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Certain Subclass of Meromorphic β-starlike Functions Associated with a Differential Operator

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Abstract

Sharp bounds for the Fekete-Szegö functional $|\nu_1 - \xi {\nu_0}^2|$ are derived for certain class of meromorphic starlike functions $\omega(z)$ of order β defined on the punctured open unit disk for which

$$1 - \frac{1}{t}\left(\frac{D^{n+1,m}_{\circ}\omega(z)}{D^{n,m}_{\circ}\omega(z)} - 1\right) \prec \chi(z) \quad (t \in \mathbb{C} - \{0\}, \eta \geq 0, \kappa > 0, n, m \in \mathbb{N}_0),$$

lie in a region starlike with respect to 1 and symmetric with respect to the real axis.

1. Introduction

Let Σ denote the class of meromorphic functions of the form

$$\omega(z) = \frac{1}{z} + \sum_{\ell=0}^{\infty} \nu_\ell z^\ell, \tag{1.1}$$

which are analytic in the open punctured unit disk $\mathfrak{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$.

A function $\omega \in \Sigma$ is meromorphic starlike of order β, denoted by $\Sigma^*\beta$, if

$$-Re\left\{\frac{z\omega'(z)}{\omega(z)}\right\} > \beta \quad (0 \leq \beta < 1; z \in \mathfrak{U}^*). \tag{1.2}$$

The class $\Sigma^*\beta$ was introduced and studied by many researchers, among whom Mogra et al. [5], Pommerenke [1] and Aouf [4].

Let $\chi(z)$ be an analytic function with positive real part on $\mathfrak{U}$ satisfies $\chi(0) = 1$ and $\chi'(0) > 0$, which maps $\mathfrak{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\Sigma^*(\chi)$ be the class of functions $\omega(z) \in \Sigma$ for which

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$$-\frac{z\omega'(z)}{\omega(z)} \prec \chi(z) \quad (z \in \mathfrak{U}^*). \tag{1.3}$$

In the class $S^*(\chi)$ and $\Sigma^*(\chi)$ that introduced and studied respectively by Ma and Minda [8], Silverman et al. [3], the authors have obtained the Fekete-Szegö inequality for the functions in these classes.

In this present paper we consider the operator defined by El-Ashwah [7]. For $\eta \geq 0,\ \kappa > 0,\ n, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the linear operator $D^{D^{n_\circ m}(\eta,\kappa):}\Sigma \to \Sigma$ was defined by

$$D^{n_\circ m}\omega(z) = \frac{1}{z} + \sum_{\ell=0}^{\infty}\left(\frac{\kappa + \eta(\ell+1)}{\kappa}\right)^m (\ell+2)^n \nu_\ell z^\ell,$$

where

$$D^m\omega(z) = \frac{1}{z} + \sum_{\ell=0}^{\infty}\left(\frac{\kappa + \eta(\ell+1)}{\kappa}\right)^m \nu_\ell z^\ell, \quad n = 0$$

$$D^{1_\circ m}\omega(z) = \frac{1}{z} + \sum_{\ell=0}^{\infty}\left(\frac{\kappa + \eta(\ell+1)}{\kappa}\right)^m (\ell+2)\nu_\ell z^\ell$$

$$= z(D^m\omega(z))' + 2D^m\omega(z)$$

$$D^{2_\circ m}\omega(z) = \frac{1}{z} + \sum_{\ell=0}^{\infty}\left(\frac{\kappa + \eta(\ell+1)}{\kappa}\right)^m (\ell+2)^2\nu_\ell z^\ell$$

$$= z^2(D^m\omega(z))'' + 5z(D^m\omega(z))' + 4D^m\omega(z).$$

It is note that

$$z(D^{n_\circ m}\omega(z))' = D^{n+1_\circ m}\omega(z) - 2D^{n_\circ m}\omega(z). \tag{1.4}$$

To define certain new subclass of meromorphic functions of complex order and obtain the Fekete-Szegö inequality for $\omega \in \sum_{t,\eta,\kappa}^{*n_\circ m}(\chi)$.

