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Hankel Determinant Problem for q-strongly Close-to-Convex Functions

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Abstract

In this paper, we introduce a new class $K_q(\alpha)$, $\quad 0 < \alpha \leq 1, \quad 0 < q < 1$, of normalized analytic functions f such that $\left|\arg \frac{D_q f(z)}{D_q g(z)}\right| \leq \alpha \frac{\pi}{2}$, where g is convex univalent in $E = \{z : |z| < 1\}$ and $D_q f$ is the q-derivative of f defined as:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0 \quad D_q f(0) = f'(0).$$

The problem of growth of the Hankel determinant $H_n(k)$ for the class $K_q(\alpha)$ is investigated. Some known interesting results are pointed out as applications of the main results.

1 Introduction and Preliminary Results

Let f be a univalent function defined for $z \in E = \{z :| z |< 1\}$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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Let $C(\beta), S^*(\beta), \quad 0 \leq \beta < 1$ be the subclasses of univalent functions which are respectively convex and starlike of order β. Let $K(\beta)$ be a class of strongly close-to-convex functions of order α in the sense of Pommerenke [19].

A function f, analytic in E and given by (1.1) belongs to $K(\alpha)$, if and only if, there exists $g \in S^*$ such that

$$\left|arg\frac{zf'f(z)}{g(z)}\right| \leq \frac{\alpha\pi}{2}, \quad \forall z \in E, \quad \alpha \geq 0. \tag{1.2}$$

It is obvious that $K(0) = C, \quad K(1) = K$, which is the class of close-to-convex functions introduced by Kaplan [6] and it consists of univalent functions, whereas $f \in K(\alpha), \quad \alpha > 1$ may be of infinite valence, see [2].

The concept of q-derivative has been introduced and studied in [5, 7] for $0 < q < 1$ as follows:

$$D_qf(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0, \quad \text{and} \quad D_qf(0) = f'(0).$$

Then, from (1.1), we have

$$D_qf(z) = \frac{1}{z}\left[z + \sum_{n=2}^{\infty}[n]_q a_n z^n\right], \tag{1.3}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \quad n = 2, 3, ...$$

When

$$q \to 1^-, \quad [n]_q \to n, \quad \text{as} \quad \lim_{q\to 1^-}\frac{1-q^n}{1-q} = n.$$

The class S_q^* of q-starlike functions was introduced in [4] and has been studied in [7, 8, 9, 10, 12, 13, 14, 15, 16, 20, 21, 22].

Agarwal and Sahoo [1] defined and considered the class $S_q^*(\beta), \quad 0 \leq \beta < 1$.

$$f \in S_q^*(\beta), \quad \text{if} \quad \left|\frac{zD_qf(z)}{f(z)} - \frac{1-\beta q}{1-q}\right| < \frac{1-\beta}{1-q}, \quad z \in E.$$

If $q \to 1^-$, then $S_q^*(\beta)$ reduces to the class $S^*(\beta)$ and also $S_q^*(0) = S_q^*$.

Definition 1.1. Let f be analytic in E and be given by (1.1). Then f is said to belong to the class $K_q(\alpha)$, $\quad 0 < \alpha \leq 1$, if there exists $g \in C$ such that

$$\left|arg \frac{D_q f(z)}{D_q g(z)}\right| \leq \frac{\alpha\pi}{2}.$$

We call the class $K_q(\alpha)$, the class of q-strongly close-to-convex functions.

When $q \to 1^-$, we have the class $K(\alpha)$ of strongly close-to-convex functions, defined by (1.2).

