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On Some Subclasses of m-fold Symmetric Bi-univalent Functions associated with the Sakaguchi Type Functions

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Abstract

In the present investigation, we introduce the subclasses $\Lambda^{\lessdot}_{\Sigma_m}(\sigma, \phi, \upsilon)$ and $\Lambda^{\lessdot}_{\Sigma_m}(\sigma, \gamma, \upsilon)$ of m-fold symmetric bi-univalent function class Σ_m, which are associated with the Sakaguchi type of functions and defined in the open unit disk. Further, we obtain estimates on the initial coefficients b_{m+1} and b_{2m+1} for the functions of these subclasses and find out connections with some of the familiar classes.

1 Introduction

Let $\mathcal{N}$ and $\mathcal{C}$ represent the sets of natural numbers and complex numbers respectively and $\mathcal{A}$ be the family of analytic functions that are defined in $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$ and have the series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathcal{U}, k \in \mathcal{N}). \tag{1.1}$$

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A function $f \in \mathcal{A}$ be a member of the family $\mathcal{S}$ if it is univalent in $\mathcal{U}$. Applying Koebe 1-quarter theorem (see [5]) we can ensure an inverse to each $f \in \mathcal{S}$ defined as

$$f^{-1}(f(z)) = z \quad \text{for} \quad z \in \mathbb{U}$$

and

$$f(f^{-1}(w)) = w \quad \text{for} \quad |w| < r_0(f) \quad \text{and} \quad r_0(f) \geq 1/4,$$

with series expansion

$$f^{-1}(w) = w + (-a_2)w^2 + (2a_2^2 - a_3)w^3 + (5a_2a_3 - 5a_2^3 - a_4)w^4 + \cdots. \quad (1.2)$$

Let $\Sigma = \{f : \mathcal{U} \to \mathcal{C}$ such that $f, f^{-1} \in \mathcal{S}\}$ be the family of bi-univalent functions. This family Σ was introduced in the year 1967 by Lewin [10] along with the result $|a_2|_{f\in\Sigma} < 1.51$. Afterverse, Brannan and Clunie [3] stated that $|a_2|_{f\in\Sigma} \leq \sqrt{2}$. Further, Goodman [7] thought that $|a_n|_{f\in\Sigma} \leq 1$ may be true for every $n \in \mathbb{N}$. However, Netanyahu [12] ensured that $max_{f\in\Sigma}|a_2| = 4/3$. But, then Styer and Wright [29] guaranteed the existence of $f \in \Sigma$ for which $|a_2| > 4/3$.

Indeed, the study of the class Σ has been accelerated greatly due to the fundamental work of Srivastava et al. [27]. After that many researchers ([1], [4], [6], [8], [13], [17], [19], [24], [20], [21], [23], [14] etc.) found coefficient estimates for functions in several subclasses of Σ. But still the coefficient bound problem of $|a_n|$, $(n = 3, 4, 5, \cdots)$ for the functions in Σ is open.

A function $p(z)$ given by

$$p(z) = z + \sum_{k=1}^{\infty} b_{(mk+1)} z^{(mk+1)} \qquad (z \in \mathcal{U},\, m \in \mathcal{N}) \quad (1.3)$$

is known as the m-fold symmetric function (see [9], [16]). Moreover, a function $u(z)$ of the form

$$u(z) = (f(z^m))^{1/m} \qquad (f \in \mathcal{S},\, z \in \mathcal{U},\, m \in \mathcal{N})$$

is univalent and maps $\mathcal{U}$ into a m-fold symmetric region. Let $\mathcal{S}_m$ be the class of univalent and m-fold symmetric functions in $\mathcal{U}$, which are of the form (1.3).

Observe that for $m = 1$, these functions become members of the class $\mathcal{S}$ and are known as symmetric univalent functions.

For each $m \in \mathcal{N}$, every $f \in \Sigma$ produces a m-fold symmetric bi-univalent function. For a function p of the form (1.3), an univalent continuation of p^{-1} to $\mathcal{U}$ is given by (see Srivastava et al. [28]) the series expansion

$$q(w) = w - b_{m+1}w^{m+1} + \left[(m+1)\, b_{m+1}^2 - b_{2m+1}\right] w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)\, b_{m+1}^3 - (3m+2)\, b_{m+1}b_{2m+1} + b_{3m+1}\right] w^{3m+1} + \cdots . \tag{1.4}$$

See that for $m = 1$, it reduces to the equation (1.2). Hence, we can generalize the class Σ to the m-fold symmetric bi-univalent class Σ_m. See [28] for further details regarding to this class Σ_m. Also, for coefficient problems of various subclasses of Σ_m see [2, 25, 26, 30, 22] etc.