Definition 1. Let $\chi(z)$ be an analytic function with positive real part on $\mathfrak{U}^*$ with $\chi(0) = 1,\ \chi'(0) > 1$ which maps the unit disk $\mathfrak{U}$ onto a region of β-starlike with respect to 1 and is symmetric with respect to the real axis. Let $\sum_{t,\eta,\kappa}^{*n_\circ m}(\chi)$ be the class of functions $\omega(z) \in \Sigma$ for which,

$$1-\frac{1}{t}\left(\frac{D^{n+1,m}\omega(z)}{D^{n,m}\omega(z)}-1\right)\prec\chi(z)\ (t\in\mathbb{C}-\{0\},\eta\geq 0,\kappa>0,n,m\in\mathbb{N}_0 \text{ and } z\in\mathfrak{U}^*).\quad (1.5)$$

It is noted that $\sum^*_{1,1,0}(\chi)=\Sigma^*(\chi)$.

Lemma 1 [2]**.** *If* $\psi(z)=1+\mathfrak{s}_1z+\mathfrak{s}_2z^2+\mathfrak{s}_3z^3+\cdots$ *is a function with positive real part in* $\mathfrak{U}^*$*, then for any complex number* $\mathcal{C}$*,*

$$|\mathfrak{s}_1-d\mathfrak{s}_0{}^2|\leq 2\max\{1,|1-2d|\}.$$

Lemma 2 [6]**.** *If* $\psi_1(z)=1+\mathfrak{s}_1z+\mathfrak{s}_2z^2+\mathfrak{s}_3z^3+\cdots$ *is a function with positive real part in* $\mathfrak{U}^*$*, then for any complex number* d*,*

$$|\mathfrak{s}_1-d\mathfrak{s}_0{}^2|\leq\begin{cases}-4d+2 & if\ d\leq 0\\ \quad 2 & if\ 0\leq d\leq 1.\\ 4d-2 & if\ d\geq 1\end{cases}$$

When $d<0$ or $d>1$, the equality holds if and only if $\psi_1(z)=\frac{1+z}{1-z}$ or one of its rotations. If $0<d<1$, then the equality holds if and only if $\psi_1(z)=\frac{1+z^2}{1-z^2}$ or one of its rotations. If $d=0$, the equality holds if and only if

$$\psi_1(z)=\left(\frac{1}{2}+\frac{1}{2}\lambda\right)\left(\frac{1+z}{1-z}\right)+\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\left(\frac{1-z}{1+z}\right)\qquad(0\leq\lambda\leq 1\,),$$

or one of its rotations. If $d=1$, the equality holds if and only if

$$\frac{1}{\psi_1(z)}=\left(\frac{1}{2}+\frac{1}{2}\lambda\right)\left(\frac{1+z}{1-z}\right)+\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\left(\frac{1-z}{1+z}\right)\qquad(0\leq\lambda\leq 1\,),$$

or one of its rotations. Also the above upper bounds is sharp and it can be improved as follows when $0<d<1$:

$$|\mathfrak{s}_2-d\mathfrak{s}_1{}^2|+d|\mathfrak{s}_1|^2\leq 2\quad\left(0<d\leq\frac{1}{2}\right),$$

and

$$|\mathfrak{s}_2-d\mathfrak{s}_1{}^2|+(1-d)|\mathfrak{s}_1|^2\leq 2\quad\left(\frac{1}{2}<d<1\right).$$

2. Fekete-Szegö Problem

By making use the above lemmas, the following bounds for the class $\sum^{*n,m}_{t,\eta,\kappa}(\chi)$ are proved.

