_1 + u_2 + \cdots)(u_0 + u_1 + u_2 + \cdots)_x + (u_0 + u_1 + u_2 + \cdots)^2 + e^{t-x}]. \tag{3.6}$$

Given that,

$$u(x,\ 0) = e^{-x}$$

$$(p^0u_0 + p^1u_1 + p^2u_2 + \cdots)(x,\ 0) = e^{-x} \tag{3.7}$$

so, $u_0 = e^{-x},\ u_1 = 0,\ u_2 = 0$ and so on.

To obtain u_0

$$p^0: (u_0)_t = 0 \tag{3.8}$$

$$u_0 = c(x), \quad u_0(x,\ 0) = e^{-x}$$

$$\Rightarrow c(x) = e^{-x}$$

$$u_0 = e^{-x} \tag{3.9}$$

To obtain u_1,

$$p^1: (u_1)_t - u_0(u_0)_x - u_0^2 - e^{t-x} = 0 \tag{3.10}$$

$$(u_1)_t = e^{-x}(e^{-x})_x + (e^{-x})^2 + e^{t-x} = 0 \tag{3.11}$$

$$(u_1)_t = -e^{-2x} + e^{-2x} + e^{t-x} \tag{3.12}$$

$$(u_1)_t = e^{t-x} \tag{3.13}$$

integrating with respect to t

$$u_1 = e^{t-x} + c(x) \tag{3.14}$$

but $u_1(x,\ 0) = 0$

$$u_1(x,\ 0) = e^{-x} + c(x) = 0 \tag{3.15}$$

$$\Rightarrow c(x) = -e^{-x}$$

$$u_1 = e^{t-x} - e^{-x}. \tag{3.16}$$

To obtain u_2

$$p^2: (u_2)_t - u_0(u_1)_x - u_1(u_0)_x - 2u_0u_1 = 0 \tag{3.17}$$

$$(u_2)_t = e^{-x}(e^{t-x} - e^{-x}) - (e^{t-x} - e^{-x})(e^{-x})_x - 2e^{-x}(e^{t-x} - e^{-x}) \tag{3.18}$$

$$(u_2)_t = 0$$

integrating

$$u_2 = c(x), \quad u_2(x,\ 0) = 0$$

$$\Rightarrow c(x) = 0$$

$$(u_2) = 0 \tag{3.19}$$

Likewise $u_3 = 0,\ u_4 = 0$ and so on.

The exact solution

$$u(x,\ t) = u_0 + u_1 + u_2 + \cdots \tag{3.20}$$

$$= e^{-x} + e^{t-x} - e^{-x} + 0 + \cdots$$

$$= e^{t-x}.$$

The required solution is

$$u(x,\ t) = e^{t-x}. \tag{3.21}$$

Example 2. Consider a nonlinear advection equation in two-dimension

$$u_{xx} - u_{yy} + uu_x - x \tag{3.22}$$

$$u(0,\ y) = cos\, y \tag{3.23}$$

$$u_x(x,\ 0) = 1. \tag{3.24}$$

By Homotopy technique we construct a homotopy for the PDE

$$H(u,\ p) = u_{xx} - (u_0)_{xx} - p\left[u_{yy} + uu_x - x\right] \tag{3.25}$$

substituting the initial condition, we have:

$$u_{xx} - (cos\, y)_x x = p\left[u_{yy} + uu_x - x\right]. \tag{3.26}$$

Then we have

$$u_{xx} = p\left[u_{yy} + uu_x - x\right]. \tag{3.27}$$

Let,

$$u = (p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots) \tag{3.28}$$

substituting Equation (3.28) into Equation (3.22) we have,

$$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)_{xx}$$

$$+p\left[(u_0 + u_1 + u_2 + \cdots)_{yy} + (u_0 + u_1 + u_2 + \cdots)(u_0 + u_1 + u_2 + \cdots)_x - x\right]. \tag{3.29}$$

Given that

$$u(0,\ y) = cos\, y$$

$$(p^0 u_0 + p^1 u_1 + p^2 u_2 + \cdots)(0,\ y) = cos\, y. \tag{3.30}$$

It then follows that,

$$u_0 = cos\, y; \quad u_1 = 0; \quad u_3 = 0.$$

To obtain u_0:

$$p^0: (u_0)_{xx} = 0 \tag{3.31}$$

$$(u_0)_x = c_1(x) \tag{3.32}$$

but

$$(u_0)_x(x,\ 0) = 1$$

from our initial condition

$$\Rightarrow c(x) = 1$$

$$(u_0)_x = 1 \tag{3.33}$$

$$u_0 = x + c_2(x) \tag{3.34}$$

$$u_0(0,\ y) = cos\, y \tag{3.35}$$

$$\Rightarrow c_2(x) = cos\, y$$

$$u_0 = x + cos\, y. \tag{3.36}$$

To obtain u_1

$$p^1: (u_1)_{xx} = (u_0)_{yy} + u_0(u_0)_x - x \tag{3.37}$$

$$(u_1)_{xx} = (x + cos\, y)_{yy} + (x + cos\, y)(x + cos\, y)_x - x \tag{3.38}$$

$$(u_1)_{xx} = -cos\, y + (x + cos\, y) - x = 0 \tag{3.39}$$

integrating

$$(u_1)_x = c_1(x); \quad (u_1)_x(x,\ 0) = 0$$

$$u_1 = 0. \tag{3.40}$$

To obtain u_2

$$p^2 = (u_2)_{xx} = (u_2)_{yy} + u_0(u_1)_x + u_1(u_0)_x \tag{3.41}$$

$$(u_2)_{xx} = (0)_{yy} + (x + cos\, y)(0)_x + 0(x + cos\, y) \tag{3.42}$$

$$u_2 = 0. \tag{3.43}$$

Likewise $u_3 = 0,\ u_4 = 0$and so on.

The exact solution is

$$u(x,\ y) = u_0 + u_1 + u_2 + u_3 + \cdots \tag{3.44}$$

$$= x + cos\, y + 0 + 0 + 0 = x + cos\, y$$

Our required solution is

$$u(x,\ y) = x + cos\, y. \tag{3.45}$$

Conclusion

In this work, we have successfully implemented the HPM for solving the advection equation with specific boundary and initial conditions. A clear conclusion can be drawn from our solution that the HPM provides highly accurate semi-analytical solutions for nonlinear equations. The obtained results show that HPM is simple and easy to implement.

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