Lemma 1.1. *[17] Let $g \in C$. Then, $\forall q \in (0,1)$, $\; zD_q g$ is in the class $S^*(\beta)$, $\quad \beta = \left(\frac{1-q}{2(1+q)}\right)$.*

Lemma 1.2. *[18] Let $\theta_1 < \theta_2 < .... < \theta_k < \theta_1 + 2\pi$ and $\lambda_1, \lambda_2, ...\lambda_k$ be real, $\lambda > 0, \quad \lambda \geq \lambda_j \quad (j = 1, 2, ..., k)$. If*

$$\Psi(z) = \Pi_{j=1}^{k}\left(1 - e^{i\theta_j} z\right)^{-\lambda_j} = \sum_{n=1}^{\infty} b_n z^n, \tag{1.4}$$

then

$$b_n = O(1) n^{\lambda-1}, \quad as \quad n \to \infty.$$

Lemma 1.3. *[8] Let $p : p(z) 1 + c_1 z + c_2 z^2 + ...$ be analytic in E with $Re\{p(z)\} > 0$, $z \in E$. Then, for $z = re^{i\theta}$,*

$$\int_0^{2\pi} \left|p(re^{i\theta})\right| d\theta < c(\lambda) \frac{1}{(1-r)^{\lambda-1}},$$

where $\lambda > 1$ and $c(\lambda)$ is a constant depending only on λ.

2 Main Results

Theorem 2.1. *Let $f \in K_q(\alpha)$ and be given by (1.1). Then*

$$[n]_q a_n = O(1) n^{\gamma}, \quad \gamma = 2(1-\beta) + \alpha - 1, \quad (n \to \infty).$$

Proof. By Cauchy Theorem, we have

$$\begin{aligned} [n]_q\big|a_n\big| &= \frac{1}{2\pi r^{n+1}}\left|\int_0^{2\pi} zD_qf(z)e^{in\theta}d\theta\right|, \quad z=re^{i\theta} \\ &\le \frac{1}{2\pi r^{n+1}}\int_0^{2\pi}\Big|zD_qg(z)h(z)^{\alpha}(z)\Big|d\theta, \end{aligned} \tag{2.1}$$

where $g \in C, \quad Re\{h(z)\} > 0$.

Now, using Lemma 1.1, together with a well-known result, we can write

$$D_qg(z) = \left(\frac{G(z)}{z}\right)^{1-\beta}, \quad G \in S^*. \tag{2.2}$$

Thus, from (2.1) and (2.2), we have

$$\begin{aligned} [n]_q\big|a_n\big| &\le \frac{1}{2\pi r^{n+1-\beta}}\int_0^{2\pi}\big|G(z)\big|^{(1-\beta)}\big|h(z)\big|^{\alpha}d\theta \\ &\le \frac{1}{r^{n+1-\beta}}\left(\frac{1}{2\pi}\int_0^{2\pi}\big|G(z)\big|^{(1-\beta)(\frac{2}{2-\alpha})}d\theta\right)^{\frac{2-\alpha}{2}}\Big(\frac{1}{2\pi}\int_0^{2\pi}\big|h(z)\big|^2d\theta\Big)^{\frac{\alpha}{2}} \\ &\le c_1\left(\frac{1}{1-r}\right)^{2(1-\beta)+\alpha-1}, \quad (r\to 1), \end{aligned} \tag{2.3}$$

where we have used Holder's inequality, Lemma 1.3 and subordination for starlike functions. Taking $r = 1-\frac{1}{n}$ in (2.3), we obtain the required result. □

We have the following special cases.

Corollary 2.1. *For $q \to 1^-$, f belongs to the class $K(\alpha)$ of strongly close-to-convex and this gives*

$$a_n = O(1)n^{\alpha-2}, \quad (n\to\infty).$$

Corollary 2.2. *Let $f \in K_q$ with $\alpha = 1$. Then it follows from Theorem 2.1,*

$$a_n = O(1)\big[(\frac{1-q}{1-q^n})\big]n^{2(1-\beta)}, \quad \beta = \frac{1-q}{2(1+q)}.$$

We now discuss the Hankel determinant problem for $f \in K_q(\alpha)$.

