Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 3, Number 1, 2020, Pages 139-153
https://doi.org/10.34198/ejms.3120.139153

Subclass of Harmonic Univalent Functions Associated with the Generalized Mittag-Leffler Type Functions

Adnan Ghazy Alamoush

Faculty of Science, Taibah University, Saudi Arabia

Abstract

In the present paper, we introduce a new subclass of harmonic functions in the unit disc U defined by using the generalized Mittag-Leffler type functions. Coefficient conditions, extreme points, distortion bounds, convex combination are studied.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply complex domain D is said to be harmonic in D. In any simply connected domain, we can write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$.

Clunie and Sheil-Small [7] introduced a class SH of complex valued harmonic maps f which are univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ and assume a normalized representation $f = h + \overline{g}$, where $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in SH$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

Received: October 14, 2019; Accepted: December 12, 2019

2010 Mathematics Subject Classification: 30C45.

Keywords and phrases: analytic functions, univalent function, differential operator, differential subordination, convex functions, structural formula.

Later on, Sheil-Small [9] investigated the class *SH* as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on *SH* and its subclasses. Connectivity of geometric functions and hypergeometric functions with harmonic functions is seen through some of these papers ([6], [4], [5], [3], [2], [1]). The Mittag-Leffler and generalized Mittag-Leffler type functions was first introduced by the Swedish mathematician Mittag-Leffler [8] and also studied by Wiman [14]. It is a special function of $z \in C$ which depends on the complex parameter α and is defined by the power series

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \; \alpha \in C, \; R(\alpha) > 0, \; z \in C. \tag{2}$$

A first generalization of $E_{\alpha}(z)$ introduced by Wiman [14], is the two-parametric M-L function of $z \in C$, defined by the series

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \; \alpha, \beta \in C, \; R(\alpha) > 0, \; R(\beta) > 0, \; z \in C. \tag{3}$$

Prabhakar [10] introduced a three-parametric generalization of $\psi_{\alpha, \beta}^{\gamma}(z)$ defined in (3) as a kernel of certain fractional differential equations in terms of the series

$$E_{\alpha, \beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\beta + \alpha k)}, \; \alpha, \beta \in C, \; R(\alpha) > 0, \; R(\beta) > 0, \; z \in C. \tag{4}$$

Due to its integral representation $E_{\alpha, \beta}^{\gamma}(z)$ is considered as a special case of Fox's H-function as well as of Wright's generalized hypergeometric ${}_p\Psi_q$, so called Fox-Wright psi function of $z \in C$. Further extensions of the M-L function to four parameters, Salim [12] introduced the function in the form $\psi_{\alpha, \beta}^{\gamma}(z)$ in the following form

$$E_{\alpha, \beta}^{\gamma, \delta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\beta + \alpha k)(\delta)_k}, \tag{5}$$

where $\alpha, \beta, \gamma, \delta \in C, \; \min(R(\alpha), R(\beta) > 0, R(\gamma), R(\delta) > 0), \; z \in C$. Recently, Salim and Faraj [13] introduced a new generalization of Mittag-Leffler function associated with Weyl fractional integral and differential operators as follow

$$E_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{\Gamma(\beta + \alpha k)(\delta)_{pk}}, \tag{6}$$

where $\alpha, \beta, \gamma, \delta \in C$, $\min(R(\alpha), R(\beta) > 0, R(\gamma), R(\delta) > 0)$, $z \in C$, with $q, p \in \mathbb{R}_+$, $q \leq \Re(\alpha) + p$, and $(\gamma)_{pn}$ denotes an extended variant of the Pochhammer symbol, defined by $(\gamma)_{qn} = \Gamma(\gamma + qn)/\Gamma(\gamma)$.

