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Fekete-Szegö Problem for Certain New Family of Bi-Univalent Functions

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Abstract

In current effort, by making use of the principle of subordination, we introduce and study a new family of holomorphic and bi-univalent functions which are defined in open unit disk and solve Fekete-Szegö problem for functions which belong to this family.

1. Introduction and Preliminaries

Indicate by $\mathcal{A}$ the family of functions $\mathfrak{U}$ that are holomorphic in the unit disk $\mathfrak{D} = \{t \in \mathbb{C} : |t| < 1\}$ which have the shape:

$$\mathfrak{U}(t) = t + \sum_{n=2}^{\infty} a_n t^n . \tag{1.1}$$

Assume that $\mathcal{S}$ stand for the subfamily of $\mathcal{A}$ containing of the shape (1.1) that are univalent in $\mathfrak{D}$.

The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for $\mathfrak{U} \in \mathcal{S}$ is well known for its rich history in the field of "Geometric Function Theory". Its origin was in the disproof by Fekete and Szegö [12] conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in many sub families of the family of univalent functions. Nowadays, it

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seems that this topic had become an interest among the researchers (see, for example, [5, 19]).

For two functions $\mathfrak{F}$ and $\mathcal{Y}$, holomorphic in the unit disk $\mathfrak{O}$, we name that the function $\mathfrak{F}(t)$ is subordinate to $\mathcal{Y}(t)$ in $\mathfrak{O}$, and write

$$\mathfrak{F}(t) \prec \mathcal{Y}(t) \ \ (t \in \mathfrak{O}),$$

if there is a Schwarz function $\mathfrak{S}(t)$, holomorphic in $\mathfrak{O}$, with

$$\mathfrak{S}(0) = 0 \ \text{ and } \ |\mathfrak{S}(t)| < 1 \quad (t \in \mathfrak{O}),$$

such that

$$\mathfrak{F}(t) = \mathcal{Y}\big(\mathfrak{S}(t)\big) \ \ (t \in \mathfrak{O}).$$

In special, if the function $\mathcal{Y}$ is univalent in $\mathfrak{O}$, then

$$\mathfrak{F}(0) = \mathcal{Y}(0) \ \text{ and } \ \mathfrak{F}(\mathfrak{O}) \subset \mathcal{Y}(\mathfrak{O}).$$

It is well known (see [11]) that each function $\mathfrak{U} \in \mathcal{S}$ has an inverse $\mathfrak{U}^{-1}$, defined by

$$\mathfrak{U}^{-1}\big(\mathfrak{U}(t)\big) = t, \ \ (t \in \mathfrak{O})$$

and

$$\mathfrak{U}\big(\mathfrak{U}^{-1}(\mathfrak{W})\big) = \mathfrak{W}, \ \ \left(|\mathfrak{W}| < r_0(\mathfrak{U}), r_0(\mathfrak{U}) \geq \frac{1}{4}\right),$$

where

$$\begin{aligned}\mathfrak{V}(\mathfrak{W}) &= \mathfrak{U}^{-1}(\mathfrak{W}) \\ &= \mathfrak{W} - a_2\mathfrak{W}^2 + (2a_2^2 - a_3)\mathfrak{W}^3 - (5a_2^3 - 5a_2a_3 + a_4)\mathfrak{W}^4 + \cdots . \quad (1.2)\end{aligned}$$

A function $\mathfrak{U} \in \mathcal{A}$ is named bi-univalent in $\mathfrak{O}$ if both $\mathfrak{U}$ and $\mathfrak{U}^{-1}$ are univalent in $\mathfrak{O}$. Let Σ denotes the family of all bi-univalent functions in $\mathfrak{O}$ that satisfy (1.1). In fact, Srivastava et al. [22] refreshed the study of holomorphic bi-univalent functions in recent years, it was followed by such works (see [1,2,4,6,8,9,10,13,14,16,17,18,20,21,23,24, 26,27]).

We shall need the next lemma to establish the desired bounds in our investigation.