Let $f \in K_q(\alpha)$ be given by (1.1). The k^{th} Hankel determinant of f is defined for $k \geq 1$, $n \geq 1$ by

$$H_k(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+k-1} & \cdots & \cdots & a_{n+2k-2} \end{vmatrix}. \tag{2.4}$$

This problem has been solved in [18] for starlike functions and investigated by Noor [11] for the class K of close-to-convex functions.

By using a modified version of Pommerenke method [18], the rate of growth of $H_k(n)$ for the class $K_q(\alpha)$ will be discussed.

Remark 2.1. From Definition 1.1 and Lemma 1.1, it follows that for $f \in K_q(\alpha)$, we can write

$$zD_q f(z) = \big(G(z)\big)^{1-\beta} h^{\alpha}(z), \quad G \in S^*, \quad Re\{h(z)\} > 0, \quad \beta = \frac{(1-q)}{2(1+q)}, \quad \alpha \in (0,1]. \tag{2.5}$$

Also, $G \in S^*$ can be represented as

$$G(z) = zexp\Big[\int_0^{2\pi} log\frac{1}{(1-ze^{it})} d\mu(t)\Big],$$

where $\mu(t)$ is an increasing function and $\mu(2\pi) - \mu(0) = 2$.

Let $\alpha_1 \geq \alpha_2 \geq ...$, be the jumps of $\mu(t)$ and $t = \theta_1, \theta_2, ...$ be the values at which these jumps occur. We may assume that $\theta_1 = 0$. Then $\alpha_1 + \alpha_2 + ... \leq 2$ and $\alpha_1 + \alpha_2 + ... + \alpha_n = 2$, for some k, if and only if, G is of the form

$$G(z) = z\Pi_{j=1}^{k}\big(1 - e^{i\theta_j} z\big)^{\frac{-2}{k}}. \tag{2.6}$$

In [18], the following there cases are considered and for each case η_m are defined as:

(i) $0 \leq \alpha_1 \leq 1$, and $\eta_m = \alpha_{m+1} (m = 0, 1, 2, ...)$.

(ii) $1 < \alpha_1 < \frac{3}{2}$ and $\eta_0 = \alpha_1, \quad \eta_1 = \max(\alpha_1 - 1, \alpha_2), \quad \eta_2 = \max(\alpha_1 - 1, \alpha_2), \quad \eta_m = \alpha_3 \quad \text{for} \quad m \geq 3.$

(iii) $\frac{3}{2} \leq \alpha_1 \leq 2$ and $\eta_0 = \alpha_1, \quad \eta_1 = \max(\alpha_1 - 1, \alpha_2), \quad \eta_m = \alpha_m (m \geq 2).$

We first prove the following.

Theorem 2.2. *Let $f \in K_q(\alpha), \quad \alpha \in (\frac{1}{2}, 1)$. Then, for $m = 0, 1, 2, ...$, there are numbers γ_m and $c_{m\mu} (\mu = 0, 1, ..., m)$ that satisfy $|c_{c0}| = |c_{mm}| = 1$, and*

$$\sum_{\nu=0}^{\infty} \gamma_\nu \leq 3, \quad 0 \leq \gamma_m \leq \frac{2}{m+1} \tag{2.7}$$

such that

$$\sum_{\mu=0}^{\infty} [n]_q c_{m\mu} a_{n+\mu} = O(1) n^{\{\gamma_m (1-\beta) + \alpha - 1\}}, \quad (n \to \infty),$$

where G in equation (2.5) is of the form equation (2.6) and $\beta = \frac{1-q}{2(1+q)}$. The bounds (2.7) are the best possible, see [18].