Theorem 1. *Let* $\chi(z) = 1 + \tau_1 z + \tau_2 z^2 + \tau_3 z^3 + \cdots$. *If* $\omega(z)$ *is given by* (1.1) *belonging to the class* $\sum_{t,\eta,\kappa}^{*n_\circ m}(\chi)$ *and* ρ *is a complex number, then*

(1) $|\nu_1 - \rho\nu_0{}^2| \leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{2^2 3^n}|\tau_1| \max\left\{1, \left|\frac{\tau_2}{\tau_1} - \left\{1 - \frac{2\rho t 3^n}{2^{2n}}\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\right\}\tau_1\right|\right\}$, $\tau_1 \neq 0$ (1.6)

(2) $|\nu_1 - \rho\nu_0{}^2| \leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{2^2 3^n}|\tau_2|$, $\tau_1 = 0$. (1.7)

Proof. If $\omega(z) \in \sum_{t,\eta,\kappa}^{*n_\circ m}(\chi)$, then there is a Schwarz function $\Theta(z)$ in $\mathfrak{U}$ with $\Theta(0) = 0$ and $|\Theta(z)| < 1$ and such that

$$1 - \frac{1}{t}\left(\frac{D^{n+1_\circ m}\omega(z)}{D^{n_\circ m}\omega(z)} - 1\right) = \chi(\Theta(z)). \tag{1.8}$$

Define the function

$$\psi_1(z) = \frac{1 + \Theta(z)}{1 - \Theta(z)}$$

that is

$$\psi_1(z) = \frac{1 + \Theta(z)}{1 - \Theta(z)} = 1 + \mathcal{L}_1 z + \mathcal{L}_2 z^2 + \mathcal{L}_3 z^3 + \cdots.$$

Since $\Theta(z)$ is Schwarz function, it is clear that $Re(\psi_1(z)) > 0$ and $\psi_1(0) = 1$, define

$$\psi_1(z) = 1 - \frac{1}{t}\left(\frac{D^{n+1_\circ m}\omega(z)}{D^{n_\circ m}\omega(z)} - 1\right) = 1 + d_1 z + d_2 z^2 + \cdots. \tag{1.9}$$

In view of (1.8) and (1.9), we have

$$\psi(z) = \chi\left(\frac{\psi_1(z)-1}{\psi_1(z)+1}\right) \tag{1.10}$$

since

$$\frac{\psi_1(z) - 1}{\psi_1(z) + 1} = \frac{1}{2}\left[\mathcal{L}_1 z + \left(\mathcal{L}_2 - \frac{\mathcal{L}_1{}^2}{2}\right)z^2 + (\mathcal{L}_3 - \mathcal{L}_1\mathcal{L}_2 + \frac{\mathcal{L}_1{}^3}{4})z^3 + \cdots\right].$$

By the function $\chi(z) = 1 + \tau_1 z + \tau_2 z^2 + \cdots$, therefore

$$\chi\left(\frac{\psi_1(z)-1}{\psi_1(z)+1}\right) = 1 + \frac{1}{2}\tau_1\mathcal{L}_1 z + \left[\frac{1}{2}\tau_1(\mathcal{L}_2 - \frac{1}{2}\mathcal{L}_1{}^2) + \frac{1}{4}\tau_2\mathcal{L}_1{}^2\right]z^2 + \cdots. \tag{1.11}$$

From (1.9) and (1.10), we obtain

$$1 + d_1 z + d_2 z^2 + \cdots = 1 + \frac{1}{2}\tau_1\mathcal{L}_1 z + \left[\frac{1}{2}\tau_1(\mathcal{L}_2 - \frac{1}{2}\mathcal{L}_1{}^2) + \frac{1}{4}\tau_2\mathcal{L}_1{}^2\right]z^2 + \cdots,$$

thus, we conclude that

$$d_1 = \tfrac{1}{2}\tau_1\mathcal{L}_1 \quad \text{and} \quad d_2 = \tfrac{1}{2}\tau_1(\mathcal{L}_2 - \tfrac{1}{2}\mathcal{L}_1{}^2) + \tfrac{1}{4}\tau_2\mathcal{L}_1{}^2.$$

