On Generalized p-Mersenne Numbers

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Abstract

In this paper, we introduce the generalized p-Mersenne sequence and deal with, in detail, two special cases, namely, p-Mersenne and p-Mersenne-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

1 Introduction

A Mersenne number, denoted by M_n, and a Mersenne-Lucas number, denoted by H_n, are numbers of the form $M_n = 2^n - 1$ and $H_n = 2^n + 1$, respectively. The Mersenne sequence $\{M_n\}_{n\geq 0}$ and the Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ can also be defined recursively by

$$M_n = 3M_{n-1} - 2M_{n-2}$$

with initial conditions $M_0 = 0, M_1 = 1$ and

$$H_n = 3H_{n-1} - 2H_{n-2}$$

with initial conditions $H_0 = 2, H_1 = 3$, respectively.

$\{M_n\}_{n\geq 0}$ is the sequence $A000225$ in the OEIS [21], whereas $\{H_n\}_{n\geq 0}$ is the id-number $A000051$ in OEIS. Note that Mersenne-Lucas numbers are also

called as Fermat numbers. In fact, there are two definitions of the Fermat numbers. The less common is a number of the form $2^n + 1$, the first few of which are $2, 3, 5, 9, 17, 33, ...$ (OEIS $A000051$). The much more commonly encountered Fermat numbers are a special case, given by the binomial number of the form $F_n = 2^{2^n} + 1$. The first few for $n = 0, 1, 2, ...$ are $3, 5, 17, 257, 65537, 4294967297, ...$ (OEIS $A000215$).

Mersenne sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to this sequence, see for example, [1,2,3,4,5,6,7,8,9,10,15,16,17,18,19,20,22,23,27,28].

Generalizations of Mersenne numbers can be obtained in various ways (see for example [5,10,17,22]). Our generalizations of Mersenne numbers in Section 2 are not Mersenne numbers in the sense of [10,22] and are Mersenne numbers in the sense of [23] which is given as: a generalized Mersenne sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1)\}_{n \geq 0}$ is defined by the second-order recurrence relation

$$W_n = 3W_{n-1} - 2W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero. For more information on generalized Mersenne numbers, see Soykan [23].

The purpose of this article is to generalize and investigate these interesting sequence of numbers (Mersenne numbers). First, we recall some properties of Fibonacci numbers and its generalizations, namely generalized Fibonacci numbers.

The Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}_{n \geq 0}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, \ F_1 = 1.$$

The generalization of Fibonacci sequence leads to several nice and interesting sequences. The generalized Fibonacci sequence (or generalized (r, s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0, W_1; r, s)\}_{n \geq 0}$ (or shortly $\{W_n\}_{n \geq 0}$) is defined (by Horadam [12]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \quad W_0 = a, W_1 = b, \quad n \geq 2 \tag{1.2}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [11,13,14] and Soykan [25].

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1: A few special case of generalized Fibonacci sequences.

Name of sequence	$W_n(a,b;r,s)$	Binet Formula	OEIS[21]
Fibonacci	$W_n(0,1;1,1) = F_n$	$\dfrac{\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n}{\sqrt5}$	A000045
Lucas	$W_n(2,1;1,1) = L_n$	$\left(\frac{1+\sqrt5}{2}\right)^n + \left(\frac{1-\sqrt5}{2}\right)^n$	A000032
Pell	$W_n(0,1;2,1) = P_n$	$\dfrac{\left(1+\sqrt2\right)^n - \left(1-\sqrt2\right)^n}{2\sqrt2}$	A000129
Pell-Lucas	$W_n(2,2;2,1) = Q_n$	$\left(1+\sqrt2\right)^n + \left(1-\sqrt2\right)^n$	A002203
Jacobsthal	$W_n(0,1;1,2) = J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2,1;1,2) = j_n$	$2^n + (-1)^n$	A014551

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

The sequence $\{W_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.2) holds for all integer n.

Now we define two special cases of the sequence $\{W_n\}$. (r, s) sequence $\{G_n(0,1;r,s)\}_{n\geq0}$ and Lucas (r, s) sequence $\{H_n(2,r;r,s)\}_{n\geq0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \qquad G_0 = 0, G_1 = 1, \tag{1.3}$$

$$H_{n+2} = rH_{n+1} + sH_n, \qquad H_0 = 2, H_1 = r. \tag{1.4}$$

The sequences $\{G_n\}_{n\geq0}$, $\{H_n\}_{n\geq0}$ and $\{E_n\}_{n\geq0}$ can be extended to negative

subscripts by defining

$$G_{-n} = -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)},$$

$$H_{-n} = -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3)-(1.4) hold for all integer n.

Some special cases of (r, s) sequence $\{G_n(0, 1; r, s)\}_{n \geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n \geq 0}$ are as follows:

1. $G_n(0, 1; 1, 1) = F_n$, Fibonacci sequence,

2. $H_n(2, 1; 1, 1) = L_n$, Lucas sequence,

3. $G_n(0, 1; 2, 1) = P_n$, Pell sequence,

4. $H_n(2, 2; 2, 1) = Q_n$, Pell-Lucas sequence,

5. $G_n(0, 1; 1, 2) = J_n$, Jacobsthal sequence,

6. $H_n(2, 1; 1, 2) = j_n$, Jacobsthal-Lucas sequence.

The following theorem shows that the generalized Fibonacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1. *For $n \in \mathbb{Z}$, for the generalized Fibonacci sequence (or generalized (r, s)-sequence or Horadam sequence or 2-step Fibonacci sequence) we have the following:*

(a)

$$W_{-n} = (-1)^{-n-1}s^{-n}(W_n - H_nW_0)$$
$$= (-1)^{n+1}s^{-n}(W_n - H_nW_0).$$

(b)

$$W_{-n} = \frac{(-1)^{n+1}s^{-n}}{-W_1^2 + sW_0^2 + rW_0W_1}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n).$$

Proof. For the proof, see Soykan [26, Theorem 3.2 and Theorem 3.3]. □

The following theorem presents sum formulas of generalized (r, s) numbers (generalized Fibonacci numbers).

Theorem 2. *Let x be a real (or complex) number. For all integers m and j, for generalized (r, s) numbers (generalized Fibonacci numbers), we have the following sum formulas:*

(a) *If $(-s)^m x^2 - xH_m + 1 \neq 0$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + W_j - (-s)^m xW_{j-m}}{(-s)^m x^2 - xH_m + 1}.$$

(1.5)

(b) *If $(-s)^m x^2 - xH_m + 1 = u(x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x(n+2)(-s)^m - (n+1)H_m)x^n W_{j+mn} + (-s)^m (n+1)x^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m x - H_m}.$$

(c) *If $(-s)^m x^2 - xH_m + 1 = u(x-c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(n+1)\left((-s)^m (n+2)x^n - nx^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m x^{n-1}W_{mn+j-m}}{2(-s)^m}.$$

Proof. It is given in Soykan [26, Theorem 4.1]. □

Note that (1.5) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + x(H_m - (-s)^m x)W_j - (-s)^m xW_{j-m}}{(-s)^m x^2 - xH_m + 1}.$$

We give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 3. *Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Fibonacci sequence $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{1.6}$$

Proof. For a proof see, [25, Lemma 1.1]. □

1.1 Binet's Formula for the Distinct Roots Case and Single Root Case

Let α and β be two roots of the quadratic equation

$$x^2 - rx - s = 0, \tag{1.7}$$

of which the left-hand side is called the characteristic polynomial (or the characteristic equation) of the recurrence relation (1.2). The following theorem presents the Binet's formula of the sequence W_n.

Theorem 4. *The general term of the sequence W_n can be presented by the following Binet formula:*

$$
\begin{aligned}
W_n &= \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n &, \quad \text{if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} &, \quad \text{if } \alpha = \beta \text{ (Single Root Case)} \end{cases} \\
&= \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n &, \quad \text{if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ \left(nW_1 - \frac{r}{2}(n-1)W_0\right)\left(\frac{r}{2}\right)^{n-1} &, \quad \text{if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.
\end{aligned}
$$

Proof. For a proof, see Soykan [25] and [26]. □

The roots of characteristic equation are

$$\alpha = \frac{r + \sqrt{\Delta}}{2}, \quad \beta = \frac{r - \sqrt{\Delta}}{2}. \tag{1.8}$$

where

$$\Delta = r^2 + 4s$$

and the followings hold

$$
\begin{aligned}
\alpha + \beta &= r, \\
\alpha\beta &= -s, \\
(\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta = r^2 + 4s.
\end{aligned}
$$

If $\Delta = r^2 + 4s \neq 0$, then $\alpha \neq \beta$, i.e., there are distinct roots of the quadratic equation (1.7) and if $\Delta = r^2 + 4s = 0$, then $\alpha = \beta$, i.e., there is a single root of the quadratic equation (1.7).

