Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 8, Number 2, 2022, Pages 237-262
https://doi.org/10.34198/ejms.8222.237262

Generating Functions of Binary Products of Tribonacci and Tribonacci Lucas Polynomials and Special Numbers

Hind Merzouk[1], Ali Boussayoud[2] and Kasi Viswanadh V. Kanuri[3,*]

[1] LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
e-mail: merzoukhind07@gmail.com

[2] LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
e-mail: aboussayoud@yahoo.fr

[3] 3669 Leatherwood Dr, Frisco, TX 75033, USA
e-mail: vis.kanuri@gmail.com

Abstract

In this paper, we introduce a new operator defined in this paper, we give some new generating functions of binary products of Tribonacci and Tribonacci Lucas polynomials and special numbers.

1 Introduction

In mathematics, orthogonal polynomials consist of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The most widely used orthogonal polynomials are the

Received: November 18, 2021; Accepted: December 24, 2021
2010 Mathematics Subject Classification: Primary 05E05; Secondary 11B39.
Keywords and phrases: Tribonacci polynomials, k-Fibonacci numbers, k-Pell numbers, generating function.
*Corresponding author

classical orthogonal polynomials (Tchebychev polynomials of first and second kinds, Fibonacci polynomials).

Further in [11], generating functions of the incomplete Fibonacci and Lucas numbers are determined. In [13], Djordjevic gave the incomplete generalized Fibonacci and Lucas numbers. In [14], Djordjevic and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal–Lucas numbers. In [12], the authors define the incomplete Fibonacci and Lucas numbers. Also the authors define the incomplete bivariate Fibonacci and Lucas p-polynomials in [17].

On the other hand, many kinds of generalizations of Fibonacci numbers have been presented in the literature. In particular, a generalization is the k-Fibonacci Numbers. For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$, is defined recurrently by:

$$\begin{cases} F_{k,0}=1, F_{k,1}=k, \\ F_{k,n+1}=kF_{k,n}+F_{k,n-1}, n\geq 1 \end{cases}. \tag{1.1}$$

The k-Pell numbers have been defined in [15] for any number k as follows:

$$\begin{cases} P_{k,0}=0, P_{k,1}=1, \\ P_{k,n+1}=2P_{k,n}+kP_{k,n-1}, n\geq 1 \end{cases}. \tag{1.2}$$

In 1973, Hoggatt and Bicknell [22] introduced Tribonacci polynomials. The Tribonacci polynomials $T_n(x)$ are defined by the recurrence relation

$$\begin{cases} T_n(x)=x^2T_{n-1}(x)+xT_{n-2}(x)+T_{n-3}(x) \\ T_0(x)=1, T_1(x)=x, T_2(x)=x^4+x \end{cases}. \tag{1.3}$$

Also, in [24], authors defined Tribonacci Lucas polynomials, incomplete Tribonacci Lucas numbers and incomplete Tribonacci Lucas polynomials. That is, Tribonacci Lucas polynomials are defined by

$$\begin{cases} K_n(x)=x^2K_{n-1}(x)+xK_{n-2}(x)+K_{n-3}(x) \\ K_0(x)=3, K_1(x)=x^2, K_2(x)=x^4+2x \end{cases}. \tag{1.4}$$

Now, we recall the notion of d-orthogonal polynomials. A remarkable propetry of the d-monic orthogonal polynomial sequence is that those sequences satisfy a $(d+1)$-order recurrence relation, written in the form

$$P_{m+d+1}(x) = (x-\beta_{m+d})\,P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x),\ m \geq 0, \tag{1.5}$$

with the initial conditions $P_0(x) = 1,\ P_{-1}(x) = 0$. So, if $d \geq 2$:

$$P_n(x) = (x-\beta_{n-1})\,P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{d-1-\nu}^{n-1-\nu} P_{n-2-\nu}(x),\ 2 \leq n \leq d, \tag{1.6}$$

and the regularity conditions $\gamma_{m+1}^{0} \neq 0$, $m \geq 0$.

Definition 1. An d-orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ is called d-classical d-orthogonal polynomial sequence if both $\{P_n\}_{n\geq 0}$ and its derivative $\{P_n^{/}\}_{n\geq 0}$ are d-orthogonal. For more details, see [25].

In this contribution, we shall define a new useful operator denoted by $\delta_{e_1 e_2}^{k}$ for which we can formulate, extend and prove new results based on our previous ones, see [16], [3], [18]. In order to determine generating functions of product of k-Fibonacci, k-Lucas, k-Pell, k-Jacobsthal and Mersenne numbers with Tribonacci and Tribonacci Lucas polynomials, we combine between our indicated past techniques and these presented polishing approaches.

In Section 2, we introduce a new symmetric function and give some properties of this symmetric function. We also give some more useful definitions which are used in the subsequent sections, we prove our main result which relates the symmetric function defined in the previous section with the symmetrizing operator. This main theorem unifies several previously known results about the generating functions. It is then used to find the product of Tribonacci polynomails with some special numbers, in Section 3. In Section 4 Generatring functions of some well-known numbers and Tribonacci Lucas polynomials.

2 Definitions and Some Properties

In this section, we introduce a new symmetric function and give some properties of this symmetric function [1, 2, 4, 5, 6]. We also give some more useful definitions from the literature which are used in the subsequent sections [8, 9, 10, 16].

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A. Taking an extra indeterminate z, one has two fundamental series.

$$\lambda_z(A) = \Pi_{a\in A}(1 + za),\ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1 - za)},$$

the expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{\infty} \Lambda_n(A)z^n,\ \sigma_z(A) = \sum_{n=0}^{\infty} S_n(A)z^n.$$

Let us now start at the following definition.

Definition 2. Let A and B be any two alphabets, then we give $S_n(A - B)$ by the following form:

$$\frac{\Pi_{b\in B}(1 - zb)}{\Pi_{a\in A}(1 - za)} = \sum_{n=0}^{\infty} S_n(A - B)z^n = \sigma_z(A - B), \tag{2.1}$$

with the condition $S_n(A - B) = 0$ for $n < 0$ [1].