Lemma 1.1 [16]. *If $\mathcal{T}(t) = 1 + \mathcal{T}_1 t + \mathcal{T}_2 t^2 + \mathcal{T}_3 t^3 + \cdots$ is an holomorphic function in $\mathfrak{O}$ with positive real part, then*

$$|\mathcal{T}_n| \leq 2 \quad (n \in \mathbb{N} = \{1,2,3,\ldots\})$$

and

$$\left|\mathcal{T}_2 - \frac{\mathcal{T}_1^2}{2}\right| \leq 2 - \frac{|\mathcal{T}_1|^2}{2}.$$

2. Main Results

In the sequel, symbolized by ϑ the holomorphic function with positive real part in $\mathfrak{O}$, satisfying $\vartheta(0) = 1$ and $\vartheta'(0) > 0$. Also, assume that $\vartheta(\mathfrak{O})$ be starlike with respect to 1 and symmetric with respect to the real axis. Such a function ϑ has the Taylor series expansion of the shape:

$$\vartheta(t) = 1 + \xi_1 t + \xi_2 t^2 + \xi_3 t^3 + \cdots, \quad \xi_1 > 0. \tag{2.1}$$

Definition 2.1. A function $\mathfrak{U} \in \Sigma$ given by (1.1) is named in the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$, $(0 \leq \gamma \leq 1, 0 \leq \eta \leq 1, 0 \leq \tau \leq 1)$ if it fulfills the next subordinations:

$$\left(\frac{t\mathfrak{U}'(t)}{\mathfrak{U}(t)}\right)^\gamma \left[(1-\tau)\frac{t\mathfrak{U}'(t)}{\mathfrak{U}(t)} + \tau\left(1 + \frac{t\mathfrak{U}''(t)}{\mathfrak{U}'(t)}\right)\right]^\eta \prec \vartheta(t)$$

and

$$\left(\frac{\mathfrak{W}\mathfrak{V}'(\mathfrak{W})}{\mathfrak{V}(\mathfrak{W})}\right)^\gamma \left[(1-\tau)\frac{\mathfrak{W}\mathfrak{V}'(\mathfrak{W})}{\mathfrak{V}(\mathfrak{W})} + \tau\left(1 + \frac{\mathfrak{W}\mathfrak{V}''(\mathfrak{W})}{\mathfrak{V}'(\mathfrak{W})}\right)\right]^\eta \prec \vartheta(\mathfrak{W}),$$

where the function $\mathfrak{V} = \mathfrak{U}^{-1}$ is given by (1.2).

Remark 2.1. It should be remarked that the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$ is a generalization of well-known families considered earlier. These families are:

(1) For $\gamma = 0$ and $\eta = 1$, the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$ reduce to the family $\mathcal{M}_\Sigma(\eta; \vartheta)$ which was introduced recently by Ali et al. [3].

(2) For $\gamma = 0$, $\eta = 1$ and $\vartheta(t) = \left(\frac{1+t}{1-t}\right)^\alpha$, $0 < \alpha \leq 1$, the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$ reduce to the family $M_\Sigma(\alpha, \tau)$ which was investigated by Liu and Wang [15].

(3) For $\gamma = 0$, $\eta = 1$ and $\vartheta(t) = \frac{1+(1-2\beta)t}{1-t}$, $0 \leq \beta < 1$, the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$ reduce to the family $B_\Sigma(\beta, \tau)$ which was studied by Liu and Wang [15].

(4) For $\gamma = \tau = 0$, $\eta = 1$ and $\vartheta(t) = \left(\frac{1+t}{1-t}\right)^\alpha$, $0 < \alpha \leq 1$, the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$ reduce to the family $S_\Sigma^*(\alpha)$ which was introduced by Brannan and Taha [7].

(5) For $\gamma = \tau = 0$, $\eta = 1$ and $\vartheta(t) = \frac{1+(1-2\beta)t}{1-t}$, $0 \leq \beta < 1$, the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$ reduce to the family $S_\Sigma^*(\beta)$ which was studied by Brannan and Taha [7].

(6) For $\tau = 1$ and $\eta = 1 - \gamma$, the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$ reduce to the family $\mathcal{L}_\Sigma(\gamma; \vartheta)$ which was investigated by Ali et al. [3].