In the case $r^2 + 4s \neq 0$ so that $\alpha \neq \beta$, for all integers n, (r, s) and Lucas (r, s) numbers (using initial conditions in Theorem 4) can be expressed using Binet's formulas as

$$
\begin{aligned}
G_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\
H_n &= \alpha^n + \beta^n,
\end{aligned}
$$

respectively. In the case $r^2 + 4s = 0$ so that $\alpha = \beta$, for all integers n, (r, s) and Lucas (r, s) numbers (using initial conditions in Theorem 4) can be expressed using Binet's formulas as

$$
\begin{aligned}
G_n &= n\alpha^{n-1}, \\
H_n &= 2\alpha^n,
\end{aligned}
$$

respectively.

2   Generalized p-Mersenne Sequence

In this paper, we consider the case $r = p, s = -(p - 1)$ where $p \in \mathbb{R}$, $p \neq 1$ (so throughout the paper we assume that $p \neq 1$ unless otherwise stated). A generalized p-Mersenne sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1)\}_{n \geq 0}$ is defined by the second-order recurrence relation

$$
W_n = pW_{n-1} - (p - 1)W_{n-2} \tag{2.1}
$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{p}{(p-1)}W_{-(n-1)} - \frac{1}{(p-1)}W_{-(n-2)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (2.1) holds for all integer n.

By Theorem 4, the Binet formula of generalized p-Mersenne numbers can be written as

$$W_n = \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n & , \quad \text{if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ \left(nW_1 - \frac{r}{2}(n-1)W_0\right)\left(\frac{r}{2}\right)^{n-1} & , \quad \text{if } \alpha = \beta \text{ (Single Root Case)} \end{cases}$$

where α and β are the roots of the quadratic equation $x^2 - px + (p-1) = 0$. Moreover

$$\alpha = p - 1$$
$$\beta = 1$$

Note that

$$\alpha + \beta = p,$$
$$\alpha\beta = p - 1,$$
$$\alpha - \beta = p - 2.$$

So

$$W_n = \begin{cases} \dfrac{W_1 - W_0}{p-2}(p-1)^n - \dfrac{W_1 - (p-1)W_0}{p-2} & , \quad \text{if } p \neq 2 \\ nW_1 - (n-1)W_0 & , \quad \text{if } p = 2 \end{cases} \qquad (2.2)$$

For the case $p \in \mathbb{R}$, $p \neq 1$, the first few generalized p-Mersenne numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2: A few generalized p-Mersenne numbers.

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{p-1}(-W_1 + pW_0)$
2	$pW_1 - (p-1)W_0$	$\frac{1}{(p-1)^2}(-pW_1 + (p^2 - p + 1)W_0)$
3	$(p^2 - p + 1)W_1 - p(p-1)W_0$	$\frac{1}{(p-1)^3}(-(p^2 - p + 1)W_1 + p(p^2 - 2p + 2)W_0)$
4	$p(p^2 - 2p + 2)W_1 - (p-1)(p^2 - p + 1)W_0$	$\frac{1}{(p-1)^4}(-p(p^2 - 2p + 2)W_1 + (p^4 - 3p^3 + 4p^2 - 2p + 1)W_0)$

For the case $p = 2$, the first few generalized 2-Mersenne numbers with positive subscript and negative subscript are given in the following Table 3.

Table 3: A few generalized 2-Mersenne numbers.

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$W_{-1} = 2W_0 - W_1$
2	$W_2 = 2W_1 - W_0$	$W_{-2} = 3W_0 - 2W_1$
3	$W_3 = 3W_1 - 2W_0$	$W_{-3} = 4W_0 - 3W_1$
4	$W_4 = 4W_1 - 3W_0$	$W_{-4} = 5W_0 - 4W_1$
5	$W_5 = 5W_1 - 4W_0$	$W_{-5} = 6W_0 - 5W_1$
6	$W_6 = 6W_1 - 5W_0$	$W_{-6} = 7W_0 - 6W_1$
7	$W_7 = 7W_1 - 6W_0$	$W_{-7} = 8W_0 - 7W_1$
8	$W_8 = 8W_1 - 7W_0$	$W_{-8} = 9W_0 - 8W_1$

Now we define two special cases of the sequence $\{W_n\}$. p-Mersenne sequence $\{M_n\}_{n \geq 0}$ and p-Mersenne-Lucas sequence $\{H_n\}_{n \geq 0}$ are defined, respectively, by the second-order recurrence relations

$$M_n = pM_{n-1} - (p-1)M_{n-2}, \qquad M_0 = 0, M_1 = 1, \qquad (2.3)$$

$$H_n = pH_{n-1} - (p-1)H_{n-2}, \qquad H_0 = 2, H_1 = p, \qquad (2.4)$$

The sequences $\{M_n\}_{n \geq 0}$ and $\{H_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -\frac{p}{(p-1)}M_{-(n-1)} - \frac{1}{(p-1)}M_{-(n-2)},$$

$$H_{-n} = -\frac{p}{(p-1)}H_{-(n-1)} - \frac{1}{(p-1)}H_{-(n-2)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (2.3)-(2.4) hold for all integer n.

If $p = 3$, then 3-Mersenne sequence $\{M_n\}$ is the well known Mersenne sequence and 3-Mersenne-Lucas sequence $\{H_n\}$ is the well known Mersenne-Lucas sequence. If $p = 2$, then 2-Mersenne sequence $\{M_n\}_{n \geq 0}$ is the sequence $A001477$ in the OEIS

[21], whereas 2-Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ is the id-number $A007395$ in OEIS.

Next, for the case $p \in \mathbb{R}$, $p \neq 1$, we present the first few values of the p-Mersenne and p-Mersenne-Lucas numbers with positive and negative subscripts:

Table 4: The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4
M_n	0	1	p	$p^2 - p + 1$	$p\left(p^2 - 2p + 2\right)$
M_{-n}	....	$-\frac{1}{p-1}$	$-\frac{p}{(p-1)^2}$	$-\frac{1}{(p-1)^3}\left(p^2 - p + 1\right)$	$-\frac{p}{(p-1)^4}\left(p^2 - 2p + 2\right)$
H_n	2	p	$p^2 - 2p + 2$	$p\left(p^2 - 3p + 3\right)$	$p^4 - 4p^3 + 6p^2 - 4p + 2$
H_{-n}	....	$\frac{p}{p-1}$	$\frac{1}{(p-1)^2}\left(p^2 - 2p + 2\right)$	$\frac{p}{(p-1)^3}\left(p^2 - 3p + 3\right)$	$\frac{1}{(p-1)^4}\left(p^4 - 4p^3 + 6p^2 - 4p + 2\right)$

Now, for the case $p = 2$, we present the first few values of the 2-Mersenne and 2-Mersenne-Lucas numbers with positive and negative subscripts:

Table 5: The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M_n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M_{-n}	....	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13	-14	-15
H_n	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
H_{-n}	....	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

For all integers n, p-Mersenne and p-Mersenne-Lucas (using initial conditions in Theorem 4) can be expressed using Binet's formulas as

$$M_n = \begin{cases} \frac{1}{p-2}(p-1)^n - \frac{1}{p-2} & , \text{ if } p \neq 2 \\ n & , \text{ if } p = 2 \end{cases}$$

and

$$H_n = \begin{cases} (p-1)^n + 1 & , \text{ if } p \neq 2 \\ 2 & , \text{ if } p = 2 \end{cases} = (p-1)^n + 1$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 5. *Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized p-Mersenne sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - pW_0)x}{1 - px + (p-1)x^2}.$$

Proof. In Lemma 3, take $r = p, s = -(p-1)$. $\qquad\square$

The previous Lemma gives the following results as particular examples.

Corollary 6. *Generated functions of p-Mersenne and p-Mersenne-Lucas numbers are*

$$\sum_{n=0}^{\infty} M_n x^n = \frac{x}{1 - px + (p-1)x^2},$$

$$\sum_{n=0}^{\infty} H_n x^n = \frac{2 - px}{1 - px + (p-1)x^2},$$

respectively.