From the other hand, since

$$1 - \frac{1}{t}\left(\frac{D^{n+1,m}\omega(z)}{D^{n,m}\omega(z)} - 1\right) = \frac{tD^{n,m}\omega(z) - z\left(D^{n,m}\omega(z)\right)' - D^{n,m}\omega(z)}{tD^{n,m}\omega(z)}$$

$$= 1 + \frac{2^n}{t}\left(\frac{\kappa+\eta}{\kappa}\right)^m \nu_0 z + \left(\frac{2^{2n}}{t}\left(\frac{\kappa+\eta}{\kappa}\right)^{2m} \nu_0{}^2 - 2\frac{3^n}{t}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\right)\nu_1 z^2 + \cdots.$$

So, we get

$$1 + d_1 z + d_2 z^2 + \cdots = 1 + \frac{2^n}{t}(t-1)\left(\frac{\kappa+\eta}{\kappa}\right)^m \nu_0 z$$

$$+\left(\tfrac{2^{2n}}{t}\left(\tfrac{\kappa+\eta}{\kappa}\right)^{2m} \nu_0{}^2 - 2\tfrac{3^n}{t}\left(\tfrac{\kappa+2\eta}{\kappa}\right)^m \nu_1\right) z^2 + \cdots.$$

By simple computation, we obtain that

$$d_1 = \tfrac{2^n}{t}\left(\tfrac{\kappa+\eta}{\kappa}\right)^m \nu_0 \quad \text{and} \quad d_2 = \left(\tfrac{2^{2n}}{t}\left(\tfrac{\kappa+\eta}{\kappa}\right)^{2m} \nu_0{}^2 - 2\tfrac{3^n}{t}\left(\tfrac{\kappa+2\eta}{\kappa}\right)^m \nu_1\right),$$

so

$$\tfrac{1}{2}\tau_1\mathcal{L}_1 = \tfrac{2^n}{t}\left(\tfrac{\kappa+\eta}{\kappa}\right)^m \nu_0 \ \text{ that is } \quad \nu_0 = \tfrac{t}{2^{n+1}}\left(\tfrac{\kappa}{\kappa+\eta}\right)^m \tau_1\mathcal{L}_1$$

and by substitute about ν_0 in d_2 and then about d_1, d_2, so we have

$$\frac{2^{2n}}{t}\left(\frac{\kappa+\eta}{\kappa}\right)^{2m} \nu_0{}^2 - 2\frac{3^n}{t}\left(\frac{\kappa+2\eta}{\kappa}\right)^m \nu_1 = \frac{1}{2}\tau_1(\mathcal{L}_2 - \frac{1}{2}\mathcal{L}_1{}^2) + \frac{1}{4}\tau_2\mathcal{L}_1{}^2.$$

That is

$$\nu_1 = \frac{-t}{23^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \left(\frac{1}{2}\tau_1(\mathcal{L}_2 - \frac{1}{2}\mathcal{L}_1{}^2) + \frac{1}{4}\tau_2\mathcal{L}_1{}^2 - \frac{2^{2n}}{t}\left(\frac{\kappa+\eta}{\kappa}\right)^{2m} \nu_0{}^2\right)$$

$$= \frac{-t}{43^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \tau_1\mathcal{L}_2 + \frac{t}{83^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \mathcal{L}_1{}^2(\tau_1 - \tau_2 + \tau_1{}^2).$$