(7) For $\tau = 1$, $\gamma = 1 - \eta$ and $\vartheta(t) = \frac{1+t}{1-t}$ the family $F_\Sigma(\gamma, \eta, \tau; \vartheta)$ reduce to the family $\mathcal{M}^\eta$ which was considered recently by Thomas [25].

Theorem 2.1. *Let* $\mathfrak{U} \in F_\Sigma(\gamma, \eta, \tau; \vartheta)$, $\mu \in \mathbb{R}$. *Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{\xi_1}{2(\gamma + \eta(2\tau + 1))}, \quad \mathfrak{U}\ or\ |\mu - 1| \leq \dfrac{1}{4(\gamma + \eta(2\tau + 1))} \times \\ \times \left| \gamma(\gamma + 1) + \eta(\tau + 1)(2(\gamma + 1) + (\eta - 1)(\tau + 1)) + 2(\gamma + \eta(\tau + 1))^2 \dfrac{\xi_1 - \xi_2}{\xi_1^2} \right|; \\ \dfrac{|\mu - 1|\xi_1^3}{\left| \frac{1}{2}[\gamma(\gamma + 1) + \eta(\tau + 1)(2(\gamma + 1) + (\eta - 1)(\tau + 1))]\xi_1^2 + (\gamma + \eta(\tau + 1))^2(\xi_1 - \xi_2) \right|}, \\ \mathfrak{U}\ or\ |\mu - 1| \geq \dfrac{1}{4(\gamma + \eta(2\tau + 1))} \times \\ \times \left| \gamma(\gamma + 1) + \eta(\tau + 1)(2(\gamma + 1) + (\eta - 1)(\tau + 1)) + 2(\gamma + \eta(\tau + 1))^2 \dfrac{\xi_1 - \xi_2}{\xi_1^2} \right|. \end{cases}$$

Proof. Assume that $\mathfrak{U} \in F_\Sigma(\gamma, \eta, \tau; \vartheta)$. Then there are two holomorphic functions $u, v: \mathfrak{D} \longrightarrow \mathfrak{D}$ with $u(0) = v(0) = 0$, $|u(t)| < 1$, $|v(\mathfrak{W})| < 1$ and $t, \mathfrak{W} \in \mathfrak{D}$ such that

$$\left(\frac{t\mathfrak{U}'(t)}{\mathfrak{U}(t)}\right)^\gamma \left[(1 - \tau)\frac{t\mathfrak{U}'(t)}{\mathfrak{U}(t)} + \tau\left(1 + \frac{t\mathfrak{U}''(t)}{\mathfrak{U}'(t)}\right)\right]^\eta = \vartheta(u(t)) \tag{2.2}$$

and

$$\left(\frac{\mathfrak{W}\mathfrak{V}'(\mathfrak{W})}{\mathfrak{V}(\mathfrak{W})}\right)^\gamma \left[(1 - \tau)\frac{\mathfrak{W}\mathfrak{V}'(\mathfrak{W})}{\mathfrak{V}(\mathfrak{W})} + \tau\left(1 + \frac{\mathfrak{W}\mathfrak{V}''(\mathfrak{W})}{\mathfrak{V}'(\mathfrak{W})}\right)\right]^\eta = \vartheta(v(\mathfrak{W})), \tag{2.3}$$

where $\mathfrak{V} = \mathfrak{U}^{-1}$.

We define the functions $\mathcal{T}$ and $\mathfrak{T}$ by

$$\mathcal{T}(t) = \frac{1 + u(t)}{1 - u(t)} = 1 + \mathcal{T}_1 t + \mathcal{T}_2 t^2 + \cdots \tag{2.4}$$

and

$$\mathfrak{T}(\mathfrak{W}) = \frac{1+v(\mathfrak{W})}{1-v(\mathfrak{W})} = 1 + \mathfrak{T}_1\mathfrak{W} + \mathfrak{T}_2\mathfrak{W}^2 + \cdots \tag{2.5}$$

It is clear that $\mathcal{T}$ and $\mathfrak{T}$ are holomorphic in $\mathfrak{O}$ and $\mathcal{T}(0) = \mathfrak{T}(0) = 1$. Since $u, v: \mathfrak{O} \longrightarrow \mathfrak{O}$, then the functions $\mathcal{T}$ and $\mathfrak{T}$ have positive real part in $\mathfrak{O}$.