Proof. In Lemma 5, take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = p$, respectively. $\qquad\square$

3 Simson's Formulas

There is a well-known Simson's Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini's Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized p-Mersenne sequence $\{W_n\}_{n\geq 0}$.

Theorem 7 (Simson's Formula of Generalized p-Mersenne Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = (p-1)^{n-1}(-W_1^2 - (p-1)W_0^2 + pW_1W_0). \qquad (3.1)$$

Proof. For a proof of Eq. (3.1), see Soykan [24], just take $s = -(p-1)$. □

The previous theorem gives the following results as particular examples.

Corollary 8. *For all integers n, p-Mersenne and p-Mersenne-Lucas numbers are given as*

$$\begin{vmatrix} M_{n+1} & M_n \\ M_n & M_{n-1} \end{vmatrix} = -(p-1)^{n-1},$$

$$\begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} = (p-1)^{n-1}(p^2 - 4p + 4),$$

respectively.

4 Some Identities

In this section, we obtain some identities of generalized p-Mersenne, p-Mersenne and p-Mersenne-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{M_n\}$.

Lemma 9. *The following equalities are true:*

(a) $(p-1)^3 W_n = (-(p^2 - p + 1)W_1 + p(p^2 - 2p + 2)W_0)M_{n+4} + (p(p^2 - 2p + 2)W_1 - (p^4 - 3p^3 + 4p^2 - 2p + 1)W_0)M_{n+3}.$

(b) $(p-1)^2 W_n = (-pW_1 + (p^2 - p + 1)W_0)M_{n+3} + ((p^2 - p + 1)W_1 - p(p^2 - 2p + 2)W_0)M_{n+2}.$

(c) $(p-1)W_n = (-W_1 + pW_0)M_{n+2} - (-pW_1 + (p^2 - p + 1)W_0)M_{n+1}$.

(d) $W_n = W_0 M_{n+1} + (W_1 - pW_0)M_n$.

(e) $W_n = W_1 M_n - (p-1)W_0 M_{n-1}$.

(f) $(p-1)^3 (W_0 - W_1)(-W_1 + (p-1)W_0)M_n = (-(p^2 - p + 1)W_1 + p(p-1)W_0)W_{n+4} + (p(p^2 - 2p + 2)W_1 - (p-1)(p^2 - p + 1)W_0)W_{n+3}$.

(g) $(p-1)^2 (W_0 - W_1)(-W_1 + (p-1)W_0)M_n = (-pW_1 + (p-1)W_0)W_{n+3} + ((p^2 - p + 1)W_1 - p(p-1)W_0)W_{n+2}$.

(h) $(p-1)(W_0 - W_1)(-W_1 + (p-1)W_0)M_n = -W_1 W_{n+2} + (pW_1 + (1-p)W_0)W_{n+1}$.

(i) $(W_0 - W_1)(-W_1 + (p-1)W_0)M_n = -W_0 W_{n+1} + W_1 W_n$.

(j) $(W_0 - W_1)(-W_1 + (p-1)W_0)M_n = (W_1 - pW_0)W_n + (p-1)W_0 W_{n-1}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times M_{n+4} + b \times M_{n+3}$$

and solving the system of equations

$$W_0 = a \times M_4 + b \times M_3$$
$$W_1 = a \times M_5 + b \times M_4$$

we find that $a = \frac{(-(p^2 - p + 1)W_1 + p(p^2 - 2p + 2)W_0)}{(p-1)^3}$, $b = \frac{(p(p^2 - 2p + 2)W_1 - (p^4 - 3p^3 + 4p^2 - 2p + 1)W_0)}{(p-1)^3}$. The other equalities can be proved similarly. $\square$

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 10. *The following equalities are true:*

(a) $(p-2)^2 (p-1)^3 W_n = -(-p(p^2 - 3p + 3)W_1 + (p^4 - 4p^3 + 6p^2 - 4p + 2)W_0)H_{n+4} + (-(p^4 - 4p^3 + 6p^2 - 4p + 2)W_1 + p(p^4 - 5p^3 + 10p^2 - 10p + 5)W_0)H_{n+3}$.

(b) $(p-1)^2 (p-2)^2 W_n = ((p^2 - 2p + 2)W_1 - p(p^2 - 3p + 3)W_0)H_{n+3} + (-p(p^2 - 3p + 3)W_1 + (p^4 - 4p^3 + 6p^2 - 4p + 2)W_0)H_{n+2}$.

(c) $(p-1)(p-2)^2 W_n = (pW_1 - (p^2 - 2p + 2)W_0)H_{n+2} + (-(p^2 - 2p + 2)W_1 + p(p^2 - 3p + 3)W_0)H_{n+1}$.

(d) $(p-2)^2 W_n = (2W_1 - pW_0)H_{n+1} + (-pW_1 + (p^2 - 2p + 2)W_0)H_n$.

(e) $(p-2)^2 W_n = (2W_0 - 2pW_0 + pW_1)H_n + (p-1)(-2W_1 + pW_0)H_{n-1}$.

(f) $(p-1)^3 (W_0 - W_1)(-W_1 + (p-1)W_0) H_n = (p(p^2 - 3p + 3)W_1 + 2W_0 - 4pW_0 + 3p^2W_0 - p^3W_0)W_{n+4} + (-(p^4 - 4p^3 + 6p^2 - 4p + 2)W_1 + p(p-1)(p^2 - 3p + 3)W_0)W_{n+3}$.

(g) $(p-1)^2 (W_0 - W_1)(-W_1 + (p-1)W_0) H_n = ((p^2 - 2p + 2)W_1 - p(p-1)W_0)W_{n+3} + (-p(p^2 - 3p + 3)W_1 + (p-1)(p^2 - 2p + 2)W_0)W_{n+2}$.

(h) $(p-1)(W_0 - W_1)(-W_1 + (p-1)W_0) H_n = (pW_1 - 2(p-1)W_0)W_{n+2} + (-(p^2 - 2p + 2)W_1 + p(p-1)W_0)W_{n+1}$.

(i) $(W_0 - W_1)(-W_1 + (p-1)W_0) H_n = (2W_1 - pW_0)W_{n+1} + (-pW_1 + 2(p-1)W_0)W_n$.

(j) $(W_0 - W_1)(-W_1 + (p-1)W_0) H_n = -(-pW_1 + (p^2 - 2p + 2)W_0)W_n + (p-1)(-2W_1 + pW_0)W_{n-1}$.

Next, we present a few basic relations between $\{M_n\}$ and $\{H_n\}$.

Lemma 11. *The following equalities are true:*

$$
\begin{aligned}
(p-1)^3\,H_n &= p\left(p^2-3p+3\right)M_{n+4}+(-p^4+4p^3-6p^2+4p-2)M_{n+3},\\
(p-1)^2\,H_n &= (p^2-2p+2)M_{n+3}-p\left(p^2-3p+3\right)M_{n+2},\\
(p-1)H_n &= pM_{n+2}+(-p^2+2p-2)M_{n+1},\\
H_n &= 2M_{n+1}-pM_n,\\
H_n &= pM_n-2\left(p-1\right)M_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
(p-2)^2\,(p-1)^3\,M_n &= p\left(p^2-3p+3\right)H_{n+4}+(-p^4+4p^3-6p^2+4p-2)H_{n+3},\\
(p-1)^2\,(p-2)^2\,M_n &= (p^2-2p+2)H_{n+3}-p\left(p^2-3p+3\right)H_{n+2},\\
(p-1)\,(p-2)^2\,M_n &= pH_{n+2}+(-p^2+2p-2)H_{n+1},\\
(p-2)^2\,M_n &= 2H_{n+1}-pH_n,\\
(p-2)^2\,M_n &= pH_n-2\left(p-1\right)H_{n-1}.
\end{aligned}
$$

We now present a few special identities for the generalized p-Mersenne sequence $\{W_n\}$.