Corresponding to $E_{\alpha, \beta, p}^{\gamma, \delta, q}(z)$, we define the function $\Theta_{\alpha, \beta, p}^{\gamma, \delta, q}(z)$ by

$$\Theta_{\alpha, \beta, p}^{\gamma, \delta, q}(z) = z * E_{\alpha, \beta, p}^{\gamma, \delta, q}(z)$$

$$= z + \sum_{k=2}^{\infty} \frac{(\gamma)_{q(k-1)}}{\Gamma[\beta + \alpha(k-1)](\delta)_{p(k-1)}} z^k.$$

Now, for $f \in A, m \in \mathbb{N}$, we define the following differential operator: $\Phi_{\gamma, \delta, q, \alpha, \beta, p}^m f : A \to A$ by

$$\Psi_{\gamma, \delta, q, \alpha, \beta, p}^0 f(z) = f(z) * \Theta_{\alpha, \beta, p}^{\gamma, \delta, q}(z),$$

$$\Psi_{\gamma, \delta, q, \alpha, \beta, p}^1 f(z) = z[f(z) * \Theta_{\alpha, \beta, p}^{\gamma, \delta, q}(z)]',$$

$$\vdots$$

$$\vdots$$

$$(m+1)\Psi_{\gamma, \delta, q, \alpha, \beta, p}^{m+1} f(z) = z[\Phi_{\gamma, \delta, q, \alpha, \beta, p}^m] + m\Psi_{\gamma, \delta, q, \alpha, \beta, p}^m, \quad z \in U.$$

Thus it is obvious to see from above that

$$\Psi_{\gamma, \delta, q, \alpha, \beta, p}^m f(z) = z + \sum_{k=2}^{\infty} \frac{(m+1)_{k-1}}{(k-1)!} \frac{(\gamma)_{q(k-1)}}{\Gamma[\beta + \alpha(k-1)](\delta)_{p(k-1)}} a_k z^k. \tag{7}$$

Note that, when $\alpha = 0$, $\beta = \gamma = \delta = 1$, we get Ruscheweyh Operator [11].

Throughout this section, unless otherwise stated, we shall use the notation

$$\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p} = \frac{(m+1)_{k-1}}{(k-1)!}\frac{(\gamma)_{q(k-1)}}{\Gamma[\beta+\alpha(k-1)](\delta)_{p(k-1)}}.$$

Involving the generalized Mittag-Leffler function as defined in (6), for $0 \le \eta < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in U$, let $SH(m, n, \eta)$ denote the family of harmonic functions *f* of the form (1.1) such that

$$\Re\left(\frac{\Psi^{m}_{\gamma,\delta,q,\alpha,\beta,p}}{\Psi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}\right) > \eta, \tag{8}$$

where $\Psi^{m}_{\gamma,\delta,q,\alpha,\beta,p} = \Psi^{m}_{\gamma,\delta,q,\alpha,\beta,p}h(z) + (-1)^{m}\overline{\Psi^{m}_{\gamma,\delta,q,\alpha,\beta,p}\, g(z)}.$

We let the subclass $\overline{S}H(m, n, \eta)$ consist of harmonic functions $f_m = h + \overline{g}_m$ in $\overline{S}H(m, n, \eta)$ so that *h* and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ g_m(z) = (-1)^{m-1}\sum_{k=1}^{\infty} b_k z^k. \tag{9}$$

The class $\overline{S}H(m, n, \eta)$ includes a variety of well-known subclasses of *SH*.

The object of this paper is to examine some generalized Mittag-Leffler function inequalities as a necessary and sufficient condition for univalent harmonic analytic functions associated with certain generalized Mittag-Leffler function to be in the function class $SH(m, n, \eta)$. The coefficient condition for the function class $SH(m, n, \eta)$ is given. Furthermore, we determine extreme points, a distortion theorem, convolution conditions and convex combinations for the functions *f* in $\overline{S}H(m, n, \eta)$.

2. Coefficient Bound

We begin with a sufficient coefficient condition for functions *f* in $SH(m, n, \eta)$.