Therefore

$$\nu_1 - \rho\nu_0{}^2 = \frac{-t}{43^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \tau_1\mathcal{L}_2 + \frac{t}{83^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \mathcal{L}_1{}^2(\tau_1 - \tau_2 + \tau_1{}^2)$$

$$-\rho \frac{t^2}{42^{2n}}\left(\frac{\kappa}{\kappa+\eta}\right)^{2m} {\tau_1}^2 {\mathcal{L}_1}^2$$

$$= \frac{-t}{43^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \tau_1 \left[\mathcal{L}_2 - {\mathcal{L}_1}^2\left(\frac{1}{2}\left\{1-\frac{\tau_2}{\tau_1}+\left\{1-\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\right\}\tau_1\right\}\right)\right]$$

$$= \frac{-t}{43^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \tau_1\left[\mathcal{L}_2 - \mathrm{H}{\mathcal{L}_1}^2\right],$$

where

$$\mathrm{H} = \frac{1}{2}\left\{1-\frac{\tau_2}{\tau_1}+\left\{1-\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\right\}\tau_1\right\}.$$

Now, by an application of Lemma 1, we have the result in (1.6). Also, if $\tau_1 = 0$, then $\nu_0 = 0$ and $\nu_1 = \frac{-t}{83^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \tau_2 {\mathcal{L}_1}^2$.

Since $\psi(z)$ has a positive real part, $|\mathcal{L}_2| \le 2$, from this, we get

$$|\nu_1 - \rho{\nu_0}^2| \le \frac{1}{83^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m \left|t\tau_2{\mathcal{L}_1}^2\right|$$

$$\le \frac{|t\tau_2|}{23^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m.$$

Now $\psi(z)$ have positive real part, $|\mathcal{L}_2| \le 2$, therefore, we have

$$|\nu_1 - \rho{\nu_0}^2| \le \frac{|t|}{3^n}\left(\frac{\kappa}{\kappa+2\eta}\right)^m.$$

The result is sharp for the functions:

$$1-\frac{1}{t}\left(\frac{D^{n+1,m}\omega(z)}{D^{n,m}\omega(z)}-1\right) = \frac{1+2z-z^2}{z(1-z^2)} \text{ and } 1-\frac{1}{t}\left(\frac{D^{n+1,m}\omega(z)}{D^{n,m}\omega(z)}-1\right) = \frac{1+z}{z(1-z)}.$$

Theorem 2. *Let* $\chi(z) = 1+\tau_1 z+\tau_2 z^2+\tau_3 z^3+\cdots\ (\tau_\ell > 0, \ell \in \mathbb{N})$. *If* $\omega(z)$ *is given by* (1.1) *belonging to the class* $\sum_{t,\eta,\kappa}^{*n,m}(\chi)$ *and* ρ *is a complex number*, *then*

$|\nu_1 - \rho{\nu_0}^2|$

$$\leq \begin{cases} \dfrac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\left\{-\tau_2+(1-\dfrac{2\rho t3^n}{2^{2n}}\left(\dfrac{\kappa}{\kappa+\eta}\right)^{2m}\left(\dfrac{\kappa+2\eta}{\kappa}\right)^m)\tau_1{}^2\right\} \\ \qquad\qquad if\ \ \rho \leq \dfrac{(\tau_1{}^2-\tau_2-\tau_1)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t\tau_1{}^2\kappa^{2m}(\kappa+2\eta)^m} \\ \dfrac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\tau_1 \\ \quad if\ \dfrac{\left(\tau_1{}^2-(\tau_2+\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t\tau_1{}^2\kappa^{2m}(\kappa+2\eta)^m} \leq \rho \leq \dfrac{\left(\tau_1{}^2-(\tau_2-\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t\tau_1{}^2\kappa^{2m}(\kappa+2\eta)^m} \\ \dfrac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\left\{\tau_2-(1-\dfrac{2\rho t3^n}{2^{2n}}\left(\dfrac{\kappa}{\kappa+\eta}\right)^{2m}\left(\dfrac{\kappa+2\eta}{\kappa}\right)^m)\tau_1{}^2\right\} \\ \qquad\qquad if\ \ \ \rho \geq \dfrac{\left(\tau_1{}^2-(\tau_2-\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t\tau_1{}^2\kappa^{2m}(\kappa+2\eta)^m} \end{cases}$$