Theorem 12. *(Catalan's identity of the generalized p-Mersenne sequence) For all integers n and m, the following identity holds:*

$$
\begin{aligned}
&W_{n+m}W_{n-m}-W_n^2\\
&=\begin{cases}
\dfrac{(p-1)^{n-m}\left((p-1)^m-1\right)^2\left(W_0-W_1\right)\left(W_1+(1-p)W_0\right)}{(p-2)^2} &,\quad \text{if } p\neq 2\\
-m^2\left(W_0-W_1\right)^2 &,\quad \text{if } p=2
\end{cases}.
\end{aligned}
$$

Proof. We use the identity (2.2). □

As special cases of the above theorem, we have the following corollary.

Corollary 13. *For all integers n and m, the following identities hold:*

(a)

$$
M_{n+m}M_{n-m}-M_n^2=\begin{cases}
\dfrac{-(p-1)^{n-m}\left((p-1)^m-1\right)^2}{(p-2)^2} &,\quad \text{if } p\neq 2\\
-m^2 &,\quad \text{if } p=2
\end{cases}.
$$

(b)

$$H_{n+m}H_{n-m} - H_n^2 = \begin{cases} (p-1)^{n-m}\left((p-1)^m - 1\right)^2 & , \quad \text{if } p \neq 2 \\ -m^2\,(p-2)^2 & , \quad \text{if } p = 2 \end{cases}.$$

Note that for $m = 1$ in Catalan's identity of the generalized p-Mersenne sequence, we get the Cassini identity for the generalized p-Mersenne sequnce.

Theorem 14. *(Cassini's identity of the generalized p-Mersenne sequence) For all integers n, the following identity holds:*

$$W_{n+1}W_{n-1} - W_n^2 = \begin{cases} (p-1)^{n-1}\,(W_0 - W_1)\,(W_1 + (1-p)W_0) & , \quad \text{if } p \neq 2 \\ -\,(W_0 - W_1)^2 & , \quad \text{if } p = 2 \end{cases}.$$

As special cases of the above theorem, we have the following corollary.

Corollary 15. *For all integers n, the following identities hold:*

(a) $M_{n+1}M_{n-1} - M_n^2 = \begin{cases} -(p-1)^{n-1} & , \quad \text{if } p \neq 2 \\ -1 & , \quad \text{if } p = 2 \end{cases}$

(b) $H_{n+1}H_{n-1} - H_n^2 = \begin{cases} (p-1)^{n-1}\,(p-2)^2 & , \quad \text{if } p \neq 2 \\ 0 & , \quad \text{if } p = 2 \end{cases}$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using (2.2). The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of generalized p-Mersenne sequence $\{W_n\}$.

Theorem 16. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*
$$W_{m+1}W_n - W_m W_{n+1}$$
$$= \begin{cases} \dfrac{\left((p-1)^m - (p-1)^n\right)(W_0 - W_1)\left((1-p)W_0 + W_1\right)}{p-2} & , \quad \text{if } p \neq 2 \\ -\,(m-n)\,(W_0 - W_1)^2 & , \quad \text{if } p = 2 \end{cases}.$$

(b) *(Gelin-Cesàro's identity)*

$$W_{n+2}W_{n+1}W_{n-1}W_{n-2} - W_n^4$$

$$= \begin{cases} \dfrac{(p-1)^{n-3}\,(W_0 - W_1)(-W_1 + (p-1)W_0)}{(p-2)^2}\Omega_1 & , \quad \text{if } p \neq 2 \\ \Omega_2 & , \quad \text{if } p = 2 \end{cases}$$

where

$\Omega_1 = (-(p-1)^{2n}(p-1)(p^2+p-1) + (p-1)^n(p^4-2p^3+4p^2-4p+2) - p^3 + 2p-1)W_1^2 + (p-1)\left(-(p-1)^{2n}(p^2+p-1) + (p-1)^n(p^4-2p^3+4p^2-4p+2) - p^4 + p^3 + 2p^2 - 3p + 1\right)W_0^2 + (2(p-1)^{2n}(p-1)(p^2+p-1) + (p-1)^n\,p(-p^4 + 2p^3 - 4p^2 + 4p - 2) + 2p^4 - 2p^3 - 4p^2 + 6p - 2)W_0 W_1$

and

$\Omega_2 = -(W_0 - W_1)^2((5n^2-4)W_1^2 + (5n^2-10n+1)W_0^2 - 2(5n^2-5n-4)W_0 W_1).$

(c) *(Melham's identity)*

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3$$

$$= \begin{cases} \dfrac{(p-1)^{n+1}\,(W_0 - W_1)(-W_1 + (p-1)W_0)}{p-2}\Delta & , \quad \text{if } p \neq 2 \\ (W_0 - W_1)^2(-(7n+15)W_1 + (7n+8)W_0) & , \quad \text{if } p = 2 \end{cases}$$

where

$\Delta = -((p-1)^n(p^3 - p + 1)(p-1)^2 - p^3 + p^2 - 2p + 1)W_1 + (p-1)((p-1)^{n+1}(p^3 - p + 1) - p^3 + p^2 - 2p + 1)W_0.$

Proof. Use the identity (2.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As special cases of the above theorem, we have the following three corollaries. First one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of p-Mersenne sequence $\{M_n\}$.

Corollary 17. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$M_{m+1}M_n - M_m M_{n+1} = \begin{cases} -\dfrac{(p-1)^m - (p-1)^n}{p-2} & , \quad if\ p \neq 2 \\ -(m-n) & , \quad if\ p = 2 \end{cases}.$$

(b) *(Gelin-Cesàro's identity)*

$$M_{n+2}M_{n+1}M_{n-1}M_{n-2} - M_n^4 = \begin{cases} \dfrac{(p-1)^{n-3}}{(p-2)^2}\Psi_1 & , \quad if\ p \neq 2 \\ -\left(5n^2 - 4\right) & , \quad if\ p = 2 \end{cases}$$

where

$$\Psi_1 = -(p-1)^{2n}(p-1)(p^2+p-1)+(p-1)^n(p^4-2p^3-4p+4p^2+2)-p^3+2p-1.$$

(c) *(Melham's identity)*

$$M_{n+1}M_{n+2}M_{n+6} - M_{n+3}^3 = \begin{cases} \dfrac{-(p-1)^{n+1}}{p-2}\Psi_2 & , \quad if\ p \neq 2 \\ -(7n+15) & , \quad if\ p = 2 \end{cases}$$

where

$$\Psi_2 = ((p-1)^{n+2}\left(p^3 - p + 1\right) - p^3 + p^2 - 2p + 1).$$

Second one presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of p-Mersenne-Lucas sequence $\{H_n\}$.

Corollary 18. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$H_{m+1}H_n - H_m H_{n+1} = \begin{cases} ((p-1)^m - (p-1)^n)\,(p-2) & , \quad if\ p \neq 2 \\ -(m-n)\,(p-2)^2 & , \quad if\ p = 2 \end{cases}.$$

(b) *(Gelin-Cesàro's identity)*

$$H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = \begin{cases} (p-1)^{n-3}\,(p-2)^2\,\Gamma_1 & , \quad if\ p \neq 2 \\ (p-2)^2\Gamma_2 & , \quad if\ p = 2 \end{cases}$$

where

$$\Gamma_1 = ((p-1)^{2n+1}(p^2+p-1)+(p-1)^n(p^4-2p^3+4p^2-4p+2)+p^3-2p+1)$$

and

$$\Gamma_2 = (-5(p-2)^2 n^2 - 20(p-2)n + 4(p^2-4p-1)).$$

(c) *(Melham's identity)*

$$H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = \begin{cases} -(p-1)^{n+1}(p-2)^2\,\Gamma_3 & , \quad \text{if } p \neq 2 \\ (p-2)^2\,\Gamma_4 & , \quad \text{if } p = 2 \end{cases}.$$

where

$$\Gamma_3 = (-(p-1)^{n+2}(p^3-p+1)-p^3+p^2-2p+1)$$

and

$$\Gamma_4 = (14n-15p-7np+16).$$

5 On the Recurrence Properties of Generalized p-Mersenne Sequence

Taking $r = p, s = -(p-1)$ in Theorem 1 (a) and (b), we obtain the following Proposition.

Proposition 19. *For $n \in \mathbb{Z}$, generalized p-Mersenne numbers (the case $r = p, s = -(p-1)$) have the following identity:*

$$
\begin{aligned}
W_{-n} &= -(p-1)^{-n}(W_n - H_n W_0) \\
&= \frac{-(p-1)^{-n}}{-W_1^2 - (p-1)W_0^2 + pW_0 W_1} \\
&\quad \times ((2W_1 - pW_0)W_0 W_{n+1} - (W_1^2 - (p-1)W_0^2)W_n)
\end{aligned}
$$

From the above Proposition, we have the following corollary which gives the connection between the special cases of generalized p-Mersenne sequence at the positive index and the negative index: for p-Mersenne and p-Mersenne-Lucas numbers: take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and take $W_n = H_n$ with $H_0 = 2, H_1 = p$, respectively. Note that in this case $H_n = H_n$.