Theorem 2.1. *Let* $f = h + \overline{g}$ *be given by* (1). *If*

$$\sum_{k=1}^{\infty}\left(\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}| a_k |\right.$$

$$+\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|b_k|\Bigg)\leq 2. \tag{10}$$

Proof. If $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|}$$

$$\geq 1-\frac{\sum_{k=1}^{\infty}\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}}{1-\sum_{k=2}^{\infty}\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|a_k|}\geq 0,$$

which proves univalence. Note that *f* is sense-preserving in *U*. This is because

$$|h'(z)|=1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1}$$

$$>1-\sum_{k=2}^{\infty}\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|a_k|$$

$$\geq \sum_{k=1}^{\infty}\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|b_k|$$

$$>\sum_{k=1}^{\infty}\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|b_k||z^k|$$

$$\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1}\geq |g'(z)|. \tag{11}$$

Using the fact that $\Re(w) > \eta$ if and only if $|1 - \eta + w| \geq |1 + \eta - w|$, it suffices to show that

$$|(1-\eta)\Psi^n_{\gamma,\delta,q,\alpha,\beta,p}(z) + \Psi^m_{\gamma,\delta,q,\alpha,\beta,p}(z)|$$
$$-|(1+\eta)\Psi^n_{\gamma,\delta,q,\alpha,\beta,p}(z) - \Psi^m_{\gamma,\delta,q,\alpha,\beta,p}(z)| > 0. \tag{12}$$

Substituting the value of $\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}(z)$ and $\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}(z)$ in (11) yields, by (9), that

$$|(1-\eta)\Psi^n_{\gamma,\delta,q,\alpha,\beta,p} + \Psi^m_{\gamma,\delta,q,\alpha,\beta,p}| - |(1+\eta)\Psi^n_{\gamma,\delta,q,\alpha,\beta,p} - \Psi^m_{\gamma,\delta,q,\alpha,\beta,p}| > 0$$

$$= |(2-\eta)z + \sum_{k=2}^{\infty}[\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}(1-\eta) + \Phi^m_{\gamma,\delta,q,\alpha,\beta,p}]a_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty}[\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}(1-\eta) + (-1)^{m-n}\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}]\overline{b_k z^k}$$

$$- |-\eta z + \sum_{k=2}^{\infty}[\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (1+\eta)\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}]a_k z^k|$$

$$- (-1)^n \sum_{k=1}^{\infty}[(-1)^{m-n}\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (1+\eta)\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}]\overline{b_k z^k}\,|$$

$$\geq 2(1-\eta)|z| - \sum_{k=2}^{\infty} 2[\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}]|a_k||z|^k$$

$$- \sum_{k=1}^{\infty}[(1+\eta)\Phi^n_{\gamma,\delta,q,\alpha,\beta,p} + (-1)^{m-n}\eta\Phi^m_{\gamma,\delta,q,\alpha,\beta,p}]|b_k||z|^k$$

$$- \sum_{k=1}^{\infty}|[(-1)^{m-n}\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (1-\eta)\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}]|b_k||z|^k$$

$$= 2(1-\eta)|z|\left\{1 - \sum_{k=2}^{\infty}\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|a_k||z|^{k-1}\right.$$

$$\left. - \sum_{k=1}^{\infty}\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|b_k||z|^{k-1}\right\}$$

$$> 2(1-\eta)|z|\left\{1 - \left(\sum_{k=2}^{\infty}\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|a_k|\right.\right.$$

$$+\sum_{k=1}^{\infty}\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|b_k|\Bigg)\Bigg)\Bigg\}.$$

This last expression is non-negative by (10), and so the proof is complete.

The harmonic function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\eta}{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}x_k z^k$$

$$+\sum_{k=1}^{\infty}\frac{1-\eta}{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}\overline{y_k z^k}, \tag{13}$$

where $m\in\mathbb{N}$, $n\in\mathbb{N}_0$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$ shows that the coefficient bound given by (10) is sharp. The functions f of the form (13) is $f\in SH(m,n,\eta)$, because

$$\sum_{k=1}^{\infty}\left(\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|a_k|+\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|b_k|\right)$$

$$=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

In the following theorem, it is shown that the condition (10) is also necessary for functions $f_m=h+\overline{g_m}$, where h and g_m are of the form (9).