Corollary 1. *Taking* $A = \{0, 0, ...0\}$ *in* (2.1) *gives*

$$\Pi_{b\in B}(1 - zb) = \sum_{n=0}^{\infty} S_n(-B)z^n = \lambda_z(-B). \tag{2.2}$$

Further, in the case $A = \{0, 0, ...0\}$ *or* $B = \{0, 0, ...0\}$*, we have*

$$\sum_{n=0}^{\infty} S_n(A - B)z^n = \sigma_z(A) \times \lambda_z(-B). \tag{2.3}$$

Thus,

$$S_n(A - B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B). \tag{2.4}$$

Definition 3. [6] Let g be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - g^\sigma(x_1, \cdots, x_i, x_{i+1}, \cdots x_n)}{x_i - x_{i+1}},$$

where g^σ is given by

$$g^\sigma(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) = g(x_1, \cdots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n).$$

Definition 4. [7] Given an alphabet $E = \{e_1, e_2\}$, the symmetrizing operator $\delta^k_{e_1 e_2}$ is defined by

$$\delta^k_{e_1 e_2}(e_1^n) = \frac{e_1^{k+n} - e_2^{k+n}}{e_1 - e_2} = S_{k+n-1}(e_1 + e_2), \text{ for all } k, n \in \mathbb{N}.$$

In this part, The following propositions [18] is one of the key tools of the proof of our main results.

Proposition 1. *Let* $A = \{a_1, -a_2\}$ *and* $E = \{e_1, e_2, e_3\}$ *two alphabets, we have*

$$\sum_{n=0}^{\infty} S_n(E)S_{n-1}(a_1 + [-a_2])z^n = \frac{\begin{array}{c} -S_1(-E)\,z - (a_1 - a_2)S_2(-E)\,z^2 \\ -\left((a_1 - a_2)^2 + a_1 a_2\right)S_3(-E)\,z^3 \end{array}}{\prod_{e \in E}(1 - e a_1 z) \prod_{e \in E}(1 + e a_2 z)}. \tag{2.5}$$

Proposition 2. *Let* $A = \{a_1, -a_2\}$ *and* $E = \{e_1, e_2, e_3\}$ *two alphabets, we have*

$$\sum_{n=0}^{\infty} S_n(E)S_n(a_1 + [-a_2])z^n = \frac{1 + a_1 a_2 S_2(-E)\,z^2 + a_1 a_2 (a_1 + a_2) S_3(-E)\,z^3}{\prod_{e \in E}(1 - e a_1 z) \prod_{e \in E}(1 + e a_2 z)}. \tag{2.6}$$

From (2.6) we deduce

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_{n-1}(A)z^n = \frac{z + a_1a_2S_2\left(-E\right)z^3 + a_1a_2(a_1-a_2)S_3\left(-E\right)z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)}. \quad (2.7)$$

Proposition 3. *Given two alphabets* $E = \{e_1, e_2, e_3\}$ *and* $A = \{a_1, -a_2\}$, *we have*

$$\sum_{n=0}^{\infty} S_{n-1}(E)S_n(A)z^n = \frac{(a_1-a_2)z - S_1\left(-E\right)a_1a_2z^2 - S_3\left(-E\right)a_1^2a_2^2z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)}. \quad (2.8)$$

From (2.8) we deduce

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_{n-1}(A)z^n = \frac{(a_1+a_2)z^2 - S_1\left(-E\right)a_1a_2z^3 - S_3\left(-E\right)a_1^2a_2^2z^5}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)}. \quad (2.9)$$

Proposition 4. *Given two alphabets* $E = \{e_1, e_2, e_3\}$ *and* $A = \{a_1, -a_2\}$, *we have*

$$\sum_{n=0}^{\infty} S_{n-2}(E)S_n(A)z^n = \frac{\left((a_1-a_2)^2 + a_1a_2\right)z^2 + (a_1-a_2)S_1\left(-E\right)z^3 + S_2\left(-E\right)a_1^2a_2^2z^4}{\prod_{e\in E}(1-ea_1z)\prod_{e\in E}(1+ea_2z)}. \quad (2.10)$$

3 Generating Functions of Binary Products of Tribonacci Polynomials and Special Numbers

In this section, we are going to create the new generating functions of products of Tribonacci polynomials and some numbers (k-Fibonacci, k-Lucas, k-Pell, k-Jacobsthal and Mersenne) based on Propositions 1, 2, 3 and 4.

Theorem 1. *For* $n \in \mathbb{N}$, *the new generating function of the product of Tribonacci polynomials and* k*-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x)F_{k,n}z^n = \frac{1 - xz^2 - kz^3}{1 - kx^2z - \left(\left(k^2+2\right)x + x^4\right)z^2 - \left(k^2+3+x^3\right)kz^3 - \left(k^2+1\right)x^2z^4 + kxz^5 - z^6}. \quad (3.1)$$

Proof. By [16], we have $F_{k,n} = S_n(a_1 + [-a_2])$. Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n &= \sum_{n=0}^{\infty} S_n(E) S_n(a_1 + [-a_2]) z^n \\
&= \frac{1}{(a_1 + a_2)} \left(a_1 \sum_{n=0}^{\infty} S_n(E)(a_1 z)^n + a_2 \sum_{n=0}^{\infty} S_n(E)(-a_2 z)^n \right) \\
&= \frac{1}{(a_1 + a_2)} \left(\frac{a_1}{1 + S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3} \right. \\
&\quad \left. + \frac{a_2}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3} \right),
\end{aligned}$$

by reduce to same denominator, we obtain the following result

$$\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n = \frac{1 + p_1(x) z^2 + p_2(x) z^3}{1 + q_1(x) z + q_2(x) z^2 + q_3(x) z^3 + q_4(x) z^4 + q_5(x) z^5 + q_6(x) z^6},$$

where

$$\begin{aligned}
p_1(x) &= a_1 a_2 S_2(-E), \\
p_2(x) &= a_1 a_2 (a_1 - a_2) S_3(-E),
\end{aligned}$$

and

$$\begin{aligned}
q_1(x) &= (a_1 - a_2) S_1(-E), \\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1 a_2 (S_1(-E)^2 - 2S_2(-E)), \\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1 a_2 (a_1 - a_2)(S_1(-E) S_2(-E) - 3S_3(-E)), \\
q_4(x) &= -a_1 a_2 (a_1 - a_2)^2 S_3(-E) S_1(-E) + a_1^2 a_2^2 (S_2(-E)^2 - 2S_3(-E) S_1(-E)), \\
q_5(x) &= a_1^2 a_2^2 S_3(-E) S_2(-E)(a_1 - a_2), \\
q_6(x) &= -S_3(-E)^2 a_1^3 a_2^3.
\end{aligned}$$

After a simple calculation, of $p_i(x)$ and $q_i(x)$ we obtain

$$\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n = \frac{1 - xz^2 - kz^3}{\begin{array}{r} 1 - kx^2 z - \left(\left(k^2 + 2\right) x + x^4 \right) z^2 - \left(k^2 + 3 + x^3\right) kz^3 \\ - \left(k^2 + 1\right) x^2 z^4 + kxz^5 - z^6 \end{array}}.$$