Corollary 20. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *p-Mersenne sequence:*

$$M_{-n} = -(p-1)^{-n} M_n = \begin{cases} -\frac{1}{p-2} + (p-1)^{-n} \frac{1}{p-2} & , \quad \text{if } p \neq 2 \\ -n & , \quad \text{if } p = 2 \end{cases}$$

(b) *p-Mersenne-Lucas sequence:*

$$H_{-n} = (p-1)^{-n} H_n = \begin{cases} 1 + (p-1)^{-n} & , \quad \text{if } p \neq 2 \\ 2 & , \quad \text{if } p = 2 \end{cases} = 1 + (p-1)^{-n}.$$

6   The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}$

In this section, we present sum formulas of generalized p-Mersenne numbers. The following theorem presents sum formulas of generalized p-Mersenne numbers (the case $r = p, s = -(p-1)$).

Theorem 21. *Let x be a real (or complex) number. For all integers m and j, for generalized p-Mersenne numbers we have the following sum formulas:*

(a) *If $(p-1)^m x^2 - x H_m + 1 \neq 0$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{\begin{matrix} ((p-1)^m x - H_m) x^{n+1} W_{mn+j} + (p-1)^m x^{n+1} W_{mn+j-m} \\ + W_j - (p-1)^m x W_{j-m} \end{matrix}}{(p-1)^m x^2 - x H_m + 1}.$$

(6.1)

(b) *If $(p-1)^m x^2 - x H_m + 1 = u(x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{\begin{matrix} (x(n+2)(p-1)^m - (n+1) H_m) x^n W_{j+mn} \\ + (p-1)^m (n+1) x^n W_{mn+j-m} - (p-1)^m W_{j-m} \end{matrix}}{2(p-1)^m x - H_m}.$$

(c) *If $(p-1)^m x^2 - x H_m + 1 = u(x-c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(n+1)\left((p-1)^m (n+2)x^n - nx^{n-1}H_m\right)W_{mn+j}}{+ n(n+1)(p-1)^m x^{n-1}W_{mn+j-m}}{2(p-1)^m}.$$

Proof. Take $r = p, s = -(p-1)$ and $H_n = H_n$ in Theorem 2. □

Note that (6.1) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((p-1)^m x - H_m)x^{n+1}W_{mn+j} + (p-1)^m x^{n+1}W_{mn+j-m}}{+ x(H_m - (p-1)^m x)W_j - (p-1)^m x W_{j-m}}{(p-1)^m x^2 - x H_m + 1}.$$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 22. *For generalized p-Mersenne numbers (the case $r = p, s = -(p-1)$) we have the following sum formulas:*

(a) $(m = 1, j = 0)$

If $(p-1)x^2 - px + 1 \neq 0$, i.e., $x \neq 1$, $x \neq \frac{1}{p-1}$, then

$$\sum_{k=0}^{n} x^k W_k = \frac{((p-1)x - p)x^{n+1}W_n + (p-1)x^{n+1}W_{n-1} + xW_1 + (1 - px)W_0}{(p-1)x^2 - px + 1}$$

and

if $(p-1)x^2 - px + 1 = 0$, i.e., $x = 1$ or $x = \frac{1}{p-1}$, then if $p \neq 2$, then

$$\sum_{k=0}^{n} x^k W_k = \frac{(x(n+2)(p-1) - (n+1)p)x^n W_n + (p-1)(n+1)x^n W_{n-1}}{+ W_1 - pW_0}{2(p-1)x - p}$$

and if $p = 2$, then

$$\sum_{k=0}^{n} x^k W_k = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right)W_n + n(n+1)x^{n-1}W_{n-1}}{2}.$$

(b) $(m = 2, j = 0)$

If $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 \neq 0$, i.e., $x \neq 1$, $x \neq \frac{1}{p^2 - 2p + 1}$, then

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{\begin{array}{c} ((p-1)^2 x - (p^2 - 2p + 2))x^{n+1}W_{2n} + (p-1)^2 x^{n+1}W_{2n-2} \\ + pxW_1 - (x(p^2 - p + 1) - 1)W_0 \end{array}}{(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1}$$

and

if $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 = 0$, i.e., $x = 1$ or $x = \frac{1}{p^2 - 2p + 1}$, then if $p \neq 0$ and $p \neq 2$, then

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{\begin{array}{c} (x(n+2)(p-1)^2 - (n+1)(p^2 - 2p + 2))x^n W_{2n} \\ + (p-1)^2 (n+1)x^n W_{2n-2} + pW_1 - (p^2 - p + 1)W_0 \end{array}}{2(p-1)^2 x - (p^2 - 2p + 2)}$$

and if $p = 0$ or $p = 2$ (so that $(p-1)^2 = 1$ and $p^2 - 2p + 2 = 2$), then

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(n+1)x^{n-1}}{2}((2x + nx - 2n)W_{2n} + nW_{2n-2}).$$

(c) $(m = 2, j = 1)$

If $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 \neq 0$, i.e., $x \neq 1$, $x \neq \frac{1}{p^2 - 2p + 1}$, then

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\begin{array}{c} ((p-1)^2 x - (p^2 - 2p + 2))x^{n+1}W_{2n+1} \\ + (p-1)^2 x^{n+1}W_{2n-1} + (px - x + 1)W_1 - p(p-1)xW_0 \end{array}}{(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1}$$

and

if $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 = 0$, i.e., $x = 1$ or $x = \frac{1}{p^2 - 2p + 1}$, then if $p \neq 0$ and $p \neq 2$, then

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\begin{array}{c} (x(n+2)(p-1)^2 - (n+1)(p^2 - 2p + 2))x^n W_{2n+1} \\ + (p-1)^2 (n+1)x^n W_{2n-1} + (p-1)(W_1 - pW_0) \end{array}}{2(p-1)^2 x - (p^2 - 2p + 2)}$$

and if $p = 0$ or $p = 2$ (so that $(p-1)^2 = 1$ and $p^2 - 2p + 2 = 2$), then

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right)W_{2n+1} + n(n+1)x^{n-1}W_{2n-1}}{2}.$$

(d) $(m = -1, j = 0)$

If $\dfrac{(x-1)(p-x-1)}{p-1} \neq 0$, *i.e.*, $x \neq 1$, $x \neq p-1$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{(x-p)x^{n+1}W_{-n} + x^{n+1}W_{-n+1} + (p-1)W_0 - xW_1}{-(x-1)(p-x-1)}$$

and

if $\dfrac{(x-1)(p-x-1)}{p-1} = 0$, *i.e.*, $x = 1$ *or* $x = p-1$, *then if* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{(x(n+2) - (n+1)p)x^n W_{-n} + (n+1)x^n W_{-n+1} - W_1}{2x - p}$$

and if $p = 2$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right)W_{-n} + n(n+1)x^{n-1}W_{-n+1}}{2}.$$

(e) $(m = -2, j = 0)$

If $\dfrac{(x-1)\left(-x + p^2 - 2p + 1\right)}{(p-1)^2} \neq 0$, *i.e.*, $x \neq 1$, $x \neq p^2 - 2p + 1$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{\begin{aligned}&x^{n+1}W_{-2n+2} + (x - (p^2 - 2p + 2))x^{n+1}W_{-2n} \\ &\quad - pxW_1 + (p-1)(x + p - 1)W_0\end{aligned}}{-(x-1)(-x + p^2 - 2p + 1)}$$

and

if $\dfrac{(x-1)\left(-x + p^2 - 2p + 1\right)}{(p-1)^2} = 0$, *i.e.*, $x = 1$ *or* $x = p^2 - 2p + 1$, *then if*
$p \neq 0$ *and* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{\begin{aligned}&(n+1)x^n W_{-2n+2} + (x(n+2) - (n+1)\left(p^2 - 2p + 2\right))x^n W_{-2n} \\ &\quad - pW_1 + (p-1)W_0\end{aligned}}{2x - (p^2 - 2p + 2)}$$

and if $p = 0$ *or* $p = 2$ *(so that* $(p-1)^2 = 1$ *and* $p^2 - 2p + 2 = 2$*), then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{n(n+1)x^{n-1}W_{-2n+2} + (n+1)\left((n+2)x^n - 2nx^{n-1}\right)W_{-2n}}{2}.$$

