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New Family of Multivalent Analytic Functions Defined on Complex Hilbert Space

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Abstract

In this article, we define a certain new class of multivalent analytic functions with negative coefficients on complex Hilbert space. We derive a number of important geometric properties, such as, coefficient estimates, radii of starlikeness and convexity, extreme points and convex set.

1. Introduction

Let $\mathcal{A}_p$ indicate the family of functions f of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Also, let S_p denote the subclass of $\mathcal{A}_p$ consisting of functions of the form

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$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (a_{n+p} \geq 0,\ p \in \mathbb{N} = \{1, 2, \ldots\}). \tag{1.2}$$

Denote by H the Hilbert space on the complex field and T by a linear operator on H. For a complex analytic function f on the open unit disk U. The operator on H, indicated by $f(T)$, is defined by the well-known Riesz-Dunford integral [2].

$$f(T) = \frac{1}{2\pi i} \int_c f(z)(zI - T)^{-1} dz,$$

where I is the identity operator on H, c is a positively oriented simple closed rectifiable contour lying in U and containing the spectrum $\sigma(T)$ of T in its interior domain [3]. Also $f(T)$ can be defined by the series

$$f(T) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} T^n,$$

which converges in the norm topology [4].

Now, we define the class $WS_p(\delta, \alpha, \beta, T)$ consisting of the functions $f \in S_p$ such that

$$\left\| \frac{2\delta(\alpha^p - \beta^p) T f'(T)}{f(\alpha T) - f(\beta T)} - p \right\| < p, \tag{1.3}$$

where $0 < \delta < 1$, $-1 \leq \beta < \alpha \leq 1$, $\alpha \neq 0$, $\beta \neq 0$, $p \in \mathbb{N}$ and for all operator T with $\| T \| < 1$, $T \neq \varnothing$ ($\varnothing$ denote the zero operator on H).

The operator on Hilbert space were considered recently by Yu [8], Joshi [6], Kim et al. [1], Ghanim and Darus [5], Selvaraj et al. [7] and Wanas and Frasin [9].

2. Main Results

Theorem 2.1. *Let* $f \in S_p$ *be given by* (1.2). *Then* $f \in WS_p(\delta, \alpha, \beta, T)$ *for all* $T \neq \varnothing$ *if and only if*

$$\sum_{n=1}^{\infty} [\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})] a_{n+p}$$

$$\leq \frac{p}{2}(\alpha^p - \beta^p)(1 - \|2\delta - 1\|). \tag{2.1}$$

where $0 < \delta < 1,\ -1 \leq \beta < \alpha \leq 1,\ \alpha \neq 0,\ \beta \neq 0,\ p \in \mathbb{N}.$

The result is sharp for the function given by

$$f(z) = z^p - \frac{p(\alpha^p - \beta^p)(1 - \|2\delta - 1\|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]} z^{n+p}, \quad n \geq 1. \tag{2.2}$$

Proof. Let the inequality (2.1) holds. It is enough to show that

$$\|2\delta(\alpha^p - \beta^p)Tf'(T) - p(f(\alpha T) - f(\beta T))\| < p\|f(\alpha T) - f(\beta T)\|.$$

We consider

$$\|2\delta(\alpha^p - \beta^p)Tf'(T) - p(f(\alpha T) - f(\beta T))\| - p\|f(\alpha T) - f(\beta T)\|$$

$$= \Bigg\| p(2\delta - 1)(\alpha^p - \beta^p)T^p$$

$$- \sum_{n=1}^{\infty}[2\delta(n+p)(\alpha^p - \beta^p) - p(\alpha^{n+p} - \beta^{n+p})]a_{n+p}T^{n+p} \Bigg\|$$

$$- p\left\| (\alpha^p - \beta^p)T^p - \sum_{n=1}^{\infty}(\alpha^{n+p} - \beta^{n+p})a_{n+p}T^{n+p} \right\|$$

$$\leq p\|2\delta - 1\|(\alpha^p - \beta^p)\|T\|^p$$

$$+ \sum_{n=1}^{\infty}[2\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]a_{n+p}\|T\|^{n+p}$$

$$- p(\alpha^p - \beta^p)\|T\|^p + p\sum_{n=1}^{\infty}(\alpha^{n+p} - \beta^{n+p})a_{n+p}\|T\|^{n+p}$$

$$\leq \sum_{n=1}^{\infty} 2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]a_{n+p}$$

$$- p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|) \le 0.$$

Therefore, $f \in WS_p(\delta, \alpha, \beta, T)$.