In order to prove our theorems, we need the following result.

Lemma 1.1. *[15] Let the function $\upsilon \in \mathcal{P}$ be given by the following series:*

$$\upsilon(z) = 1 + \upsilon_1 z + \upsilon_2 z^2 + \cdots , \quad (z \in \mathcal{U}).$$

The following sharp estimate holds true:

$$|\upsilon_n| \leq 2 \quad (n \in \mathcal{N}).$$

If this $\upsilon(z)$ belongs to Σ_m (see [16]), it takes the form

$$\upsilon(z) = 1 + \upsilon_m z^m + \upsilon_{2m} z^{2m} + \upsilon_{3m} z^{3m} + \cdots , \quad (z \in \mathbb{U}).$$

In the present work, with reference to the Sakaguchi type function classes defined by Lokesh and Keerthi [11], we obtain estimates on initial coefficients $|b_{m+1}|$ and $|b_{2m+1}|$ for functions belong to the new subclasses $\Lambda_{\Sigma_m}^{\lessdot}(\sigma, \phi, \upsilon)$ and $\Lambda_{\Sigma_m}^{\lessdot}(\sigma, \gamma, \upsilon)$ of the function class Σ_m. Also, we have pointed out connections with certain familiar subclasses of the class Σ.

2 Coefficient Bounds for the Function Class $\Lambda_{\Sigma_m}^{\sphericalangle}(\sigma,\phi,\upsilon)$

Definition 2.1. A function $p(z)\in\Sigma_m$ given by (1.3) is said to be in the class $\Lambda_{\Sigma_m}^{\sphericalangle}(\sigma,\phi,\upsilon)$ if the following conditions are fulfilled:

$$\left|\arg\left[\frac{((1-\sigma)z)^{1-\upsilon}(p'(z))^{\sphericalangle}}{(p(z)-p(\sigma z))^{1-\upsilon}}\right]\right|<\frac{\phi\pi}{2}, \tag{2.1}$$

$$(0<\phi\leq 1, 0\leq\upsilon<1, |\sigma|\leq 1, \sphericalangle\geq 1 \textit{ but } \sigma\neq 1, z\in\mathcal{U})$$

and

$$\left|\arg\left[\frac{((1-\sigma)w)^{1-\upsilon}(q'(w))^{\sphericalangle}}{(q(w)-q(\sigma w))^{1-\upsilon}}\right]\right|<\frac{\phi\pi}{2}, \tag{2.2}$$

$$(0<\phi\leq 1, 0\leq\upsilon<1, |\sigma|\leq 1, \sphericalangle\geq 1 \textit{ but } \sigma\neq 1, w\in\mathcal{U})$$

where the function $q=p^{-1}$ is of the form (1.4).

Theorem 2.2. *Let p given by (1.3) be in the class $\Lambda_{\Sigma_m}^{\sphericalangle}(\sigma,\phi,\upsilon)$, $0<\phi\leq 1$. Then*

$$|b_{m+1}|\leq\min\left\{\frac{2\phi}{\sphericalangle(m+1)-(1-\upsilon)\nu_{m+1}},\ \frac{2\phi}{\sqrt{\begin{array}{r}(m+1)[\phi\sphericalangle(2m+1)-\sphericalangle^2(m+1)(\phi-1)-(1-\upsilon)(\phi\nu_{2m+1}\\ +2\sphericalangle\nu_{m+1})]+(1-\upsilon)(1-\upsilon+\phi)\nu_{m+1}^2\end{array}}}\right\} \tag{2.3}$$

and

$$|b_{2m+1}|\leq\frac{2\phi^2(m+1)}{[\sphericalangle(m+1)-(1-\upsilon)\nu_{m+1}]^2}+\frac{2\phi}{[(2m+1)\sphericalangle-(1-\upsilon)\nu_{2m+1}]}, \tag{2.4}$$

where $\nu_{m+1}=1+\sigma+\cdots+\sigma^m$, $\nu_{2m+1}=1+\sigma+\cdots+\sigma^{2m}$.