(f) $(m = -2, j = 1)$

If $\dfrac{(x-1)(-x+p^2-2p+1)}{(p-1)^2} \neq 0$, i.e., $x \neq 1$, $x \neq p^2 - 2p + 1$, then

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{\begin{array}{c} x^{n+1}W_{-2n+3} + (x - (p^2 - 2p + 2))x^{n+1}W_{-2n+1} + (-p^2 x \\ + px - x + p^2 - 2p + 1)W_1 + px(p-1)W_0 \end{array}}{-(x-1)(-x+p^2-2p+1)}$$

and

if $\dfrac{(x-1)(-x+p^2-2p+1)}{(p-1)^2} = 0$, i.e., $x = 1$ or $x = p^2 - 2p + 1$, then if $p \neq 0$ and $p \neq 2$, then

$\sum_{k=0}^{n} x^k W_{-2k+1}$

$$= \frac{\begin{array}{c} (n+1)x^n W_{-2n+3} + (x(n+2) - (n+1)(p^2 - 2p + 2))x^n W_{-2n+1} \\ - (p^2 - p + 1)W_1 + p(p-1)W_0 \end{array}}{2x - (p^2 - 2p + 2)}$$

and if $p = 0$ or $p = 2$ (so that $(p-1)^2 = 1$ and $p^2 - 2p + 2 = 2$), then

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{n(n+1)x^{n-1}W_{-2n+3} + (n+1)\left((n+2)x^n - 2nx^{n-1}\right)W_{-2n+1}}{2}.$$

From the above proposition, we have the following corollary which gives sum formulas of p-Mersenne numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1$).

Corollary 23. *For $n \geq 0$, p-Mersenne numbers have the following properties:*

(a) $(m = 1, j = 0)$

If $(p-1)x^2 - px + 1 \neq 0$, i.e., $x \neq 1$, $x \neq \frac{1}{p-1}$, then

$$\sum_{k=0}^{n} x^k M_k = \frac{((p-1)x - p)x^{n+1}M_n + (p-1)x^{n+1}M_{n-1} + x}{(p-1)x^2 - px + 1}$$

and

if $(p-1)x^2 - px + 1 = 0$, i.e., $x = 1$ or $x = \frac{1}{p-1}$, then if $p \neq 2$, then

$$\sum_{k=0}^{n} x^k M_k = \frac{(x(n+2)(p-1) - (n+1)p)x^n M_n + (p-1)(n+1)x^n M_{n-1} + 1}{2(p-1)x - p}$$

and if $p = 2$, then

$$\sum_{k=0}^{n} x^k M_k = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right)M_n + n(n+1)x^{n-1}M_{n-1}}{2}.$$

(b) $(m = 2, j = 0)$

If $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 \neq 0$, i.e., $x \neq 1$, $x \neq \frac{1}{p^2 - 2p + 1}$, then

$$\sum_{k=0}^{n} x^k M_{2k} = \frac{((p-1)^2 x - (p^2 - 2p + 2))x^{n+1}M_{2n} + (p-1)^2 x^{n+1}M_{2n-2} + px}{(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1}$$

and

if $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 = 0$, i.e., $x = 1$ or $x = \frac{1}{p^2 - 2p + 1}$, then if $p \neq 0$ and $p \neq 2$, then

$$\sum_{k=0}^{n} x^k M_{2k} = \frac{\begin{array}{c}(x(n+2)(p-1)^2 - (n+1)(p^2 - 2p + 2))x^n M_{2n} \\ + (p-1)^2 (n+1)x^n M_{2n-2} + p\end{array}}{2(p-1)^2 x - (p^2 - 2p + 2)}$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} x^k M_{2k} = \frac{(n+1)x^{n-1}}{2}((2x + nx - 2n)M_{2n} + nM_{2n-2}).$$

(c) $(m = 2, j = 1)$

If $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 \neq 0$, i.e., $x \neq 1$, $x \neq \frac{1}{p^2 - 2p + 1}$, then

$$\sum_{k=0}^{n} x^k M_{2k+1} = \frac{\begin{array}{c}((p-1)^2 x - (p^2 - 2p + 2))x^{n+1}M_{2n+1} \\ + (p-1)^2 x^{n+1}M_{2n-1} + (px - x + 1)\end{array}}{(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1}$$

and

if $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 = 0$, i.e., $x = 1$ or $x = \frac{1}{p^2 - 2p + 1}$, then if $p \neq 0$ and $p \neq 2$, then

$$\sum_{k=0}^{n} x^k M_{2k+1} = \frac{\begin{array}{c}(x(n+2)(p-1)^2 - (n+1)(p^2 - 2p + 2))x^n M_{2n+1} \\ + (p-1)^2 (n+1)x^n M_{2n-1} + p - 1\end{array}}{2(p-1)^2 x - (p^2 - 2p + 2)}$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} x^k M_{2k+1} = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right)M_{2n+1} + n(n+1)x^{n-1}M_{2n-1}}{2}.$$

(d) $(m = -1, \ j = 0)$

If $\dfrac{(x-1)(p-x-1)}{p-1} \neq 0$, *i.e.,* $x \neq 1$, $x \neq p-1$, *then*

$$\sum_{k=0}^{n} x^k M_{-k} = \frac{(x-p)x^{n+1}M_{-n} + x^{n+1}M_{-n+1} - x}{-(x-1)(p-x-1)}$$

and

if $\dfrac{(x-1)(p-x-1)}{p-1} = 0$, *i.e.,* $x = 1$ *or* $x = p-1$, *then if* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k M_{-k} = \frac{(x(n+2)-(n+1)p)x^n M_{-n} + (n+1)x^n M_{-n+1} - 1}{2x-p}$$

and if $p = 2$, then

$$\sum_{k=0}^{n} x^k M_{-k} = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right)M_{-n} + n(n+1)x^{n-1}M_{-n+1}}{2}.$$

(e) $(m = -2, \ j = 0)$

If $\dfrac{(x-1)(-x+p^2-2p+1)}{(p-1)^2} \neq 0$, *i.e.,* $x \neq 1$, $x \neq p^2 - 2p + 1$, *then*

$$\sum_{k=0}^{n} x^k M_{-2k} = \frac{x^{n+1}M_{-2n+2} + (x-(p^2-2p+2))x^{n+1}M_{-2n} - px}{-(x-1)(-x+p^2-2p+1)}$$

and

if $\dfrac{(x-1)(-x+p^2-2p+1)}{(p-1)^2} = 0$, *i.e.,* $x = 1$ *or* $x = p^2 - 2p + 1$, *then if* $p \neq 0$ *and* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k M_{-2k} = \frac{(n+1)x^n M_{-2n+2} + (x(n+2)-(n+1)\left(p^2-2p+2\right))x^n M_{-2n} - p}{2x-(p^2-2p+2)}$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} x^k M_{-2k} = \frac{n(n+1)x^{n-1}M_{-2n+2} + (n+1)\left((n+2)x^n - 2nx^{n-1}\right)M_{-2n}}{2}.$$

(f) $(m = -2, j = 1)$

If $\dfrac{(x-1)\left(-x+p^2-2p+1\right)}{(p-1)^2} \neq 0$, *i.e.,* $x \neq 1$, $x \neq p^2 - 2p + 1$, *then*

$$\sum_{k=0}^{n} x^k M_{-2k+1} = \frac{\begin{array}{c} x^{n+1}M_{-2n+3} + (x - (p^2 - 2p + 2))x^{n+1}M_{-2n+1} \\ + (-p^2x + px - x + p^2 - 2p + 1) \end{array}}{- (x-1)\left(-x+p^2-2p+1\right)}$$

and

if $\dfrac{(x-1)\left(-x+p^2-2p+1\right)}{(p-1)^2} = 0$, *i.e.,* $x = 1$ *or* $x = p^2 - 2p + 1$, *then if* $p \neq 0$ *and* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k M_{-2k+1} = \frac{\begin{array}{c} (n+1)x^n M_{-2n+3} + (x(n+2) \\ - (n+1)\left(p^2 - 2p + 2\right))x^n M_{-2n+1} - (p^2 - p + 1) \end{array}}{2x - (p^2 - 2p + 2)}$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} x^k M_{-2k+1} = \frac{n(n+1)x^{n-1}M_{-2n+3} + (n+1)\left((n+2)x^n - 2nx^{n-1}\right)M_{-2n+1}}{2}.$$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = p$ in the last proposition, we have the following corollary which presents sum formulas of p-Mersenne-Lucas numbers.