Theorem 2.2. *Let* $f_m=h+\overline{g_m}$ *be given by* (9). *Then* $f_m\in\overline{S}H(m,n,\eta)$ *if and only if*

$$\sum_{k=1}^{\infty}[(\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p})|a_k|+(\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p})|b_k|]$$

$$\le 2(1-\eta). \tag{14}$$

Proof. Since $\overline{S}H(m,n,\eta)\subset\overline{S}H(m,n,\eta)$ we only need to prove the "only if" part of Theorem 2.2. To this end, for functions f of the form (9), we notice that the condition

(8) is equivalent to

$$\Re\left\{\frac{\begin{array}{c}(1-\eta)z-\sum_{k=2}^{\infty}(\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{m}-\eta\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{n})a_k z^k \\ +(-1)^{2m-1}\sum_{k=1}^{\infty}(\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{m}-(-1)^{m-n}\eta\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{n})b_k\bar{z}^k\end{array}}{z-\sum_{k=2}^{\infty}\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{n}a_k z^k+(-1)^{m+n-1}\sum_{k=1}^{\infty}\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{n}b_k\bar{z}^k}\right\}\geq 0. \quad (15)$$

The above condition (15) must hold for all values of z on the positive real axes, where, $0 \leq |z| = \mu < 1$, we have

$$\frac{\begin{array}{c}1-\eta-\sum_{k=2}^{\infty}(\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{m}-\eta\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{n})a_k\mu^{k-1} \\ -\sum_{k=1}^{\infty}(\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{m}-(-1)^{m-n}\eta\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{n})b_k\mu^{k-1}\end{array}}{1-\sum_{k=2}^{\infty}\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{n}a_k\mu^{k-1}+(-1)^{m-n}\sum_{k=1}^{\infty}\Phi_{\gamma,\delta,q,\alpha,\beta,p}^{n}b_k\mu^{k-1}}\geq 0. \quad (16)$$

If the condition (14) does not hold, then the numerator in (17) is negative for sufficiently close to 1. Hence there exists a $z_0 = \mu_0$ in $(0, 1)$ for which the quotient in (17) is negative. This contradicts the condition for $f_m \in \overline{S}H(m, n, \eta)$ and so the proof is complete.

3. Distortion Bounds

In this section, we obtain distortion bounds for functions f in $\overline{S}H(m, n, \eta)$.

Theorem 3.1. *Let $f_m \in \overline{S}H(m, n, \eta)$. Then for $|z| < 1$, we have*

$$|f_m(z)| \leq (1+|b_1|)\mu + \frac{1}{[\Upsilon_2]^m}\left(\frac{(1-\eta)}{[\Upsilon_2]^{m-n}-\eta} - \frac{1-(-1)^{m-n}\eta}{[\Upsilon_2]^{m-n}-\eta}|b_1|\right)\mu^2,$$

and

$$|f_m(z)| \geq (1-|b_1|)\mu - \frac{1}{[\Upsilon_2]^m}\left(\frac{(1-\eta)}{[\Upsilon_2]^{m-n}-\eta} - \frac{1-(-1)^{m-n}\eta}{[\Upsilon_2]^{m-n}-\eta}|b_1|\right)\mu^2,$$

where $[\Upsilon_2]^m = (m+1)\left(\dfrac{(\gamma)_q}{\Gamma[\beta+\alpha](\delta)_p}\right).$

Proof. We only prove the left-hand inequality. The proof for the right-hand inequality is similar and is thus omitted. Let $f_m \in \overline{S}H_{\alpha,\beta,p,n}^{\gamma,\delta,q,m}(\eta)$. Taking the absolute value of f_m, we have

$$|f_m(z)| = \left| z - \sum_{k=2}^{\infty} a_k z^k + (-1)^k \sum_{k=1}^{\infty} b_k \bar{z}^k \right|$$