This completes the proof. □

Theorem 2. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials and k-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x) L_{k,n} z^n = \frac{2 - kx^2 z - \left(2+k^2\right) xz^2 - k\left(k^2+3\right) z^3}{1 - kx^2 z - \left(\left(k^2+2\right) x + x^4\right) z^2 - \left(k^2+3+x^3\right) kz^3 - \left(k^2+1\right) x^2 z^4 + kxz^5 - z^6}. \tag{3.2}$$

Proof. We have [16]

$$L_{k,n} = 2S_n\left(a_1 + [-a_2]\right) - kS_{n-1}\left(a_1 + [-a_2]\right).$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_n(x) L_{k,n} z^n \\
&= \sum_{n=0}^{\infty} S_n(E)\left(2S_n\left(a_1+[-a_2]\right) - kS_{n-1}\left(a_1+[-a_2]\right)\right) z^n \\
&= 2\sum_{n=0}^{\infty} S_n(E)\, S_n\left(a_1+[-a_2]\right) z^n - k\sum_{n=0}^{\infty} S_n(E)\, S_{n-1}\left(a_1+[-a_2]\right) z^n \\
&= 2\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n - \frac{k}{(a_1+a_2)}\left(\sum_{n=0}^{\infty} S_n(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_n(E)(-a_2 z)^n\right) \\
&= 2\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n - \frac{k}{(a_1+a_2)}\left(\frac{1}{1+S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
&\quad \left. - \frac{1}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right),
\end{aligned}$$

after some calculations we find

$$\sum_{n=0}^{\infty} T_n(x) L_{k,n} z^n = 2\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n - k\left(\frac{p_1(x)z - p_2(x)z^2 - p_3(x)z^3}{1 + q_1(x)z + q_2(x)z^2 + q_3(x)z^3 + q_4(x)z^4 + q_5(x)z^5 + q_6(x)z^6}\right),$$

where

$$\begin{aligned}
p_1(x) &= -S_1(-E), \\
p_2(x) &= (a_1 - a_2) S_2(-E), \\
p_3(x) &= ((a_1 - a_2)^2 + a_1 a_2) S_3(-E),
\end{aligned}$$

and

$$
\begin{aligned}
q_1(x) &= (a_1 - a_2)S_1(-E), \\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1a_2(S_1(-E)^2 - 2S_2(-E)), \\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1a_2(a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E)), \\
q_4(x) &= -a_1a_2(a_1 - a_2)^2S_3(-E)S_1(-E) + a_1^2a_2^2(S_2(-E)^2 - 2S_3(-E)S_1(-E)), \\
q_5(x) &= a_1^2a_2^2S_3(-E)S_2(-E)(a_1 - a_2), \\
q_6(x) &= -S_3(-E)^2a_1^3a_2^3.
\end{aligned}
$$

After a simple calculation, of $p_i(x)$ and $q_i(x)$ we obtain (3.2). □

Theorem 3. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials and k-Pell numbers is given by*

$$
\sum_{n=0}^{\infty} T_n(x)P_{k,n}z^n = \frac{x^2z + 2xz^2 + (4+k)\,z^3}{\begin{array}{r}1 - 2x^2z - \left(2x\,(2+k) + kx^4\right)z^2 - 2\left(3k + 4 + kx^3\right)z^3 \\ - k\,(k+4)\,x^2z^4 + 2k^2xz^5 - k^3z^6\end{array}}. \tag{3.3}
$$

Proof. By referred to [16], we have

$$
P_{k,n} = S_{n-1}\left(a_1 + [-a_2]\right).
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} T_n(x)P_{k,n}z^n &= \sum_{n=0}^{\infty} S_n\left(E\right)S_{n-1}\left(a_1 + [-a_2]\right)z^n \\
&= \frac{1}{(a_1+a_2)}\left(\sum_{n=0}^{\infty} S_n(E)(a_1z)^n - \sum_{n=0}^{\infty} S_n(E)(-a_2z)^n\right) \\
&= \frac{1}{(a_1+a_2)}\left(\frac{1}{1 + S_1(-E)a_1z + S_2(-E)a_1^2z^2 + S_3(-E)a_1^3z^3}\right. \\
&\qquad \left. - \frac{1}{1 - S_1(-E)a_2z + S_2(-E)a_2^2z^2 - S_3(-E)a_2^3z^3}\right),
\end{aligned}
$$

make some calculations, we get

$$
\sum_{n=0}^{\infty} T_n(x)P_{k,n}z^n = \frac{p_1(x)z - p_2(x)z^2 - p_3(x)z^3}{1 + q_1(x)z + q_2(x)z^2 + q_3(x)z^3 + q_4(x)z^4 + q_5(x)z^5 + q_6(x)z^6},
$$

where

$$\begin{aligned} p_1(x) &= -S_1(-E), \\ p_2(x) &= (a_1 - a_2) S_2(-E), \\ p_3(x) &= ((a_1 - a_2)^2 + a_1 a_2) S_3(-E), \end{aligned}$$

and

$$\begin{aligned} q_1(x) &= (a_1 - a_2) S_1(-E), \\ q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1 a_2 (S_1(-E)^2 - 2S_2(-E)), \\ q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1 a_2 (a_1 - a_2)(S_1(-E) S_2(-E) - 3S_3(-E)), \\ q_4(x) &= -a_1 a_2 (a_1 - a_2)^2 S_3(-E) S_1(-E) + a_1^2 a_2^2 (S_2(-E)^2 - 2S_3(-E) S_1(-E)), \\ q_5(x) &= a_1^2 a_2^2 S_3(-E) S_2(-E)(a_1 - a_2), \\ q_6(x) &= -S_3(-E)^2 a_1^3 a_2^3. \end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} T_n(x) P_{k,n} z^n = \frac{x^2 z + 2xz^2 + (4+k) z^3}{1 - 2x^2 z - \left(2x(2+k) + kx^4\right) z^2 - 2\left(3k + 4 + kx^3\right) z^3 - k(k+4) x^2 z^4 + 2k^2 x z^5 - k^3 z^6}.$$

This completes the proof. □

Theorem 4. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x) J_{k,n} z^n = \frac{x^2 z + kxz^2 + \left(2 + k^2\right) z^3}{1 - kx^2 z - \left(\left(k^2 + 4\right) x + 2x^4\right) z^2 - k\left(k^2 + 6 + 2x^3\right) z^3 - 2\left(k^2 + 2\right) x^2 z^4 + 4kxz^5 - 8z^6}. \tag{3.4}$$