To show the converse, let $f \in WS_p(\delta, \alpha, \beta, T)$. Then

$$\| 2\delta(\alpha^p - \beta^p)Tf'(T) - p(f(\alpha T) - f(\beta T)) \| < p \| f(\alpha T) - f(\beta T) \|,$$

gives

$$\left\| p(2\delta - 1)(\alpha^p - \beta^p)T^p - \sum_{n=1}^{\infty} [2\delta(n+p)(\alpha^p - \beta^p) - p(\alpha^{n+p} - \beta^{n+p})] a_{n+p} T^{n+p} \right\|$$

$$< p \left\| (\alpha^p - \beta^p)T^p - \sum_{n=1}^{\infty} (\alpha^{n+p} - \beta^{n+p}) a_{n+p} T^{n+p} \right\|.$$

Setting $T = rI$ $(0 < r < 1)$ in the above inequality, we have

$$\frac{p|2\delta - 1|(\alpha^p - \beta^p)r^p + \sum_{n=1}^{\infty} [2\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})] a_{n+p} r^{n+p}}{(\alpha^p - \beta^p)r^p - \sum_{n=1}^{\infty} (\alpha^{n+p} - \beta^{n+p}) a_{n+p} r^{n+p}} < p. \tag{2.3}$$

Upon clearing denominator in (2.3) and letting $r \to 1$, we obtain

$$p|2\delta - 1|(\alpha^p - \beta^p) + \sum_{n=1}^{\infty} [2\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})] a_{n+p}$$

$$< p(\alpha^p - \beta^p)r^p - p \sum_{n=1}^{\infty} (\alpha^{n+p} - \beta^{n+p}) a_{n+p}.$$

Thus

$$\sum_{n=1}^{\infty} [\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})] a_{n+p} \le \frac{p}{2}(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|),$$

which completes the proof.

Corollary 2.1. *If* $f \in WS_p(\delta, \alpha, \beta, T)$, *then*

$$a_{n+p} \le \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}, \quad n \ge 1.$$

Theorem 2.2. *If* $f \in WS_p(\delta, \alpha, \beta, T)$ *and* $\| T \| < 1, T \neq \emptyset$, *then*

$$\| T \|^p - \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(p+1)(\alpha^p - \beta^p) + p(\alpha^{p+1} - \beta^{p+1})]} \| T \|^{p+1} \le \| f(T) \|$$

$$\le \| T \|^p + \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(p+1)(\alpha^p - \beta^p) + p(\alpha^{p+1} - \beta^{p+1})]} \| T \|^{p+1}$$

and

$$p\| T \|^{p-1} - \frac{p(n+p)(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]} \| T \|^p \le \| f'(T) \|$$

$$\le p\| T \|^{p-1} + \frac{p(n+p)(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]} \| T \|^p.$$

Proof. According to Theorem 2.1, we have

$$\sum_{n=1}^{\infty} a_{n+p} \le \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(p+1)(\alpha^p - \beta^p) + p(\alpha^{p+1} - \beta^{p+1})]}.$$

Thus

$$\| f(T) \| \ge \| T \|^p - \sum_{n=1}^{\infty} a_{n+p} \| T \|^{n+p} \ge \| T \|^p - \| T \|^{p+1} - \sum_{n=1}^{\infty} a_{n+p}$$

$$\ge \| T \|^p - \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(p+1)(\alpha^p - \beta^p) + p(\alpha^{p+1} - \beta^{p+1})]} \| T \|^{p+1}.$$

Also,

$$\| f(T) \| \le \| T \|^p + \sum_{n=1}^{\infty} a_{n+p} \| T \|^{n+p}$$

$$\leq \|T\|^p + \frac{p(\alpha^p - \beta^p)(1 - \|2\delta - 1\|)}{2[\delta(p+1)(\alpha^p - \beta^p) + p(\alpha^{p+1} - \beta^{p+1})]}\|T\|^{p+1}.$$

In the light of Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty}(n+p)a_{n+p} \leq \frac{p(n+p)(\alpha^p - \beta^p)(1 - \|2\delta - 1\|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}.$$

Hence

$$\|f'(T)\| \geq p\|T\|^{p-1} - \sum_{n=1}^{\infty}(n+p)a_{n+p}\|T\|^{n+p-1}$$

$$\geq p\|T\|^{p-1} - \|T\|^p \sum_{n=1}^{\infty}(n+p)a_{n+p}$$

$$\geq p\|T\|^{p-1} - \frac{p(n+p)(\alpha^p - \beta^p)(1 - \|2\delta - 1\|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}\|T\|^p$$

and

$$\|f'(T)\| \leq p\|T\|^{p-1} + \|T\|^p \sum_{n=1}^{\infty}(n+p)a_{n+p}$$

$$\leq p\|T\|^{p-1} + \frac{p(n+p)(\alpha^p - \beta^p)(1 - \|2\delta - 1\|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}\|T\|^p.$$

Therefore the proof is complete.

Theorem 2.3. *If* $f \in WS_p(\delta, \alpha, \beta, T)$, *then f will be multivalent starlike of order* $\theta(0 \leq \theta < p)$ *in the disk* $|z| < r_1$, *where*

$$r_1 = \inf_n \left\{ \frac{2(p-\theta)[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}{p(n+p-\theta)(\alpha^p - \beta^p)(1 - \|2\delta - 1\|)} \right\}^{\frac{1}{n}}, \quad (n \geq 1).$$

The result is sharp for the function f given by (2.2).