$$\leq (1+|b_1|)\mu + \sum_{k=2}^{\infty} (|a_k| + |b_k|)\mu^n$$

$$\leq (1+|b_1|)\mu + \sum_{k=2}^{\infty} (|a_k| + |b_k|)\mu^2$$

$$= (1+|b_1|)\mu + \frac{1-\eta}{[\Upsilon_2]^m[\Upsilon_2]^{m-n} - \eta}$$

$$\times \sum_{k=2}^{\infty} \frac{[\Upsilon_2]^m([\Upsilon_2]^{m-n} - \eta)}{1-\eta}(|a_k| + |b_k|)\mu^2$$

$$\leq (1+|b_1|)\mu + \frac{(1-\eta)\mu^2}{[\Upsilon_2]^m([\Upsilon_2]^{m-n} - \eta)}$$

$$\times \sum_{k=2}^{\infty} \left(\frac{\Phi_{\gamma,\delta,q,\alpha,\beta,p}^m - \eta\Phi_{\gamma,\delta,q,\alpha,\beta,p}^n}{1-\eta} |a_k| \right.$$

$$\left. + \frac{\Phi_{\gamma,\delta,q,\alpha,\beta,p}^m - (-1)^{m-n}\eta\Phi_{\gamma,\delta,q,\alpha,\beta,p}^n}{1-\eta} |b_k| \right)$$

$$\leq (1+|b_1|)\mu + \frac{(1-\eta)}{[\Upsilon_2]^m([\Upsilon_2]^{m-n} - \eta)} \left(\frac{\Phi_{\gamma,\delta,q,\alpha,\beta,p}^m - \eta\Phi_{\gamma,\delta,q,\alpha,\beta,p}^n}{1-\eta} |a_k| \right.$$

$$\left. + \frac{\Phi_{\gamma,\delta,q,\alpha,\beta,p}^m - (-1)^{m-n}\eta\Phi_{\gamma,\delta,q,\alpha,\beta,p}^n}{1-\eta} |b_k| \right) \mu^2.$$

The following covering result follows from the left hand inequality in Theorem 3.1.

4. Convolution, Convex Combinations and Extreme Points

In this section, we show the class $SH(m, n, \eta)$ is invariant under convolution and convex combination.

For harmonic functions *f* of the form

$$f_m(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{m-1} \sum_{k=1}^{\infty} |b_k| \bar{z}^k$$

and

$$F_m(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^{m-1} \sum_{k=1}^{\infty} |B_k| \bar{z}^k$$

we define the convolution of $f_m(z)$ and $F_m(z)$ as

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{k=2}^{\infty} |a_k||A_k| z^k + (-1)^k \sum_{k=1}^{\infty} |b_k||B_k| \bar{z}^k. \quad (17)$$

Theorem 4.1. *For* $0 \le \rho \le \eta < 1,$ *let* $f_m \in \bar{S}H(m, n, \eta)$ *and* $F_m \in \bar{S}(m, n, \rho).$ *Then the convolution*

$$f_m * F_m \in \bar{S}H(m, n, \eta) \subset \bar{S}(m, n, \rho).$$

Proof. Then the convolution $f_m * F_m$ is given by (17). We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 4.1. For $F_m \in \bar{S}H(m, n, \rho),$ we note that $|A_k| \le 1$ and $|B_k| \le 1.$ Now, for the convolution function $f_m * F_m,$ we obtain

$$\sum_{k=2}^{\infty} \frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \rho\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\rho} |a_k||A_k|$$

$$+ \sum_{k=1}^{\infty} \frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (-1)^{m-n}\rho\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\rho} |b_k||B_k|$$

$$\le \sum_{k=2}^{\infty} \frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \rho\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\rho} |a_k|$$

$$+\sum_{k=1}^{\infty}\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\rho\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\rho}|b_k|$$

$$\leq\sum_{k=2}^{\infty}\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|a_k|$$

$$+\sum_{k=1}^{\infty}\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|b_k|$$

$$\leq 1.$$

Since $0 \leq \rho \leq \eta < 1$, and $f_m \in \overline{S}H(m, n, \eta)$, $f_m * F_m \in \overline{S}H(m, n, \eta) \subset \overline{S}H(m, n, \rho)$.