Proof. Recall that, we have [16]

$$J_{k,n} = S_{n-1}(a_1 + [-a_2]).$$

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n(x) J_{k,n} z^n &= \sum_{n=0}^{\infty} S_n(E)\, S_{n-1}(a_1 + [-a_2])\, z^n \\
&= \frac{1}{(a_1+a_2)}\left(\sum_{n=0}^{\infty} S_n(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_n(E)(-a_2 z)^n\right) \\
&= \frac{1}{(a_1+a_2)}\left(\frac{1}{1+S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
&\quad \left. - \frac{1}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right),
\end{aligned}$$

by reduce to same denominator, we obtain the following result

$$\sum_{n=0}^{\infty} T_n(x) J_{k,n} z^n = \frac{p_1(x)z - p_2(x)z^2 - p_3(x)z^3}{1 + q_1(x)z + q_2(x)z^2 + q_3(x)z^3 + q_4(x)z^4 + q_5(x)z^5 + q_6(x)z^6},$$

where

$$\begin{aligned}
p_1(x) &= -S_1(-E), \\
p_2(x) &= (a_1 - a_2) S_2(-E), \\
p_3(x) &= ((a_1 - a_2)^2 + a_1 a_2) S_3(-E),
\end{aligned}$$

and

$$\begin{aligned}
q_1(x) &= (a_1 - a_2) S_1(-E), \\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1 a_2 (S_1(-E)^2 - 2S_2(-E)), \\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1 a_2 (a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E)), \\
q_4(x) &= -a_1 a_2 (a_1 - a_2)^2 S_3(-E) S_1(-E) + a_1^2 a_2^2 (S_2(-E)^2 - 2S_3(-E)S_1(-E)), \\
q_5(x) &= a_1^2 a_2^2 S_3(-E) S_2(-E)(a_1 - a_2), \\
q_6(x) &= -S_3(-E)^2 a_1^3 a_2^3.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} T_n(x) J_{k,n} z^n = \frac{x^2 z + kxz^2 + \left(2 + k^2\right) z^3}{\begin{array}{r} 1 - kx^2 z - \left(\left(k^2+4\right)x + 2x^4\right) z^2 - k\left(k^2 + 6 + 2x^3\right) z^3 \\ - 2\left(k^2+2\right) x^2 z^4 + 4kxz^5 - 8z^6 \end{array}}.$$

This completes the proof. □

Theorem 5. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci polynomials and Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} T_n(x) M_n z^n = \frac{x^2 z + 3xz^2 + 7z^3}{1 - 3x^2 z - (5x - 2x^4) z^2 - 3(3 - 2x^3) z^3 + 14x^2 z^4 + 12xz^5 + 8z^6}. \tag{3.5}$$

Proof. Recall that, we have [3]

$$M_n = S_{n-1}(a_1 + [-a_2]).$$

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n(x) M_n z^n &= \sum_{n=0}^{\infty} S_n(E) S_{n-1}(a_1 + [-a_2]) z^n \\
&= \frac{1}{(a_1 + a_2)} \left(\sum_{n=0}^{\infty} S_n(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_n(E)(-a_2 z)^n \right) \\
&= \frac{1}{(a_1 + a_2)} \left(\frac{1}{1 + S_1(-E) a_1 z + S_2(-E) a_1^2 z^2 + S_3(-E) a_1^3 z^3} \right. \\
&\quad \left. - \frac{1}{1 - S_1(-E) a_2 z + S_2(-E) a_2^2 z^2 - S_3(-E) a_2^3 z^3} \right),
\end{aligned}$$

by reduce to same denominator, we obtain the following result

$$\sum_{n=0}^{\infty} T_n(x) M_n z^n = \frac{p_1(x) z - p_2(x) z^2 - p_3(x) z^3}{1 + q_1(x) z + q_2(x) z^2 + q_3(x) z^3 + q_4(x) z^4 + q_5(x) z^5 + q_6(x) z^6},$$

where

$$\begin{aligned}
p_1(x) &= -S_1(-E), \\
p_2(x) &= (a_1 - a_2) S_2(-E), \\
p_3(x) &= ((a_1 - a_2)^2 + a_1 a_2) S_3(-E),
\end{aligned}$$

and

$$\begin{aligned}
q_1(x) &= (a_1 - a_2)S_1(-E), \\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1 a_2(S_1(-E)^2 - 2S_2(-E)), \\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1 a_2(a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E)), \\
q_4(x) &= -a_1 a_2(a_1 - a_2)^2 S_3(-E)S_1(-E) + a_1^2 a_2^2(S_2(-E)^2 - 2S_3(-E)S_1(-E)), \\
q_5(x) &= a_1^2 a_2^2 S_3(-E)S_2(-E)(a_1 - a_2), \\
q_6(x) &= -S_3(-E)^2 a_1^3 a_2^3.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} T_n(x) M_n z^n = \frac{x^2 z + 3xz^2 + 7z^3}{1 - 3x^2 z - (5x - 2x^4)z^2 - 3(3 - 2x^3)z^3 + 14x^2 z^4 + 12xz^5 + 8z^6}.$$

This completes the proof. □

4 Generating Functions of Binary Products of Tribonacci Lucas Polynomials and Well-Known Numbers

Based on Propositions 1, 2, 3 and 4, we can state the following theorems which represent the new generating functions of products of Tribonacci Lucas polynomials and some well-known numbers.

Theorem 6. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci Lucas polynomials and k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x) F_{k,n} z^n = \frac{3 - 2kx^2 z - \left(k^2 + 4 + 2x^3\right) xz^2 + k\left(x^3 - 3\right) z^3 - x^2 z^4}{1 - kx^2 z - \left(\left(k^2 + 2\right) x + x^4\right) z^2 - \left(k^2 + 3 + x^3\right) kz^3 - \left(k^2 + 1\right) x^2 z^4 + kxz^5 - z^6}. \tag{4.1}$$