Proof. Let $\rho \leq \frac{\left(\tau_1{}^2-(\tau_2+\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t\tau_1{}^2\kappa^{2m}(\kappa+2\eta)^m}$. Then

$$|\nu_1-\rho\nu_0{}^2| \leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\tau_1\left\{-\frac{\tau_2}{\tau_1}+\left(1-\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\right)\tau_1\right\}$$

$$\leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\left\{-\tau_2+\left(1-\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\right)\tau_1{}^2\right\}.$$

On the other hand, if

$$\frac{\left(\tau_1{}^2-(\tau_2+\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t\tau_1{}^2\kappa^{2m}(\kappa+2\eta)^m} \leq \rho \leq \frac{\left(\tau_1{}^2-(\tau_2-\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t\tau_1{}^2\kappa^{2m}(\kappa+2\eta)^m}.$$

We get

$$|\nu_1-\rho\nu_0{}^2|$$

$$\leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\tau_1\left\{-\frac{\tau_2}{\tau_1}+\left(1-\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\right)\tau_1\right\}$$

$$\leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\tau_1\left\{-\frac{\tau_2}{\tau_1}+\left(1-2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\frac{(\tau_1{}^2-\tau_2-\tau_1)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t\tau_1{}^2\kappa^{2m}(\kappa+2\eta)^m}\right)\tau_1\right\}$$

$$\leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\tau_1\left\{-\frac{\tau_2}{\tau_1}+\left(1-\frac{({\tau_1}^2-\tau_2-\tau_1)}{{\tau_1}^2}\right)\tau_1\right\}$$

$$\leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\tau_1.$$

Finally, if $\rho \geq \frac{\left({\tau_1}^2-(\tau_2-\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t{\tau_1}^2\kappa^{2m}(\kappa+2\eta)^m}$, then

$$|\nu_1-\rho{\nu_0}^2| \leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\tau_1\left\{\frac{\tau_2}{\tau_1}-\left(1-\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\right)\tau_1\right\}$$

$$\leq \frac{t\left(\frac{\kappa}{\kappa+2\eta}\right)^m}{23^n}\left\{\tau_2-\left(1-\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\right){\tau_1}^2\right\}.$$

To show that the bounds are sharp, we define the functions $\vartheta_{y_l}(l \geq 2)$ by

$$1-\frac{1}{t}\left(\frac{D^{n+1,m}\vartheta_{y_l}(z)}{D^{n,m}\vartheta_{y_l}(z)}-1\right)=\vartheta\left(z^{l-1}\right),\ D^{n,m}\vartheta_{y_l}(0)=0=D^{n+1,m}\vartheta_{y_l}(0)-1.$$

If

$$\rho < \frac{({\tau_1}^2-(\tau_2+\tau_1))2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t{\tau_1}^2\kappa^{2m}(\kappa+2\eta)^m} \text{ or } \rho > \frac{\left({\tau_1}^2-(\tau_2-\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t{\tau_1}^2\kappa^{2m}(\kappa+2\eta)^m},$$

then the equality holds if and only if ϑ_{y_l} or one of its rotation ,when

$$\frac{\left({\tau_1}^2-(\tau_2+\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t{\tau_1}^2\kappa^{2m}(\kappa+2\eta)^m} < \rho < \frac{\left({\tau_1}^2-(\tau_2-\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t{\tau_1}^2\kappa^{2m}(\kappa+2\eta)^m},$$

then the equality holds if and only if $\omega(z)$ is ϑ_{y_3} or one of its rotations. If

$$\rho = \frac{\left({\tau_1}^2-(\tau_2+\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t{\tau_1}^2\kappa^{2m}(\kappa+2\eta)^m},$$

then the equality holds if and only if $\omega(z)$ is ϑ_s or one of its rotations. If $\rho = \frac{\left({\tau_1}^2-(\tau_2-\tau_1)\right)2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n t{\tau_1}^2\kappa^{2m}(\kappa+2\eta)^m}$, then the equality holds if and only if $\omega(z)$ is ϑ_r or one of its rotations.