Proof. Given $p\in\Lambda_{\Sigma_m}^{\sphericalangle}(\sigma,\phi,\upsilon)$. Thus,

$$\left[\frac{((1-\sigma)z)^{1-\upsilon}(p'(z))^{\sphericalangle}}{(p(z)-p(\sigma z))^{1-\upsilon}}\right]=[v(z)]^{\phi}, \tag{2.5}$$

$$\left[\frac{((1-\sigma)w)^{1-\upsilon}(q'(w))^{\leftthreetimes}}{(q(w)-q(\sigma w))^{1-\upsilon}}\right] = [y(w)]^{\phi}, \tag{2.6}$$

where $\upsilon(z)$ and $y(w)$ in $\mathcal{P}$ with the series representation

$$\upsilon(z) = 1 + \upsilon_m z^m + \upsilon_{2m} z^{2m} + \cdots, \quad y(z) = 1 + y_m z^m + y_{2m} z^{2m} + \cdots.$$

Now, equating the coefficients in (2.5) and (2.6), we have

$$[\leftthreetimes(m+1) - (1-\upsilon)\nu_{m+1}]b_{m+1} = \phi \upsilon_m, \tag{2.7}$$

$$[\leftthreetimes(2m+1) - (1-\upsilon)\nu_{2m+1}]b_{2m+1} + \frac{1}{2}\Bigg[(1-\upsilon)\left((2-\upsilon)\nu_{m+1} - 2\leftthreetimes(m+1)\right) \nu_{m+1} + \leftthreetimes(\leftthreetimes-1)(m+1)^2\Bigg] b_{m+1}^2 = \phi \upsilon_{2m} + \frac{\phi(\phi-1)}{2}\upsilon_m^2, \tag{2.8}$$

$$-[\leftthreetimes(m+1) - (1-\upsilon)\nu_{m+1}]b_{m+1} = \phi y_m, \tag{2.9}$$

$$[\leftthreetimes(2m+1) - (1-\upsilon)\nu_{2m+1}][(m+1)b_{m+1}^2 - b_{2m+1}] + \frac{1}{2}\Bigg[(1-\upsilon)((2-\upsilon)\nu_{m+1} - 2\leftthreetimes(m+1))\nu_{m+1} + \leftthreetimes(\leftthreetimes-1)(m+1)^2\Bigg] b_{m+1}^2 = \phi y_{2m} + \frac{\phi(\phi-1)}{2} y_m^2. \tag{2.10}$$

From (2.7) and (2.9) we obtain

$$\upsilon_m = -y_m, \tag{2.11}$$

$$2[\leftthreetimes(m+1) - (1-\upsilon)\nu_{m+1}]^2 b_{m+1}^2 = \phi^2[\upsilon_m^2 + y_m^2]. \tag{2.12}$$

Also from (2.8), (2.10) and (2.12) we have

$$\{(m+1)[\leftthreetimes(2m+1) - (1-\upsilon)\nu_{2m+1}] + (1-\upsilon)\nu_{m+1}((2-\upsilon)\nu_{m+1} - 2\leftthreetimes(m+1)) + \leftthreetimes(\leftthreetimes-1)(m+1)^2\} b_{m+1}^2 = \phi(\upsilon_{2m} + y_{2m}) + \frac{\phi(\phi-1)[\leftthreetimes(m+1) - (1-\upsilon)\nu_{m+1}]^2}{\phi^2} b_{m+1}^2. \tag{2.13}$$

Therefore, we have

$$b_{m+1}^2 = \frac{\phi^2(v_{2m}+y_{2m})}{(m+1)[\phi\curlywedge(2m+1)-\curlywedge^2(m+1)(\phi-1)-(1-\upsilon)(\phi\nu_{2m+1}+2\curlywedge\nu_{m+1})]+(1-\upsilon)(1-\upsilon+\phi)\nu_{m+1}^2}. \tag{2.14}$$

Thus, we obtain from relations (2.12) and (2.14) that

$$|b_{m+1}^2| \leq \frac{\phi^2[|v_m^2|+|y_m^2|]}{2[\curlywedge(m+1)-(1-\upsilon)\nu_{m+1}]^2} \tag{2.15}$$

and

$$|b_{m+1}^2| \leq \frac{\phi^2(|v_{2m}|+|y_{2m}|)}{(m+1)[\phi\curlywedge(2m+1)-\curlywedge^2(m+1)(\phi-1)-(1-\upsilon)(\phi\nu_{2m+1}+2\curlywedge\nu_{m+1})]+(1-\upsilon)(1-\upsilon+\phi)\nu_{m+1}^2}, \tag{2.16}$$

respectively. Applying Lemma 1.1 for the coefficients v_m, v_{2m}, y_m, y_{2m} we have the desired estimates on $|b_{m+1}|$ as given in (2.3). Next, in order to find the bound on $|b_{2m+1}|$, by subtracting (2.10) from (2.8), we obtain

$$2[\curlywedge(2m+1)-(1-\upsilon)\nu_{2m+1}]b_{2m+1}-[(2m+1)\curlywedge-(1-\upsilon)\nu_{2m+1}](m+1)b_{m+1}^2 = \phi[v_{2m}-y_{2m}]+\frac{\phi(\phi-1)}{2}[v_m^2-y_m^2]. \tag{2.17}$$