Proof. We write

$$\phi_m(z) = \sum_{\mu=0}^{m} c_{m\mu} z^{m-\mu},$$

and

$$\phi_m(z)\big(zD_q f(z)\big) = \sum_{n=1}^{m} b_{mn} z^{m+n} + \sum_{n=1}^{\infty} [n+m]_q a_{mn} z^{m+n}, \tag{2.8}$$

where

$$\begin{aligned} b_{mn} &= \sum_{\nu=0}^{n} [n+\nu]_q c_{m-\nu} a_{n-\nu}, \\ a_{mn} &= \sum_{\mu=0}^{m} c_{m\mu} a_{n+\mu}, \quad |c_{m0}| = |c_{mm}| = 1. \end{aligned}$$

We shall consider the case, where g in (2.5) is of the form given in (2.6), that is, $\alpha_1 + \alpha_2 + ... + \alpha_k = 2$ with $\gamma_m = \eta_m$. It follows that $\gamma_m = \frac{2}{m+1}$, $\gamma_0 + \gamma_1 + ... + \gamma_k \leq 3$ and $\gamma_m = \frac{2}{m+1}$ implies that $m = k - 1$, $\alpha_1 = \alpha_2 = ... + \alpha_k$.

Now, from (2.5), (2.8), Cauchy integral formula and $z = re^{i\theta}$, we have

$$\begin{aligned} \left[n+m\right]_q \left|a_{mn}\right| &\leq \frac{1}{2\pi r^{n+m}} \int_0^{2\pi} \left|\phi_m(z) G(z)^{1-\beta)} h^{\alpha}(z)\right| d\theta \\ &\leq \frac{(4)^{\beta}}{r^{n+m+\beta}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|\phi(z) G(z) h^{\alpha}(z)\right| d\theta \right), \end{aligned} \tag{2.9}$$

where we have used the well-known distortion result for the starlike function G.

Applying Schwartz inequality, it follows from (2.9) that

$$\left[n+m\right]\left|a_{mn}\right| \leq \frac{(4)^{\beta}}{r^{n+m+\beta}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|\phi_m(z) G(z)\right|^2 d\theta \right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|h(z)\right|^{2\alpha} d\theta \right)^{\frac{1}{2}}. \tag{2.10}$$

When, we write $\left[\phi_m(z) G(z)\right]^2$ in the form 1.4, the exponent $(-\lambda_j)$ satisfies $\lambda_j \leq 2(1-\beta)\gamma_m$ $(j = 1, 2, ..., k)$. Hence by using Lemma 1.2, we have

$$\int_0^{2\pi} \left|\phi_m(z) G(z)\right|^2 d\theta \leq c_1 n^{\{2(1-\beta)\gamma_m - 1\}}. \tag{2.11}$$

From (2.11), (2.10) and Lemma 1.3, we have

$$\left[n+m\right]_q \left|a_{mn}\right| \leq c_2 n^{\{\gamma_m(1-\beta)+\alpha-1\}}, \quad (n \to \infty).$$

where c_1 and c_2 are constants. This proves the result. □

The following result for the Hankel determinant $H_k(n)$ can now easily be proved.

Theorem 2.3. *Let* $f \in K_q(\alpha)$, $\alpha \in (\frac{1}{2}, 1]$, $q \in (0, 1)$. *Then*

$$H_k(n) = O(1) \left[\left(\frac{1-q}{1-q^n} \right)^k \right] n^{\{2(1-\beta) - k(1-\alpha)\}}, \quad k \geq 1,$$

where G *in* (2.5) *is of the form* (2.6).

Following well-known results can obtained as special cases of Theorem 2.3.

(i) When $q \to 1^-$, $H_k(n) = O(1)n^{2\{2-k(2-\alpha)\}}$.

(ii) With $\alpha = 1$, $q \to 1^-$, $f \in K$, we have $H_k(n) = O(1)n^{(2-k)}$, $(n \to \infty)$. This result has been proved in [11].

(iii) Also, for $k = 2$, $q \to 1^-$, we have $H_2(n) = O(1)n^{2(\alpha-1)}$.

Conclusion

In this paper, we have introduced a new class $K_q(\alpha)$ of q-strongly close-to-convex functions using the q-differential operator. We have investigated the problem of the rate of growth of the Hankel determinant for the class $K_q(\alpha)$. Several special cases are discussed as applications of the derived results. The ideas and techniques of this paper may stimulate further research in this interesting field.

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