Proof. By [16], we have $F_{k,n} = S_n(a_1 + [-a_2])$. Then, we can see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} K_n(x) F_{k,n} z^n \\
=& \sum_{n=0}^{\infty} \left(3S_n(E) - 2x^2 S_{n-1}(E) - x S_{n-2}(E)\right) S_n(a_1 + [-a_2]) z^n \\
=& 3\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n - \frac{2x^2}{(a_1+a_2)} \left(a_1 \sum_{n=0}^{\infty} S_{n-1}(E)(a_1) z^n + a_2 \sum_{n=0}^{\infty} S_{n-1}(E)(-a_2 z)^n \right) \\
& - \frac{x}{(a_1+a_2)} \left(a_1 \sum_{n=0}^{\infty} S_{n-2}(E)(a_1 z)^n + a_2 \sum_{n=0}^{\infty} S_{n-2}(E)(-a_2 z)^n \right) \\
=& 3\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n - \frac{2x^2}{(a_1+a_2)} \left(\frac{a_1^2 z}{1 + S_1(-E) a_1 z + S_2(-E) a_1^2 z^2 + S_3(-E) a_1^3 z^3} \right. \\
& \left. - \frac{a_2^2 z}{1 - S_1(-E) a_2 z + S_2(-E) a_2^2 z^2 - S_3(-E) a_2^3 z^3} \right) \\
& - \frac{x}{(a_1+a_2)} \left(\frac{a_1^3 z^2}{1 + S_1(-E) a_1 z + S_2(-E) a_1^2 z^2 + S_3(-E) a_1^3 z^3} \right. \\
& \left. + \frac{a_2^3 z^2}{1 - S_1(-E) a_2 z + S_2(-E) a_2^2 z^2 - S_3(-E) a_2^3 z^3} \right),
\end{aligned}$$

by using Proposition 1

then, by reduce to same denominator, we obtain the following result

$$\sum_{n=0}^{\infty} K_n(x) F_{k,n} z^n = 3\sum_{n=0}^{\infty} T_n(x) F_{k,n} z^n - \frac{p_1(x) z + p_2(x) z^2 + p_3(x) z^3 + p_4(x) z^4}{\begin{array}{c}1 + q_1(x) z + q_2(x) z^2 + q_3(x) z^3 \\ + q_4(x) z^4 + q_5(x) z^5 + q_6(x) z^6\end{array}},$$

where

$$\begin{aligned}
p_1(x) &= 2(a_1 - a_2) x^2, \\
p_2(x) &= -x \left(2x S_1(-E) a_1 a_2 - (a_1 - a_2)^2 - a_1 a_2 \right), \\
p_3(x) &= x (a_1 - a_2) S_1(-E), \\
p_4(x) &= -x \left(2x a_1^2 a_2^2 S_3(-E) - a_1^2 a_2^2 S_2(-E) \right),
\end{aligned}$$

and

$$\begin{aligned}
q_1(x) &= (a_1 - a_2)S_1(-E), \\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1a_2(S_1(-E)^2 - 2S_2(-E)), \\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1a_2(a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E)), \\
q_4(x) &= -a_1a_2(a_1 - a_2)^2S_3(-E)S_1(-E) + a_1^2a_2^2(S_2(-E)^2 - 2S_3(-E)S_1(-E)), \\
q_5(x) &= a_1^2a_2^2S_3(-E)S_2(-E)(a_1 - a_2), \\
q_6(x) &= -S_3(-E)^2a_1^3a_2^3.
\end{aligned}$$

After a simple calculation, of $p_i(x)$ and $q_i(x)$ we obtain

$$\sum_{n=0}^{\infty} K_n(x)F_{k,n}z^n = \frac{3 - 2kx^2z - \left(k^2 + 4 + 2x^3\right)xz^2 + k\left(x^3 - 3\right)z^3 - x^2z^4}{1 - kx^2z - \left(\left(k^2 + 2\right)x + x^4\right)z^2 - \left(k^2 + 3 + x^3\right)kz^3 - \left(k^2 + 1\right)x^2z^4 + kxz^5 - z^6}.$$

This completes the proof. □

Theorem 7. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci Lucas polynomials and k-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)L_{k,n}z^n = \frac{6 - 5kx^2z - 2\left(2k^2 + 4 + 2x^3\right)xz^2 - k\left(3k^2 + 9 - x^3\right)z^3 + 2\left(1 + k^2\right)x^2z^4 + kxz^5}{1 - kx^2z - \left(\left(k^2 + 2\right)x + x^4\right)z^2 - \left(k^2 + 3 + x^3\right)kz^3 - \left(k^2 + 1\right)x^2z^4 + kxz^5 - z^6}. \tag{4.2}$$

Proof. We have [16]

$$L_{k,n} = 2S_n\left(a_1 + [-a_2]\right) - kS_{n-1}\left(a_1 + [-a_2]\right).$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} K_n(x) L_{k,n} z^n \\
=\ & \sum_{n=0}^{\infty} \left(3S_n(E) - 2x^2 S_{n-1}(E) - xS_{n-2}(E)\right)\left(2S_n(a_1+[-a_2]) - kS_{n-1}(a_1+[-a_2])\right) z^n \\
=\ & 2\sum_{n=0}^{\infty} \left(3S_n(E) - 2x^2 S_{n-1}(E) - xS_{n-2}(E)\right) S_n(a_1+[-a_2]) z^n \\
& -k\sum_{n=0}^{\infty} \left(3S_n(E) - 2x^2 S_{n-1}(E) - xS_{n-2}(E)\right) S_{n-1}(a_1+[-a_2]) z^n \\
=\ & 2\sum_{n=0}^{\infty} K_n(x) F_{k,n} z^n - \frac{3k}{(a_1+a_2)}\left(\sum_{n=0}^{\infty} S_n(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_n(E)(-a_2 z)^n\right) \\
& + \frac{2kx^2}{(a_1+a_2)}\left(\sum_{n=0}^{\infty} S_{n-1}(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-1}(E)(-a_2 z)^n\right) \\
& + \frac{kx}{(a_1+a_2)}\left(\sum_{n=0}^{\infty} S_{n-2}(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-2}(E)(-a_2 z)^n\right) \\
=\ & 2\sum_{n=0}^{\infty} K_n(x) F_{k,n} z^n + \frac{3k}{(a_1+a_2)}\left(\frac{1}{1+S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
& \left. - \frac{1}{1-S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right) \\
& + \frac{2kx^2}{(a_1+a_2)}\left(\frac{a_1 z}{1+S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
& \left. + \frac{a_2 z}{1-S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right) \\
& + \frac{kx}{(a_1+a_2)}\left(\frac{a_1^2 z^2}{1+S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
& \left. - \frac{a_2^2 z^2}{1-S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right),
\end{aligned}$$

after Lemma 1, by reducing to the same denominator, and make some calculations, we find

$$\sum_{n=0}^{\infty} K_n(x) L_{k,n} z^n = 2\sum_{n=0}^{\infty} K_n(x) F_{k,n} z^n - k\left(\frac{p_1(x)z + p_2(x)z^2 + p_3(x)z^3 + p_4(x)z^4 + p_5(x)z^5}{1 + q_1(x)z + q_2(x)z^2 + q_3(x)z^3 + q_4(x)z^4 + q_5(x)z^5 + q_6(x)z^6}\right),$$