Next, we discuss the convex combinations of the class $\overline{S}H(m, n, \eta)$.

Theorem 4.2. *The family* $\overline{S}H(m, n, \eta)$ *is closed under convex combination.*

Proof. For $i = 1, 2, \ldots,$ suppose that $f_{m,j} \in \overline{S}H(m, n, \eta)$, where

$$f_{m,j}(z) = z - \sum_{k=2}^{\infty}|a_{k,j}|z^k + (-1)^{m-1}\sum_{k=1}^{\infty}|b_{k,j}|\bar{z}^k. \tag{18}$$

Then by Theorem 2.2,

$$\sum_{k=1}^{\infty}\left(\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}a_{k,j}\right.$$

$$\left.+\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}b_{k,j}\right)\leq 2. \tag{19}$$

For $\sum_{j=1}^{\infty}t_j = 1$, $0 \leq t_j < 1$, the convex combination of $f_{m,j}$ may be written as

$$\sum_{j=1}^{\infty}t_j f_{m,j}(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{\infty}t_j|a_{k,j}|\right)z^k$$

$$+(-1)^{m-1}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}t_j|b_{k,j}|\right)\bar{z}^k.$$

Then by (4.3),

$$\sum_{k=1}^{\infty}\left(\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}\sum_{j=1}^{\infty}t_j a_{k,j}\right.$$

$$\left.+\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}\sum_{j=1}^{\infty}t_j b_{k,j}\right)$$

$$=\sum_{i=1}^{\infty}t_i\left\{\sum_{k=1}^{\infty}\left(\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}a_{k,j}\right.\right.$$

$$\left.\left.+\frac{\Phi^{m}_{\gamma,\delta,q,\alpha,\beta,p}-(-1)^{m-n}\eta\Phi^{n}_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}b_{k,j}\right)\right\}$$

$$\leq 2\sum_{j=1}^{\infty}t_j = 2,$$

and therefore

$$\sum_{j=1}^{\infty}t_j f_{m,j}(z)\in \bar{S}H(m, n, \eta).$$

Corollary 4.3. *The class* $\bar{S}H(m, n, \eta)$ *is closed under convex linear combinations.*

Proof. Let the functions $f_{m,j}(z)$ $(j = 1, 2, ..., m)$ defined by (4.2) be in the class $\bar{S}H(m, n, \eta)$. Then the function $\varpi(z)$ defined by

$$\varpi(z) = \mu f_{m,j}(z) + (1-\mu) f_{m,j}(z),\ \ 0 \leq \mu \leq 1, \tag{20}$$

is in the class $\bar{S}H(m, n, \eta)$.

Next, we determine the extreme points of closed convex hulls of $SH(m, n, \eta)$, denoted by *clco* $SH(m, n, \eta)$.

Theorem 4.4. *Let* f_m *be given by* (10). *Then* $f_m \in \bar{S}H(m, n, \eta)$ *if and only if*

$$f_m(z) = \sum_{k=1}^{\infty}[X_k h_k(z) + Y_k g_{m_k}(z)], \tag{21}$$

where

$$h_1(z) = z,\ h_k(z) = z - \frac{1-\eta}{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}} z^k,\ \ (k = 2, \ldots),$$

$$g_{m_k}(z) = z + (-1)^{m-1} \frac{1-\eta}{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}} \bar{z}^k,\ \ (k = 1, 2, \ldots),$$

$X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \geq 0,\ X_k \geq 0,\ Y_k \geq 0.$ *In particular, the extreme points of* $SH(m, n, \eta)$ *are* $\{h_k\}$ *and* $\{g_{m_k}\}$.