Theorem 3. *Let* $\chi(z) = 1 + \tau_1 z + \tau_2 z^2 + \tau_3 z^3 + \cdots, \quad (\tau_\ell > 0, \ell \in \mathbb{N})$ *and* $\mathscr{M} = \frac{(\tau_1{}^2 - \tau_2) 2^{2n} (\kappa+\eta)^{2m} \kappa^m}{2 3^n t \tau_1{}^2 \kappa^{2m} (\kappa+2\eta)^m}$. *If* $\omega(z)$ *is given by* (1.1) *belonging to the class* $\sum_{t,\eta,\kappa}^{*n_\circ m}(\chi)$ *and* ρ *is a real number, then we have*

(1) *if* $\frac{(\tau_1{}^2 - (\tau_2+\tau_1)) 2^{2n} (\kappa+\eta)^{2m} \kappa^m}{2 3^n t \tau_1{}^2 \kappa^{2m} (\kappa+2\eta)^m} \le \rho \le \frac{(\tau_1{}^2 - \tau_2) 2^{2n} (\kappa+\eta)^{2m} \kappa^m}{2 3^n t \tau_1{}^2 \kappa^{2m} (\kappa+2\eta)^m}$, *then*

$$|\nu_1 - \rho \nu_0{}^2| + \frac{2^{2n}(\kappa+\eta)^{2m}\kappa^m}{2 3^n\, t\, \tau_1{}^2 \kappa^{2m} (\kappa+2\eta)^m} \times \left\{\tau_2 + \tau_1 + \left(\rho 2t \frac{3^n}{2^{2n}} \left(\frac{\kappa+2\eta}{\kappa}\right)^m \left(\frac{\kappa}{\kappa+\eta}\right)^{2m} - 1\right)\tau_1{}^2\right\} |\nu_0{}^2|$$

$$= \frac{t\kappa^m}{2^2 3^n (\kappa+2\eta)^m} \tau_1 \left(\mathcal{L}_1{}^2 \left\{-\frac{1}{2}\frac{\tau_2}{\tau_1} + \frac{1}{2}\tau_1 - \left(\rho 2t \frac{3^n}{2\, 2^{2n}} \left(\frac{\kappa+2\eta}{\kappa}\right)^m \left(\frac{\kappa}{\kappa+\eta}\right)^{2m} \tau_1\right)\right\}\right)$$

$$+ \frac{2^{2n}(\kappa+\eta)^{2m}\kappa^m}{2 3^n\, t\, \tau_1{}^2 \kappa^{2m} (\kappa+2\eta)^m} \left(\frac{\kappa^{2m} t^2 \tau_1{}^2 \mathcal{L}_1{}^2}{2^{2n+2}\, t \tau_1{}^2 (\kappa+\eta)^{2m}}\right) \times \left\{\tau_2 + \tau_1 + \left(\rho 2t \frac{3^n}{2^{2n}} \left(\frac{\kappa+2\eta}{\kappa}\right)^m \left(\frac{\kappa}{\kappa+\eta}\right)^{2m} - 1\right)\tau_1{}^2\right\}$$

$$= \frac{t\kappa^m}{2^3 3^n (\kappa+2\eta)^m} \tau_1{}^2 \mathcal{L}_1{}^2 - \frac{3^n t^2 \kappa^{2m} \rho}{2^{2n+2} (\kappa+\eta)^{2m}} \tau_1{}^2 \mathcal{L}_1{}^2 + \frac{t\kappa^m}{2^3 3^n (\kappa+2\eta)^m} \mathcal{L}_1{}^2 \tau_1 + \frac{\rho t^2 \kappa^{2m}}{2^{3+2n} (\kappa+\eta)^{2m}} \tau_1{}^2 \mathcal{L}_1{}^2 - \frac{t\, \kappa^m}{2^3 3^n (\kappa+2\eta)^m} \tau_1{}^2 \mathcal{L}_1{}^2$$