where

$$\begin{aligned}
p_1(x) &= -3S_1(-E) - 2x^2,\\
p_2(x) &= -\left(3(a_1 - a_2)S_2(-E) + x(a_1 - a_2)\right),\\
p_3(x) &= -\left(3((a_1 - a_2)^2 + a_1a_2)S_3(-E) + 2x^2 a_1 a_2 S_2(-E) - x a_1 a_2 S_1(-E)\right),\\
p_4(x) &= -2x^2(a_1 - a_2)a_1a_2S_3(-E),\\
p_5(x) &= x a_1^2 a_2^2 S_3(-E),
\end{aligned}$$

and

$$\begin{aligned}
q_1(x) &= (a_1 - a_2)S_1(-E),\\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1a_2(S_1(-E)^2 - 2S_2(-E)),\\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1a_2(a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E)),\\
q_4(x) &= -a_1a_2(a_1 - a_2)^2S_3(-E)S_1(-E) + a_1^2a_2^2(S_2(-E)^2 - 2S_3(-E)S_1(-E)),\\
q_5(x) &= a_1^2a_2^2S_3(-E)S_2(-E)(a_1 - a_2),\\
q_6(x) &= -S_3(-E)^2a_1^3a_2^3.
\end{aligned}$$

After a simple calculation, of $p_i(x)$ and $q_i(x)$ we obtain (4.2). □

Theorem 8. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci Lucas polynomials and k-Pell numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x) P_{k,n} z^n = \frac{x^2z + 4xz^2 + \left(12 + 3k + kx^3\right)z^3 + 4kx^2z^4 - k^2xz^5}{1 - 2x^2z - \left(2x(2+k) + kx^4\right)z^2 - 2\left(3k + 4 + kx^3\right)z^3 - k(k+4)x^2z^4 + 2k^2xz^5 - k^3z^6}. \tag{4.3}$$

Proof. By referred to [16], we have

$$P_{k,n} = S_{n-1}\left(a_1 + [-a_2]\right).$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} K_n(x) P_{k,n} z^n \\
= &\sum_{n=0}^{\infty} \left(3S_n\left(E\right) - 2x^2 S_{n-1}\left(E\right) - xS_{n-2}\left(E\right)\right) S_{n-1}\left(a_1 + [-a_2]\right) z^n \\
= &3\sum_{n=0}^{\infty} T_n(x) P_{k,n} z^n - \frac{2x^2}{(a_1+a_2)}\left(\sum_{n=0}^{\infty} S_{n-1}(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-1}(E)(-a_2 z)^n\right) \\
&-\frac{x}{(a_1+a_2)}\left(\sum_{n=0}^{\infty} S_{n-2}(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-2}(E)(-a_2 z)^n\right) \\
= &3\sum_{n=0}^{\infty} T_n(x) P_{k,n} z^n - \frac{2x^2}{(a_1+a_2)}\left(\frac{a_1 z}{1 + S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
&\left.-\frac{a_2 z}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right) \\
&-\frac{x}{(a_1+a_2)}\left(\frac{a_1^2 z^2}{1 + S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
&\left.-\frac{a_2^2 z^2}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right),
\end{aligned}$$

after doing a simple calculation, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} K_n(x) P_{k,n} z^n = &\, 3\sum_{n=0}^{\infty} T_n(x) P_{k,n} z^n \\
&- \frac{p_1(x)z + p_2(x)z^2 + p_3(x)z^3 + p_4(x)z^4 + p_5(x)z^5}{1 + q_1(x)z + q_2(x)z^2 + q_3(x)z^3 + q_4(x)z^4 + q_5(x)z^5 + q_6(x)z^6},
\end{aligned}$$

where

$$\begin{aligned}
p_1(x) &= 2x^2, \\
p_2(x) &= x(a_1 - a_2), \\
p_3(x) &= 2x^2 a_1 a_2 S_2\left(-E\right) - x a_1 a_2 S_1(-E), \\
p_4(x) &= 2x^2 a_1 a_2 \left(a_1 - a_2\right) S_3(-E), \\
p_5(x) &= -x a_1^2 a_2^2 S_3(-E),
\end{aligned}$$

and

$$\begin{aligned}
q_1(x) &= (a_1 - a_2)S_1(-E), \\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1a_2(S_1(-E)^2 - 2S_2(-E)), \\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1a_2(a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E)), \\
q_4(x) &= -a_1a_2(a_1 - a_2)^2S_3(-E)S_1(-E) + a_1^2a_2^2(S_2(-E)^2 - 2S_3(-E)S_1(-E)), \\
q_5(x) &= a_1^2a_2^2S_3(-E)S_2(-E)(a_1 - a_2), \\
q_6(x) &= -S_3(-E)^2a_1^3a_2^3.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} K_n(x)P_{k,n}z^n = \frac{x^2z + 4xz^2 + \left(12 + 3k + kx^3\right)z^3 + 4kx^2z^4 - k^2xz^5}{\begin{array}{r}1 - 2x^2z - \left(2x\left(2+k\right) + kx^4\right)z^2 - 2\left(3k + 4 + kx^3\right)z^3 \\ - k\left(k+4\right)x^2z^4 + 2k^2xz^5 - k^3z^6\end{array}}.$$

This completes the proof. □

Theorem 9. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci Lucas polynomials and k-Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x)J_{k,n}z^n = \frac{x^2z + 2kxz^2 + \left(2x^3 + 3k^2 + 6\right)z^3 + 4kx^2z^4 - 4xz^5}{\begin{array}{r}-kx^2z - \left(\left(k^2+4\right)x + 2x^4\right)z^2 - k\left(k^2 + 6 + 2x^3\right)z^3 \\ 1 - 2\left(k^2+2\right)x^2z^4 + 4kxz^5 - 8z^6\end{array}}. \tag{4.4}$$