Corollary 24. *For $n \geq 0$, p-Mersenne-Lucas numbers have the following properties:*

(a) $(m = 1, j = 0)$

If $(p-1)x^2 - px + 1 \neq 0$, *i.e.,* $x \neq 1$, $x \neq \frac{1}{p-1}$, *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{((p-1)x - p)x^{n+1}H_n + (p-1)x^{n+1}H_{n-1} + 2 - px}{(p-1)x^2 - px + 1}$$

and

if $(p-1)x^2 - px + 1 = 0$, *i.e.,* $x = 1$ *or* $x = \frac{1}{p-1}$, *then if* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{(x(n+2)(p-1) - (n+1)p)x^n H_n + (p-1)(n+1)x^n H_{n-1} - p}{2(p-1)x - p}$$

and if $p = 2$ *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right)H_n + n(n+1)x^{n-1}H_{n-1}}{2}.$$

(b) $(m = 2, j = 0)$

If $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 \neq 0$, *i.e.,* $x \neq 1$, $x \neq \frac{1}{p^2 - 2p + 1}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{\begin{aligned}((p-1)^2 x - (p^2 - 2p + 2))x^{n+1}H_{2n} + (p-1)^2 x^{n+1}H_{2n-2}\\ - x(p^2 - 2p + 2) + 2\end{aligned}}{(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1}$$

and

if $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 = 0$, *i.e.,* $x = 1$ *or* $x = \frac{1}{p^2 - 2p + 1}$, *then if* $p \neq 0$ *and* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{\begin{aligned}(x(n+2)(p-1)^2 - (n+1)(p^2 - 2p + 2))x^n H_{2n}\\ + (p-1)^2 (n+1)x^n H_{2n-2} - p^2 + 2p - 2\end{aligned}}{2(p-1)^2 x - (p^2 - 2p + 2)}$$

and if $p = 0$ *or* $p = 2$, *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{(n+1)x^{n-1}}{2}((2x + nx - 2n)H_{2n} + nH_{2n-2}).$$

(c) $(m = 2, j = 1)$

If $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 \neq 0$, *i.e.,* $x \neq 1$, $x \neq \frac{1}{p^2 - 2p + 1}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{\begin{aligned}((p-1)^2 x - (p^2 - 2p + 2))x^{n+1}H_{2n+1} + (p-1)^2 x^{n+1}H_{2n-1}\\ + p(x - px + 1)\end{aligned}}{(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1}$$

and

if $(p-1)^2 x^2 - (p^2 - 2p + 2)x + 1 = 0$, *i.e.*, $x = 1$ *or* $x = \frac{1}{p^2 - 2p + 1}$, *then if*
$p \neq 0$ *and* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{\begin{array}{c}(x(n+2)(p-1)^2 - (n+1)(p^2 - 2p + 2))x^n H_{2n+1} \\ + (p-1)^2 (n+1)x^n H_{2n-1} - p(p-1)\end{array}}{2(p-1)^2 x - (p^2 - 2p + 2)}$$

and if $p = 0$ *or* $p = 2$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right) H_{2n+1} + n(n+1)x^{n-1} H_{2n-1}}{2}.$$

(d) $(m = -1, j = 0)$

If $\dfrac{(x-1)(p-x-1)}{p-1} \neq 0$, *i.e.*, $x \neq 1$, $x \neq p - 1$, *then*

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{(x-p)x^{n+1} H_{-n} + x^{n+1} H_{-n+1} - px + 2p - 2}{-(x-1)(p-x-1)}$$

and
if $\dfrac{(x-1)(p-x-1)}{p-1} = 0$, *i.e.*, $x = 1$ *or* $x = p - 1$, *then if* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{(x(n+2) - (n+1)p)x^n H_{-n} + (n+1)x^n H_{-n+1} - p}{2x - p}$$

and if $p = 2$, *then*

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{(n+1)\left((n+2)x^n - 2nx^{n-1}\right) H_{-n} + n(n+1)x^{n-1} H_{-n+1}}{2}.$$

(e) $(m = -2, j = 0)$

If $\dfrac{(x-1)(-x + p^2 - 2p + 1)}{(p-1)^2} \neq 0$, *i.e.*, $x \neq 1$, $x \neq p^2 - 2p + 1$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{\begin{array}{c}x^{n+1} H_{-2n+2} + (x - (p^2 - 2p + 2))x^{n+1} H_{-2n} \\ - x(p^2 - 2p + 2) + 2p^2 - 4p + 2\end{array}}{-(x-1)(-x + p^2 - 2p + 1)}$$

and

if $\dfrac{(x-1)\left(-x+p^2-2p+1\right)}{(p-1)^2} = 0$, *i.e.,* $x = 1$ *or* $x = p^2 - 2p + 1$, *then if* $p \neq 0$ *and* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{(n+1)x^n H_{-2n+2} + \left(x(n+2) - (n+1)\left(p^2-2p+2\right)\right) x^n H_{-2n} - p^2 + 2p - 2}{2x - (p^2 - 2p + 2)}$$

and if $p = 0$ *or* $p = 2$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{n(n+1)x^{n-1} H_{-2n+2} + (n+1)\left((n+2)x^n - 2nx^{n-1}\right) H_{-2n}}{2}.$$

(f) $(m = -2, j = 1)$

If $\dfrac{(x-1)\left(-x+p^2-2p+1\right)}{(p-1)^2} \neq 0$, *i.e.,* $x \neq 1$, $x \neq p^2 - 2p + 1$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{x^{n+1} H_{-2n+3} + (x - \left(p^2 - 2p + 2\right))x^{n+1} H_{-2n+1} - p(p^2 x - 3px + 3x - p^2 + 2p - 1)}{-(x-1)\left(-x+p^2-2p+1\right)}$$

and

if $\dfrac{(x-1)\left(-x+p^2-2p+1\right)}{(p-1)^2} = 0$, *i.e.,* $x = 1$ *or* $x = p^2 - 2p + 1$, *then if* $p \neq 0$ *and* $p \neq 2$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{(n+1)x^n H_{-2n+3} + (x(n+2) - (n+1)\left(p^2-2p+2\right))x^n H_{-2n+1} - p(p^2 - 3p + 3)}{2x - (p^2 - 2p + 2)}$$

and if $p = 0$ *or* $p = 2$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{n(n+1)x^{n-1} H_{-2n+3} + (n+1)\left((n+2)x^n - 2nx^{n-1}\right) H_{-2n+1}}{2}.$$

Taking $x = 1$ in the last two corollaries we get the following corollaries.

Corollary 25. *Let $x = 1$. For $n \geq 0$, p-Mersenne numbers have the following properties:*

(a) *If $p \neq 2$, then*

$$\sum_{k=0}^{n} M_k = \frac{1}{p-2}((p-n-2)M_n + (p-1)(n+1)M_{n-1} + 1)$$

and if $p = 2$, then

$$\sum_{k=0}^{n} M_k = \frac{1}{2}(-(n-2)(n+1)M_n + n(n+1)M_{n-1}).$$

(b) *If $p \neq 0$ and $p \neq 2$, then*

$$\sum_{k=0}^{n} M_{2k} = \frac{1}{p(p-2)}((p^2 - 2p - n)M_{2n} + (p-1)^2(n+1)M_{2n-2} + p)$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} M_{2k} = \frac{(n+1)}{2}((2-n)M_{2n} + nM_{2n-2}).$$

(c) *If $p \neq 0$ and $p \neq 2$, then*

$$\sum_{k=0}^{n} M_{2k+1} = \frac{1}{p(p-2)}((p^2 - 2p - n)M_{2n+1} + (p-1)^2(n+1)M_{2n-1} + p - 1)$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} M_{2k+1} = \frac{1}{2}(-(n+1)(n-2)M_{2n+1} + n(n+1)M_{2n-1}).$$