Proof. For functions f_m of the form (21) we have

$$f_m(z) = \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_{m_k}(z)]$$

$$= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1-\eta}{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}} X_k z^k$$

$$+ (-1)^{m-1} \sum_{n=1}^{\infty} \frac{1-\eta}{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}} Y_k \bar{z}^k.$$

Then

$$\sum_{k=2}^{\infty} \left(\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta} \right) \frac{1-\eta}{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}} X_k -$$

$$\sum_{k=1}^{\infty} \left(\frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta} \right) \frac{1-\eta}{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}} Y_k$$

$$= \sum_{k=2}^{\infty} X_n - \sum_{k=1}^{\infty} Y_n = 1 - X_1 \leq 1, \tag{22}$$

and so $f_m \in clco\, SH(m, n, \eta)$.

Conversely, suppose that $f_m \in clco\, SH(m, n, \eta)$. Setting

$$X_k = \frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - \eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta} |a_k|,\ \ 0 \leq X_k \leq 1,\ \ k = 2, \ldots,$$

$$Y_k = \frac{\Phi^m_{\gamma,\delta,q,\alpha,\beta,p} - (-1)^{m-n}\eta\Phi^n_{\gamma,\delta,q,\alpha,\beta,p}}{1-\eta}|b_k|, \ 0 \le Y_k \le 1, \ k = 1, 2, \ldots,$$

and $X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$, and note that, by Theorem 2.2, $X_1 \ge 0$. Consequently, we obtain $f_m(z) = \sum_{k=1}^{\infty}(h_k(z)X_k + g_{m_k}(z)Y_k)$, as required.

Using Corollary 4.3 we have *clco* $SH(m, n, \eta) = \overline{G}_S H(m, n, \eta)$. Then the statement of Theorem 4.4 is true for $f \in \overline{G}_S H(m, n, \rho)$.

5. Conclusion

In this paper, using the Hadamard product or convolution to define a new differential operator involving generalized Mittag-Leffler function. Also, we defined new subclass of univalent functions and established some interesting properties.

References

[1] O. P. Ahuja, Harmonic starlikeness and convexity of integral operators generated by hypergeometric series, *Integral Transforms Spec. Funct.* 20(8) (2009), 629-641. https://doi.org/10.1080/10652460902734124

[2] O. P. Ahuja, Planar harmonic convolution operators generated by hypergeometric functions, *Integral Transforms Spec. Funct.* 18(3) (2007), 165-177. https://doi.org/10.1080/10652460701210227

[3] O. P. Ahuja, Planar harmonic univalent and related mappings, *J. Inequal. Pure Appl. Math.* 6(4) (2005), 1-18.

[4] A. G. Al Amoush and M. Darus, Harmonic starlike functions with respect to symmetric points, *Matematika* 32 (2016), 121-131.

[5] A. G. Alamoush and M. Darus, On subclass of harmonic univalent functions associated with convolution of derivative operator, *Bull. Calcutta Math. Soc.* 106(3) (2014), 153-168.

[6] Y. Avcı and E. Złotkiewicz, On harmonic univalent mappings, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 44 (1990), 1-7.

[7] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn.: Ser. A I Math.* 9(3) (1984), 3-25. https://doi.org/10.5186/aasfm.1984.0905

[8] G. Mittag-Leffler, Sur la Nouvelle function $E_\alpha(x)$, *Comp. Rend. de l'Acad. des Sci.* 137 (1903), 554-558.

[9] T. Sheil-Small, Constants for planar harmonic mappings, *J. Lond. Math. Soc.* 42(2) (1990), 237-248. https://doi.org/10.1112/jlms/s2-42.2.237

[10] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.* 19 (1971), 7-15.

[11] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* 49 (1975), 109-115. https://doi.org/10.1090/S0002-9939-1975-0367176-1

[12] Tariq O. Salim, Some properties relating to the generalized Mitta-Leffler function*, Adv. Appl. Math. Anal.* 4 (2009), 21-30.

[13] Tariq O. Salim and Ahmad W. Faraj, A generalization of Mittag-Leffler function and Integral operator associated with fractional calculus, *J. Frac. Calc. Appl.* 3(5) (2012), 1-13.

[14] A. Wiman, Über den fundamentalsatz in der Teorie der funktionen $E_a(x)$, *Acta Math.* 29 (1905), 191-201. https://doi.org/10.1007/BF02403202