Proof. Recall that, we have [16]

$$J_{k,n} = S_{n-1}\left(a_1 + [-a_2]\right).$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} K_n(x) J_{k,n} z^n \\
&= \sum_{n=0}^{\infty} \left(3S_n(E) - 2x^2 S_{n-1}(E) - xS_{n-2}(E)\right) S_{n-1}(a_1 + [-a_2]) z^n \\
&= 3\sum_{n=0}^{\infty} T_n(x) J_{k,n} z^n - \frac{2x^2}{(a_1+a_2)} \left(\sum_{n=0}^{\infty} S_{n-1}(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-1}(E)(-a_2 z)^n \right) \\
&\quad - \frac{x}{(a_1+a_2)} \left(\sum_{n=0}^{\infty} S_{n-2}(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-2}(E)(-a_2 z)^n \right) \\
&= 3\sum_{n=0}^{\infty} T_n(x) J_{k,n} z^n - \frac{2x^2}{(a_1+a_2)} \left(\frac{a_1 z}{1 + S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3} \right. \\
&\quad \left. - \frac{a_2 z}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3} \right) \\
&\quad - \frac{x}{(a_1+a_2)} \left(\frac{a_1^2 z^2}{1 + S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3} \right. \\
&\quad \left. - \frac{a_2^2 z^2}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3} \right),
\end{aligned}$$

by reducing to the same denominator, we obtain the following result

$$\begin{aligned}
\sum_{n=0}^{\infty} K_n(x) J_{k,n} z^n &= 3\sum_{n=0}^{\infty} T_n(x) J_{k,n} z^n \\
&\quad + \frac{p_1(x)z + p_2(x)z^2 + p_3(x)z^3 + p_4(x)z^4 + p_5(x)z^5}{1 + q_1(x)z + q_2(x)z^2 + q_3(x)z^3 + q_4(x)z^4 + q_5(x)z^5 + q_6(x)z^6},
\end{aligned}$$

where

$$\begin{aligned}
p_1(x) &= -2x^2, \\
p_2(x) &= -x(a_1 - a_2), \\
p_3(x) &= -\left(2x^2 a_1 a_2 S_2(-E) - a_1 a_2 x S_1(-E)\right), \\
p_4(x) &= -2x^2 a_1 a_2 (a_1 - a_2) S_3(-E), \\
p_5(x) &= x a_1^2 a_2^2 S_3(-E),
\end{aligned}$$

and

$$\begin{aligned}
q_1(x) &= (a_1 - a_2)S_1(-E), \\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1 a_2 (S_1(-E)^2 - 2S_2(-E)), \\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1 a_2 (a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E)), \\
q_4(x) &= -a_1 a_2 (a_1 - a_2)^2 S_3(-E)S_1(-E) + a_1^2 a_2^2 (S_2(-E)^2 - 2S_3(-E)S_1(-E)), \\
q_5(x) &= a_1^2 a_2^2 S_3(-E)S_2(-E)(a_1 - a_2), \\
q_6(x) &= -S_3(-E)^2 a_1^3 a_2^3.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} K_n(x) J_{k,n} z^n = \frac{x^2 z + 2kxz^2 + \left(2x^3 + 3k^2 + 6\right) z^3 + 4kx^2 z^4 - 4xz^5}{1 - kx^2 z - \left(\left(k^2 + 4\right) x + 2x^4\right) z^2 - k\left(k^2 + 6 + 2x^3\right) z^3 - 2\left(k^2 + 2\right) x^2 z^4 + 4kxz^5 - 8z^6}.$$

This completes the proof. □

Theorem 10. *For $n \in \mathbb{N}$, the new generating function of the product of Tribonacci Lucas polynomials and Mersenne numbers is given by*

$$\sum_{n=0}^{\infty} K_n(x) M_n z^n = \frac{x^2 z + 6xz^2 + \left(21 - 2x^3\right) z^3 - 12x^2 z^4 - 4xz^5}{1 - 3x^2 z - \left(5x - 2x^4\right) z^2 - 3\left(3 - 2x^3\right) z^3 + 14x^2 z^4 + 12xz^5 + 8z^6}. \tag{4.5}$$

Proof. Recall that, we have [3]

$$M_n = S_{n-1}\left(a_1 + [-a_2]\right).$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} K_n(x) M_n z^n \\
&= \sum_{n=0}^{\infty} \left(3S_n(E) - 2x^2 S_{n-1}(E) - x S_{n-2}(E)\right) S_{n-1}(a_1 + [-a_2]) z^n \\
&= 3\sum_{n=0}^{\infty} T_n(x) M_n z^n - \frac{2x^2}{(a_1+a_2)} \left(\sum_{n=0}^{\infty} S_{n-1}(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-1}(E)(-a_2 z)^n\right) \\
&\quad - \frac{x}{(a_1+a_2)} \left(\sum_{n=0}^{\infty} S_{n-2}(E)(a_1 z)^n - \sum_{n=0}^{\infty} S_{n-2}(E)(-a_2 z)^n\right) \\
&= 3\sum_{n=0}^{\infty} T_n(x) M_n z^n - \frac{2x^2}{(a_1+a_2)} \left(\frac{a_1 z}{1 + S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
&\quad \left. - \frac{a_2 z}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right) \\
&\quad - \frac{x}{(a_1+a_2)} \left(\frac{a_1^2 z^2}{1 + S_1(-E)a_1 z + S_2(-E)a_1^2 z^2 + S_3(-E)a_1^3 z^3}\right. \\
&\quad \left. - \frac{a_2^2 z^2}{1 - S_1(-E)a_2 z + S_2(-E)a_2^2 z^2 - S_3(-E)a_2^3 z^3}\right),
\end{aligned}$$

by reducing to the same denominator, we obtain the following result

$$\begin{aligned}
\sum_{n=0}^{\infty} K_n(x) M_n z^n &= 3\sum_{n=0}^{\infty} T_n(x) M_n z^n \\
&\quad + \frac{p_1(x)z + p_2(x)z^2 + p_3(x)z^3 + p_4(x)z^4 + p_5(x)z^5}{1 + q_1(x)z + q_2(x)z^2 + q_3(x)z^3 + q_4(x)z^4 + q_5(x)z^5 + q_6(x)z^6},
\end{aligned}$$

where

$$\begin{aligned}
p_1(x) &= -2x^2, \\
p_2(x) &= -x(a_1 - a_2), \\
p_3(x) &= -\left(2x^2 a_1 a_2 S_2(-E) - x a_1 a_2 S_1(-E)\right), \\
p_4(x) &= -2x^2 a_1 a_2 (a_1 - a_2) S_3(-E), \\
p_5(x) &= x a_1^2 a_2^2 S_3(-E),
\end{aligned}$$

and

$$\begin{aligned}
q_1(x) &= (a_1 - a_2)S_1(-E), \\
q_2(x) &= S_2(-E)(a_1 - a_2)^2 - a_1a_2(S_1(-E)^2 - 2S_2(-E)), \\
q_3(x) &= S_3(-E)(a_1 - a_2)^3 - a_1a_2(a_1 - a_2)(S_1(-E)S_2(-E) - 3S_3(-E)), \\
q_4(x) &= -a_1a_2(a_1 - a_2)^2S_3(-E)S_1(-E) + a_1^2a_2^2(S_2(-E)^2 - 2S_3(-E)S_1(-E)), \\
q_5(x) &= a_1^2a_2^2S_3(-E)S_2(-E)(a_1 - a_2), \\
q_6(x) &= -S_3(-E)^2a_1^3a_2^3.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} K_n(x)M_nz^n = \frac{x^2z + 6xz^2 + \left(21 - 6x^3\right)z^3 - 12x^2z^4 - 4xz^5}{1 - 3x^2z - (5x - 2x^4)z^2 - 3(3 - 2x^3)z^3 + 14x^2z^4 + 12xz^5 + 8z^6}.$$

This completes the proof. □

5 Conclusion

In this paper, the new theorems has been proposed in order to determine the generating functions. The proposed theorems is based on the symmetric fonctions. The obtained results agree with the results obtained in some previous works.