(d) *If $p \neq 2$, then*

$$\sum_{k=0}^{n} M_{-k} = \frac{1}{(p-2)}((np - n + p - 2)M_{-n} - (n+1)M_{-n+1} + 1)$$

and if $p = 2$, then

$$\sum_{k=0}^{n} M_{-k} = \frac{1}{2}(-(n+1)(n-2)M_{-n} + n(n+1)M_{-n+1}).$$

(e) *If $p \neq 0$ and $p \neq 2$, then*

$$\sum_{k=0}^{n} M_{-2k} = \frac{1}{p(p-2)}(-(n+1)M_{-2n+2} + (n(p-1)^2 + p(p-2))M_{-2n} + p)$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} M_{-2k} = \frac{1}{2}(n(n+1)M_{-2n+2} - (n-2)(n+1)M_{-2n}).$$

(f) *If $p \neq 0$ and $p \neq 2$, then*

$$\sum_{k=0}^{n} M_{-2k+1}$$

$$= \frac{1}{p(p-2)}(-(n+1)M_{-2n+3} + (n(p-1)^2 + p(p-2))M_{-2n+1} + (p^2-p+1))$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} M_{-2k+1} = \frac{1}{2}(n(n+1)M_{-2n+3} - (n-2)(n+1)M_{-2n+1}).$$

p-Mersenne-Lucas numbers have the following properties.

Corollary 26. *Let $x = 1$. For $n \geq 0$, p-Mersenne-Lucas numbers have the following properties:*

(a) *If $p \neq 2$, then*

$$\sum_{k=0}^{n} H_k = \frac{1}{p-2}((p-n-2)H_n + (p-1)(n+1)H_{n-1} - p)$$

and if $p = 2$, then

$$\sum_{k=0}^{n} H_k = \frac{1}{2}(-(n-2)(n+1)H_n + n(n+1)H_{n-1}).$$

(b) *If $p \neq 0$ and $p \neq 2$, then*

$$\sum_{k=0}^{n} H_{2k} = \frac{1}{p(p-2)}((p^2-2p-n)H_{2n} + (p-1)^2(n+1)H_{2n-2} - p^2 + 2p - 2)$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} H_{2k} = \frac{(n+1)}{2}((2-n)H_{2n} + nH_{2n-2}).$$

(c) *If $p \neq 0$ and $p \neq 2$, then*

$$\sum_{k=0}^{n} H_{2k+1} = \frac{1}{p(p-2)}((p^2 - 2p - n)H_{2n+1} + (p-1)^2(n+1)H_{2n-1} - p(p-1))$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} H_{2k+1} = \frac{1}{2}(-(n-2)(n+1)H_{2n+1} + n(n+1)H_{2n-1}).$$

(d) *If $p \neq 2$, then*

$$\sum_{k=0}^{n} H_{-k} = \frac{1}{(p-2)}((n(p-1) + p - 2)H_{-n} - (n+1)H_{-n+1} + p)$$

and if $p = 2$, then

$$\sum_{k=0}^{n} H_{-k} = \frac{1}{2}(-(n-2)(n+1)H_{-n} + n(n+1)H_{-n+1}).$$

(e) *If $p \neq 0$ and $p \neq 2$, then*

$$\sum_{k=0}^{n} H_{-2k} = \frac{1}{p(p-2)}(-(n+1)H_{-2n+2} + (n(p-1)^2 + p(p-2))H_{-2n} + p^2 - 2p + 2)$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} H_{-2k} = \frac{1}{2}(n(n+1)H_{-2n+2} - (n-2)(n+1)H_{-2n}).$$

(f) *If $p \neq 0$ and $p \neq 2$, then*

$$\sum_{k=0}^{n} H_{-2k+1}$$
$$= \frac{1}{p(p-2)}(-(n+1)H_{-2n+3} + (n(p-1)^2 + p(p-2))H_{-2n+1} + p(p^2 - 3p + 3))$$

and if $p = 0$ or $p = 2$, then

$$\sum_{k=0}^{n} H_{-2k+1} = \frac{1}{2}(n(n+1)H_{-2n+3} - (n-2)(n+1)H_{-2n+1}).$$

7 Matrices Related with Generalized p-Mersenne Numbers

We define the square matrix A of order 2 as:

$$A = \begin{pmatrix} p & -(p-1) \\ 1 & 0 \end{pmatrix}$$

such that $\det A = p - 1$. Then, we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} p & -(p-1) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n \\ W_{n-1} \end{pmatrix} \tag{7.1}$$

and

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} p & -(p-1) \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = M_n$ in (7.1) we have

$$\begin{pmatrix} M_{n+1} \\ M_n \end{pmatrix} = \begin{pmatrix} p & -(p-1) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_n \\ M_{n-1} \end{pmatrix}. \tag{7.2}$$

We also define

$$M_n = \begin{pmatrix} M_{n+1} & -(p-1)M_n \\ M_n & -(p-1)M_{n-1} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -(p-1)W_n \\ W_n & -(p-1)W_{n-1} \end{pmatrix}.$$

Theorem 27. *For all integers m, n, we have*

(a) $M_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n M_m = M_m C_n$.

Proof. Take $r = p, s = -(p-1)$ in Soykan [25, Theorem 5.1]. $\square$

Corollary 28. *For all integers n, we have the following formulas for the p-Mersenne and p-Mersenne-Lucas numbers.*

(a) *p-Mersenne Numbers.*

$$A^n = \begin{pmatrix} p & -(p-1) \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} M_{n+1} & -(p-1)M_n \\ M_n & -(p-1)M_{n-1} \end{pmatrix}.$$

(b) *p-Mersenne-Lucas Numbers. If $p \neq 2$, then*

$$A^n = \begin{pmatrix} p & -(p-1) \\ 1 & 0 \end{pmatrix}^n = \frac{1}{(p-2)^2} \begin{pmatrix} 2H_{n+2} - pH_{n+1} & -(p-1)(2H_{n+1} - pH_n) \\ 2H_{n+1} - pH_n & -(p-1)(2H_n - pH_{n-1}) \end{pmatrix}$$

and if $p = 2$ (so that $M_n = n$, $H_n = 2$), then

$$A^n = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \frac{n+1}{2}H_{n+1} & -\frac{n}{2}H_n \\ \frac{n}{2}H_n & -\frac{n-1}{2}H_{n-1} \end{pmatrix} = \begin{pmatrix} n+1 & -n \\ n & -(n-1) \end{pmatrix}.$$

Proof. **(a)** It is given in Theorem 27 (a).

(b) Note that if $p = 2$, then $M_n = n$, $H_n = 2$ and so

$$M_n = n = \frac{n}{2} \times 2 = \frac{n}{2}H_n. \tag{7.3}$$

Note also that, from Lemma 11, we have

$$(p-2)^2 M_n = 2H_{n+1} - pH_n. \tag{7.4}$$

Using (7.3), (7.4) and (a), we get required result.

□

Theorem 29. *For all integers m, n, we have*

$$W_{n+m} = W_n M_{m+1} - (p-1)W_{n-1}M_m \tag{7.5}$$

Proof. Take $r = p, s = -(p-1)$ in Soykan [25, Theorem 5.2.].

□

By Lemma 9, we know that

$$(W_0 - W_1)\left(-W_1 + (p-1)W_0\right) M_n = -W_0 W_{n+1} + W_1 W_n,$$

so (7.5) can be written in the following form

$$W_{n+m} = \frac{\begin{array}{c} W_n \left(W_{m+1}\left(W_1 - pW_0\right) + W_m W_0\left(p-1\right)\right) \\ + W_{n-1}\left(p-1\right)\left(W_0 W_{m+1} - W_1 W_m\right) \end{array}}{\left(W_0 - W_1\right)\left(-W_1 + (p-1)W_0\right)}.$$

Corollary 30. *For all integers m, n, we have*

$$M_{n+m} = M_n M_{m+1} - (p-1)M_{n-1}M_m,$$
$$H_{n+m} = H_n M_{m+1} - (p-1)H_{n-1}M_m,$$

and (if $p \neq 2$, then)

$$H_{n+m} = \frac{H_n\left(-pH_{m+1} + 2\left(p-1\right)H_m\right) + H_{n-1}\left(p-1\right)\left(2H_{m+1} - pH_m\right)}{\left(p-2\right)^2}.$$

Note that if $p = 2$ (so that for all integers n, $H_n = 2$), then for all integers m, n, $H_{n+m} = 2$.

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