Proof. It is sufficient to show that

$$\left\| \frac{Tf'(T)}{f(T)} - p \right\| \leq p - \theta. \tag{2.4}$$

We get

$$\left\| \frac{Tf'(T)}{f(T)} - p \right\| \leq \frac{\sum_{n=1}^{\infty} n a_{n+p} \| T \|^n}{1 - \sum_{n=1}^{\infty} a_{n+p} \| T \|^n}.$$

Hence (2.4) will be satisfied if

$$\sum_{n=1}^{\infty} \left(\frac{n + p - \theta}{p - \theta} \right) a_{n+p} \| T \|^n \leq 1. \tag{2.5}$$

In the light of Theorem 2.1, if $f \in WS_p(\delta, \alpha, \beta, T)$, then

$$\sum_{n=1}^{\infty} \frac{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)} a_{n+p} \leq 1. \tag{2.6}$$

By making use of (2.6), we see that (2.5) holds true if

$$\frac{n + p - \theta}{p - \theta} \| T \|^n \leq \frac{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}$$

or equivalently

$$\| T \| \leq \left\{ \frac{2(p - \theta)[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}{p(n + p - \theta)(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)} \right\}^{\frac{1}{n}}.$$

This gives the desired result.

Theorem 2.4. *If* $f \in WS_p(\delta, \alpha, \beta, T)$, *then* f *is multivalent convex of order* θ $(0 \leq \theta < p)$ *in the disk* $|z| < r_2$, *where*

$$r_2 = \inf_n \left\{ \frac{2(p-\theta)[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}{(n+p)(n+p-\theta)(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)} \right\}^{\frac{1}{n}}, \ (n \geq 1).$$

The result is sharp for the function f given by (2.2).

Proof. It is sufficient to show that

$$\left\| \frac{Tf''(T)}{f'(T)} + 1 - p \right\| \leq p - \theta.$$

The result follows by application of arguments similar to the proof of Theorem 2.3.

Theorem 2.5. *Let* $f_0(z) = z^p$ *and*

$$f_n(z) = z^p - \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]} z^{n+p}, \ n \geq 1.$$

Then $f \in WS_p(\delta, \alpha, \beta, T)$ *if and only if it can be expressed in the form*:

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z), \tag{2.7}$$

where $\lambda_n \geq 0$ *and* $\sum_{n=0}^{\infty} \lambda_n = 1$.

Proof. Assume that *f* can be expressed by (2.7). Then, we find that

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \lambda_0 f_0(z) + \sum_{n=0}^{\infty} \lambda_n f_n(z)$$

$$= z^p - \sum_{n=1}^{\infty} \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]} \lambda_n z^{n+p}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}$$

$$\times \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]} \lambda_n$$

$$= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1,$$

and so $f \in WS_p(\delta, \alpha, \beta, T)$.

Conversely, suppose that f given by (1.2) is in the class $WS_p(\delta, \alpha, \beta, T)$. Then by Corollary 2.1, we have

$$a_{n+p} \leq \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}.$$

Putting

$$\lambda_n = \frac{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]}{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)} a_n, \quad n \geq 1,$$

and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$. Then

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$$

$$= z^p - \sum_{n=1}^{\infty} \frac{p(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)}{2[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]} \lambda_n z^{n+p}$$

$$= z^p - \sum_{n=1}^{\infty} (z^p - f_n(z))\lambda_n = \left(1 - \sum_{n=1}^{\infty} \lambda_n\right) z^p + \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

$$= \lambda_0 f_0(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z),$$

which completes the proof.

Theorem 2.6. *The class* $WS_p(\delta, \alpha, \beta, T)$ *is a convex set.*

Proof. Let f_1 and f_2 be the arbitrary elements of $WS_p(\delta, \alpha, \beta, T)$. Then for every $t\,(0 \leq t \leq 1)$, we show that $(1-t)f_1 + tf_2 \in WS_p(\delta, \alpha, \beta, T)$. Thus, we obtain

$$(1-t)f_1 + tf_2 = z^p - \sum_{n=1}^{\infty}((1-t)a_{n+p} + tb_{n+p})z^{n+p}.$$

Hence

$$\sum_{n=1}^{\infty}[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]((1-t)a_{n+p} + tb_{n+p})$$

$$= (1-t)\sum_{n=1}^{\infty}[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]a_{n+p}$$

$$+ t\sum_{n=1}^{\infty}[\delta(n+p)(\alpha^p - \beta^p) + p(\alpha^{n+p} - \beta^{n+p})]b_{n+p}$$

$$\leq \frac{p}{2}(1-t)(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|) + \frac{p}{2}t(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|)$$

$$= \frac{p}{2}(\alpha^p - \beta^p)(1 - \| 2\delta - 1 \|).$$

This completes the proof.

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