Acknowledgment. Sincere thanks are due to the referee for his/her careful reading of the manuscript and for his/her valuable comments. This work was supported by Directorate General for Scientific Research and Technological Development (DGRSDT), Algeria.

References

[1] A. Boussayoud and S. Boughaba, On some identities and generating functions for k-Pell sequences and Chebyshev polynomials, *Online J. Anal. Comb.* 14(3) (2019), 1-13.

[2] A. Boussayoud and A. Abderrezzak, Complete homogeneous symmetric functions and Hadamard product, *Ars Comb.* 144 (2019), 81-90.

[3] A. Boussayoud, M. Chelgham and S. Boughaba, On some identities and generating functions for Mersenne numbers and polynomials, *Turkish Journal of Analysis and Number Theory* 6(3) (2018), 93-97. https://doi.org/10.12691/tjant-6-3-5

[4] A. Boussayoud, M. Bolyer and M. Kerada, On some identities and symmetric functions for Lucas and Pell numbers, *Electron. J. Math. Analysis Appl.* 5 (2017), 202-207.

[5] A. Boussayoud, M. Kerada and N. Harrouche, On the k-Lucas numbers and Lucas polynomials, *Turkish Journal of Analysis and Number* 5 (2017), 121-125. https://doi.org/10.12691/tjant-5-4-1

[6] A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers, *Online. J. Anal. Comb.* 12(1) (2017), 1-10.

[7] A. Boussayoud, M. Kerada and M. Boulyer, A simple and accurate method for determination of some generalized sequence of numbers, *Int. J. Pure Appl. Math.* 108 (2016), 503-511.

[8] A. Boussayoud, M. Kerada and R. Sahali, Symmetrizing operations on some orthogonal polynomials, *Int. Electron. J. Pure Appl. Math.* 9 (2015), 191-199. http://dx.doi.org/10.12732/iejpam.v9i3.8

[9] A. Boussayoud and R. Sahali, The application of the operator $L_{b_1 b_2}^{-k}$ in the series $\sum_{j=0}^{\infty} a_j b_1^j z^j$, *J. Adv. Res. Appl. Math.* 7 (2015), 68-75.

[10] A. Boussayoud and M. Kerada, Symmetric and generating functions, *Int. Electron. J. Pure Appl. Math.* 7 (2014), 195-203.

[11] A. Pintér and H.M. Srivastava, Generating functions of the incomplete Fibonacci and Lucas numbers, *Rend. Circ. Mat. Palermo* 48 (1999), 591-596. https://doi.org/10.1007/BF02844348

[12] D. Tasci and M. Cetin Firengiz, Incomplete Fibonacci and Lucas p-numbers, *Math. Comput. Model.* 52 (2010), 1763-1770. https://doi.org/10.1016/j.mcm.2010.07.003

[13] G.B. Djordjevic, Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers, *Fibonacci Q.* 42 (2004), 106-113.

[14] G.B. Djordjevic and H.M. Srivastava, Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers, *Math. Comput. Model.* 42 (2005), 1049-1056. https://doi.org/10.1016/j.mcm.2004.10.026

[15] P. Catarino, On some identities and generating functions for k-Pell numbers, *Int. J. Math. Anal.* 7 (2013), 1877-1884. http://dx.doi.org/10.12988/ijma.2013.35131

[16] S. Boughaba, A. Boussayoud and M. Kerada, Construction of symmetric functions of generalized Fibonacci numbers, *Tamap Journal of Mathematics and Statistics* 3 (2019), 1-7. https://doi.org/10.29371/2019.16.SI01

[17] S. Falcon and A. Plaza, On the Fibonacci k-numbers, *Chaos Solit. Fract.* 32 (2007), 1615-1624. https://doi.org/10.1016/j.chaos.2006.09.022

[18] H. Merzouk, A. Boussayoud and M. Chelgham, Symmetric functions of binary products of Tribonacci Lucas numbers and orthogonal polynomials, *J. Sci. Arts* 2(55) (2021), 461-478. https://doi.org/10.46939/J.Sci.Arts-21.2-a13

[19] H. Merzouk, B. Aloui and A. Boussayoud, Generating functions of the products of 2-orthogonal Chebyshev polynomials with some numbers and the other Chebyshev polynomials, *Issues of Analysis* 10(28)(1) (2021), 17-33. https://doi.org/10.15393/j3.art.2021.8530

[20] H. Merzouk, A. Boussayoud and M. Chelgham, Generating functions of generalized Tribonacci and Tricobsthal polynomials, *Montes Taurus J. Pure Appl. Math.* 2(2) (2020), 7-37.

[21] H. Merzouk, A. Boussayoud and M. Chelgham, Symmetric functions of generalized polynomials of second order, *Turkish Journal of Analysis and Number* 7(5) (2019), 135. https://doi.org/10.12691/tjant-7-5-2

[22] Jr. V.E. Hoggatt and M. Bicknell, Generalized Fibonacci polynomials, *Fibonacci Quarterly* 11 (1973), 457-465.

[23] N. Saba and A. Boussayoud, Complete homogeneous symmetric functions of Gauss Fibonacci polynomials and bivariate Pell polynomials, *Open J. Math. Sci.* 4 (2020), 179-185. https://doi.org/10.30538/oms2020.0108

[24] N. Yilmaz and N. Taskara, Incomplete Tribonacci-Lucas numbers and polynomials, arXiv:1404.4168v1 [math.NT] (16 Apr 2014).

[25] Y. Ben Cheikh and N. Ben Romdhane, d-orthogonal polynomial sets of Chebyshev type, in: *Proceedings of the International Conference on Difference Equations, Special Functions and Orthogonal Polynomials*, S. Elaydi et al. (Ed.), Munich, Germany, 25-30 July 2005, World Scientific.
https://doi.org/10.1142/9789812770752_0008