It follows from (2.4) and (2.5) that

$$u(t) = \frac{\mathcal{T}(t)-1}{\mathcal{T}(t)+1} = \frac{1}{2}\mathcal{T}_1 t + \frac{1}{2}\left(\mathcal{T}_2 - \frac{1}{2}\mathcal{T}_1^{\,2}\right)t^2 + \cdots \tag{2.6}$$

and

$$v(\mathfrak{W}) = \frac{\mathfrak{T}(\mathfrak{W})-1}{\mathfrak{T}(\mathfrak{W})+1} = \frac{1}{2}\mathfrak{T}_1\mathfrak{W} + \frac{1}{2}\left(\mathfrak{T}_2 - \frac{1}{2}\mathfrak{T}_1^2\right)\mathfrak{W}^2 + \cdots . \tag{2.7}$$

Using (2.6) and (2.7) together with (2.1), we easily obtain

$$\vartheta\big(u(t)\big) = 1 + \frac{1}{2}\xi_1\mathcal{T}_1 t + \frac{1}{2}\left(\frac{1}{2}\xi_2\mathcal{T}_1^{\,2} + \xi_1\left(\mathcal{T}_2 - \frac{1}{2}\mathcal{T}_1^{\,2}\right)\right)t^2 + \cdots \tag{2.8}$$

and

$$\vartheta\big(v(\mathfrak{W})\big) = 1 + \frac{1}{2}\xi_1\mathfrak{T}_1\mathfrak{W} + \frac{1}{2}\left(\frac{1}{2}\xi_2\mathfrak{T}_1^2 + \xi_1\left(\mathfrak{T}_2 - \frac{1}{2}\mathfrak{T}_1^2\right)\right)\mathfrak{W}^2 + \cdots . \tag{2.9}$$

Since

$$\left(\frac{t\mathfrak{U}'(t)}{\mathfrak{U}(t)}\right)^{\gamma}\left[(1-\tau)\frac{t\mathfrak{U}'(t)}{\mathfrak{U}(t)} + \tau\left(1 + \frac{t\mathfrak{U}''(t)}{\mathfrak{U}'(t)}\right)\right]^{\eta} = 1 + \big(\gamma + \eta(\tau+1)\big)a_2 t$$

$$+\big[2\big(\gamma + \eta(2\tau+1)\big)a_3$$

$$+\frac{1}{2}\big[\gamma(\gamma-1) + \eta(\tau+1)\big(2\gamma + (\eta-1)(\tau+1)\big) - 2\big(\gamma + \eta(3\tau+1)\big)\big]a_2^2\big]t^2 + \cdots$$

and

$$\left(\frac{\mathfrak{W}\mathfrak{V}'(\mathfrak{W})}{\mathfrak{V}(\mathfrak{W})}\right)^{\gamma}\left[(1-\tau)\frac{\mathfrak{W}\mathfrak{V}'(\mathfrak{W})}{\mathfrak{V}(\mathfrak{W})} + \tau\left(1 + \frac{\mathfrak{W}\mathfrak{V}''(\mathfrak{W})}{\mathfrak{V}'(\mathfrak{W})}\right)\right]^{\eta} = 1 - \big(\gamma + \eta(\tau+1)\big)a_2\mathfrak{W}$$

$$+[\ \frac{1}{2}\big[\gamma(\gamma-1) + \eta(\tau+1)\big(2\gamma + (\eta-1)(\tau+1)\big) + 2\big(3\gamma + \eta(5\tau+3)\big)\big]a_2^2$$