$$= \frac{t\kappa^m}{2^3 3^n (\kappa+2\eta)^m} \mathcal{L}_1{}^2 \tau_1, \quad |\mathcal{L}_1| \le 2$$

$$= \frac{t\kappa^m}{2 3^n (\kappa+2\eta)^m} \tau_1$$

(2) *if* $\mathscr{M} \le \rho \le \frac{(\tau_1{}^2 - (\tau_2-\tau_1)) 2^{2n} (\kappa+\eta)^{2m} \kappa^m}{2 3^n t \tau_1{}^2 \kappa^{2m} (\kappa+2\eta)^m}$, *then*

$$|\nu_1 - \rho \nu_0{}^2| + \frac{2^{2n}(\kappa+\eta)^{2m}\kappa^m}{2 3^n\, t\, \tau_1{}^2 \kappa^{2m} (\kappa+2\eta)^m} \times \left\{\tau_1 - \tau_2 + \left(1 - \rho 2t \frac{3^n}{2^{2n}} \left(\frac{\kappa+2\eta}{\kappa}\right)^m \left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\right)\tau_1{}^2\right\} |\nu_0{}^2|$$

$$= \frac{t\kappa^m}{2^2 3^n(\kappa+2\eta)^m}\tau_1\mathcal{L}_2 - \frac{t\kappa^m}{2^3 3^n(\kappa+2\eta)^m}\tau_1{\mathcal{L}_1}^2$$

$$\times\left(1-\frac{\tau_2}{\tau_1}+\tau_1-\left(\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}\tau_1\right)\right)$$

$$+\frac{2^{2n}(\kappa+\eta)^{2m}\kappa^m}{23^n\, t\,{\tau_1}^2\kappa^{2m}(\kappa+2\eta)^m}\left(\frac{\kappa^{2m}t^2{\tau_1}^2{\mathcal{L}_1}^2}{2^{2n+2}\,\kappa^{2m}(\kappa+\eta)^m}\right)$$

$$\times\left\{\tau_1-\tau_2+{\tau_1}^2-\rho 2t\frac{3^n}{2^{2n}}\left(\frac{\kappa+2\eta}{\kappa}\right)^m\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}{\tau_1}^2\right\}$$

$$= \frac{t\kappa^m}{2^2 3^n(\kappa+2\eta)^m}\tau_1\mathcal{L}_2 - \frac{t\kappa^m}{2^3 3^n(\kappa+2\eta)^m}\tau_1{\mathcal{L}_1}^2 + \frac{t\kappa^m}{2^3 3^n(\kappa+2\eta)^m}\tau_2{\mathcal{L}_1}^2$$

$$-\frac{t\kappa^m}{2^3 3^n(\kappa+2\eta)^m}{\tau_1}^2{\mathcal{L}_1}^2 + \frac{\rho t^2\kappa^{2m}}{2^{2n+2}(\kappa+\eta)^{2m}}{\mathcal{L}_1}^2{\tau_1}^2$$

$$+\frac{t\kappa^m}{2^3 3^n(\kappa+2\eta)^m}\tau_1{\mathcal{L}_1}^2 - \frac{t\kappa^m}{2^3 3^n(\kappa+2\eta)^m}\tau_2{\mathcal{L}_1}^2 + \frac{t\,\kappa^m}{2^3 3^n(\kappa+2\eta)^m}{\tau_1}^2{\mathcal{L}_1}^2$$

$$-\rho t^2\frac{1}{2^{2n+2}}\left(\frac{\kappa}{\kappa+\eta}\right)^{2m}{\tau_1}^2{\mathcal{L}_1}^2$$

$$= \frac{t\kappa^m}{2^2 3^n(\kappa+2\eta)^m}\tau_1\mathcal{L}_2,\quad |\mathcal{L}_2|\le 2$$

$$= \frac{t\kappa^m}{23^n(\kappa+2\eta)^m}\tau_1.$$

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