$-2(\gamma+\eta(2\tau+1))a_3]\mathfrak{W}^2+\cdots,$

then (2.2), (2.3), (2.8) and (2.9) yields

$$(\gamma+\eta(\tau+1))a_2=\frac{1}{2}\xi_1\mathcal{T}_1, \tag{2.10}$$

$$\begin{aligned}&2(\gamma+\eta(2\tau+1))a_3\\&+\frac{1}{2}\left[\gamma(\gamma-1)+\eta(\tau+1)(2\gamma+(\eta-1)(\tau+1))-2(\gamma+\eta(3\tau+1))\right]a_2^2\\&=\frac{1}{2}\left(\frac{1}{2}\xi_2\mathcal{T}_1^{\,2}+\xi_1\left(\mathcal{T}_2-\frac{1}{2}\mathcal{T}_1^{\,2}\right)\right),\end{aligned} \tag{2.11}$$

$$-(\gamma+\eta(\tau+1))a_2=\frac{1}{2}\xi_1\mathfrak{T}_1 \tag{2.12}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\gamma(\gamma-1)+\eta(\tau+1)(2\gamma+(\eta-1)(\tau+1))+2(3\gamma+\eta(5\tau+3))\right]a_2^2\\&\quad-2(\gamma+\eta(2\tau+1))a_3=\frac{1}{2}\left(\frac{1}{2}\xi_2\mathfrak{T}_1^2+\xi_1\left(\mathfrak{T}_2-\frac{1}{2}\mathfrak{T}_1^2\right)\right).\end{aligned} \tag{2.13}$$

In view of (2.10) and (2.12) that

$$\mathcal{T}_1=-\mathfrak{T}_1 \tag{2.14}$$

and

$$8(\gamma+\eta(\tau+1))^2a_2^2=\xi_1^2(\mathcal{T}_2^{\,2}+\mathfrak{T}_2^2). \tag{2.15}$$

By subtracting (2.11) from (2.13) and using (2.14), we find that

$$a_3=a_2^2+\frac{\xi_1(\mathcal{T}_2-\mathfrak{T}_2)}{8(\gamma+\eta(2\tau+1))}\,. \tag{2.16}$$

If we add (2.11) to (2.13), we deduce that

$$\begin{aligned}&\left[\gamma(\gamma+1)+\eta(\tau+1)(2(\gamma+1)+(\eta-1)(\tau+1))\right]a_2^2\\&=\frac{1}{2}\left[\xi_1(\mathcal{T}_2+\mathfrak{T}_2)-\frac{1}{2}(\xi_1-\xi_2)(\mathcal{T}_1^{\,2}+\mathfrak{T}_1^2)\right].\end{aligned} \tag{2.17}$$

By substitute the value of $\mathcal{T}_1^{\,2}+\mathfrak{T}_1^2$ from (2.15) in the right hand side of (2.17), we get

$$a_2^2 = \frac{\xi_1^3(\mathcal{J}_2 + \mathfrak{T}_2)}{2[\gamma(\gamma+1)+\eta(\tau+1)(2(\gamma+1)+(\eta-1)(\tau+1))]\xi_1^2 + 4(\gamma+\eta(\tau+1))^2(\xi_1-\xi_2)}. \tag{2.18}$$

From (2.16) and (2.18), we obtain

$$a_3 - \mu a_2^2 = \xi_1 \left[\left(\psi(\mu) + \frac{1}{8(\gamma+\eta(2\tau+1))}\right)\mathcal{J}_2 + \left(\psi(\mu) - \frac{1}{8(\gamma+\eta(2\tau+1))}\right)\mathfrak{T}_2\right],$$

where

$$\psi(\mu) = \frac{(1-\mu)\xi_1^2}{2[\gamma(\gamma+1)+\eta(\tau+1)(2(\gamma+1)+(\eta-1)(\tau+1))]\xi_1^2 + 4(\gamma+\eta(\tau+1))^2(\xi_1-\xi_2)}.$$

According to Lemma 1.1 and (2.1), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{\xi_1}{2(\gamma+\eta(2\tau+1))}, & 0 \leq |\psi(\mu)| \leq \dfrac{1}{8(\gamma+\eta(2\tau+1))}; \\ 4\xi_1|\psi(\mu)|, & |\psi(\mu)| \geq \dfrac{1}{8(\gamma+\eta(2\tau+1))}, \end{cases}$$

After some computations, we deduce that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{\xi_1}{2(\gamma+\eta(2\tau+1))}, \quad for\ |\mu-1| \leq \dfrac{1}{4(\gamma+\eta(2\tau+1))} \times \\ \times \left|\gamma(\gamma+1)+\eta(\tau+1)(2(\gamma+1)+(\eta-1)(\tau+1)) + 2(\gamma+\eta(\tau+1))^2 \dfrac{\xi_1-\xi_2}{\xi_1^2}\right|; \\ \dfrac{|\mu-1|\xi_1^3}{\left|\frac{1}{2}[\gamma(\gamma+1)+\eta(\tau+1)(2(\gamma+1)+(\eta-1)(\tau+1))]\xi_1^2 + (\gamma+\eta(\tau+1))^2(\xi_1-\xi_2)\right|}, \\ for\ |\mu-1| \geq \dfrac{1}{4(\gamma+\eta(2\tau+1))} \times \\ \times \left|\gamma(\gamma+1)+\eta(\tau+1)(2(\gamma+1)+(\eta-1)(\tau+1)) + 2(\gamma+\eta(\tau+1))^2 \dfrac{\xi_1-\xi_2}{\xi_1^2}\right|. \end{cases}$$

Taking $\mu = 1$ in Theorem 2.1, we conclude the next outcome:

Corollary 2.1. *If* $\mathfrak{U} \in F_\Sigma(\gamma, \eta, \tau; \vartheta)$, *then*

$$\left|a_3 - a_2^2\right| \le \frac{\xi_1}{2\left(\gamma + \eta(2\tau + 1)\right)}.$$

Taking $\mu = 0$ in Theorem 2.1, we conclude the next outcome:

Corollary 2.2. *If* $\mathfrak{U} \in F_{\Sigma}(\gamma, \eta, \tau; \vartheta)$, *then*

$$|a_3| \le \begin{cases} \dfrac{\xi_1}{2\left(\gamma + \eta(2\tau+1)\right)}, \\ for\, \dfrac{\xi_1 - \xi_2}{\xi_1^2} \in \left(-\infty, -\dfrac{4\left(\gamma + \eta(2\tau+1)\right) + \gamma(\gamma+1) + \eta(\tau+1)\left(2(\gamma+1) + (\eta-1)(\tau+1)\right)}{2\left(\gamma + \eta(\tau+1)\right)^2}\right] \\ \quad \cup \left[\dfrac{4\left(\gamma + \eta(2\tau+1)\right) - \gamma(\gamma+1) - \eta(\tau+1)\left(2(\gamma+1) + (\eta-1)(\tau+1)\right)}{2\left(\gamma + \eta(\tau+1)\right)^2}, \infty\right); \\ \\ \dfrac{\xi_1^3}{\left|\frac{1}{2}\left[\gamma(\gamma+1) + \eta(\tau+1)\left(2(\gamma+1) + (\eta-1)(\tau+1)\right)\right]\xi_1^2 + \left(\gamma + \eta(\tau+1)\right)^2(\xi_1 - \xi_2)\right|}, \\ for\, \dfrac{\xi_1 - \xi_2}{\xi_1^2} \in \left[-\dfrac{4\left(\gamma + \eta(2\tau+1)\right) + \gamma(\gamma+1) + \eta(\tau+1)\left(2(\gamma+1) + (\eta-1)(\tau+1)\right)}{2\left(\gamma + \eta(\tau+1)\right)^2},\right. \\ \quad \left. -\dfrac{\gamma(\gamma+1) + \eta(\tau+1)\left(2(\gamma+1) + (\eta-1)(\tau+1)\right)}{2\left(\gamma + \eta(\tau+1)\right)^2}\right) \\ \quad \cup \left(-\dfrac{\gamma(\gamma+1) + \eta(\tau+1)\left(2(\gamma+1) + (\eta-1)(\tau+1)\right)}{2\left(\gamma + \eta(\tau+1)\right)^2},\right. \\ \quad \left. \dfrac{4\left(\gamma + \eta(2\tau+1)\right) - \gamma(\gamma+1) - \eta(\tau+1)\left(2(\gamma+1) + (\eta-1)(\tau+1)\right)}{2\left(\gamma + \eta(\tau+1)\right)^2}\right]. \end{cases}$$

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