Then, in view of (2.11) and (2.12), and applying Lemma 1.1 for the coefficients v_m, v_{2m}, y_m, y_{2m}, we have

$$|b_{2m+1}| \leq \frac{2\phi^2(m+1)}{[\curlywedge(m+1)-(1-\upsilon)\nu_{m+1}]^2}+\frac{2\phi}{[(2m+1)\curlywedge-(1-\upsilon)\nu_{2m+1}]}.$$

This completes the proof of Theorem 2.2. □

3 Coefficient Bounds for the Function Class $\Lambda_{\Sigma_m}^{\curlywedge}(\sigma,\gamma,\upsilon)$

Definition 3.1. A function $p(z)\in\Sigma_m$ given by (1.3) is said to be in the class $\Lambda_{\Sigma_m}^{\curlywedge}(\sigma,\gamma,\upsilon)$ if the following conditions are fulfilled:

$$\left[\frac{((1-\sigma)z)^{1-\upsilon}(p'(z))^{\curlywedge}}{(p(z)-p(\sigma z))^{1-\upsilon}}\right] > \gamma, \tag{3.1}$$

$$(0 \leq \gamma < 1, 0 \leq \upsilon < 1, |\sigma| \leq 1, \leftthreetimes \geq 1 \text{ but } \sigma \neq 1, z \in \mathcal{U})$$

and

$$\left[\frac{((1-\sigma)w)^{1-\upsilon}(q'(w))^{\leftthreetimes}}{(q(w)-q(\sigma w))^{1-\upsilon}}\right] > \gamma, \tag{3.2}$$

$$(0 \leq \gamma < 1, 0 \leq \upsilon < 1, |\sigma| \leq 1, \leftthreetimes \geq 1 \text{ but } \sigma \neq 1, w \in \mathcal{U})$$

where the function $q = p^{-1}$ is of the form (1.4).

Theorem 3.2. *Let $p(z)$ given by (1.3) be in the class $\Lambda^{\leftthreetimes}_{\Sigma_m}(\sigma, \gamma, \upsilon)$, $0 \leq \gamma < 1$. Then*

$$|b_{m+1}| \leq \min\left\{ \frac{2(1-\gamma)}{\leftthreetimes(m+1)-(1-\upsilon)\nu_{m+1}}, \right.$$
$$\left. \frac{2\sqrt{1-\gamma}}{\sqrt{\begin{array}{r}(m+1)[\leftthreetimes(2m+1)-(1-\upsilon)\nu_{2m+1}]+(1-\upsilon)((2-\upsilon)\nu_{m+1} \\ -2\leftthreetimes(m+1))\nu_{m+1}+\leftthreetimes(\leftthreetimes-1)(m+1)^2\end{array}}} \right\} \tag{3.3}$$

and

$$|b_{2m+1}| \leq \frac{2(1-\gamma)^2(m+1)}{[\leftthreetimes(m+1)-(1-\upsilon)\nu_{m+1}]^2} + \frac{2(1-\gamma)}{(2m+1)\leftthreetimes-(1-\upsilon)\nu_{2m+1}}, \tag{3.4}$$

where $\nu_{m+1} = 1+\sigma+\cdots+\sigma^m$, $\nu_{2m+1} = 1+\sigma+\cdots+\sigma^{2m}$.

Proof. Given $p \in \Lambda^{\leftthreetimes}_{\Sigma_m}(\sigma, \gamma, \upsilon)$. Thus,

$$\left[\frac{((1-\sigma)z)^{1-\upsilon}(p'(z))^{\leftthreetimes}}{(p(z)-p(\sigma z))^{1-\upsilon}}\right] = \gamma + (1-\gamma)v(z), \tag{3.5}$$

$$\left[\frac{((1-\sigma)w)^{1-\upsilon}(q'(w))^{\leftthreetimes}}{(q(w)-q(\sigma w))^{1-\upsilon}}\right] = \gamma + (1-\gamma)y(w), \tag{3.6}$$

where $v, y \in \mathcal{P}$ and $q = p^{-1}$.

It follows from (3.5) and (3.6) that

$$[\leftthreetimes(m+1)-(1-\upsilon)\nu_{m+1}]b_{m+1} = (1-\gamma)v_m, \tag{3.7}$$

$$[\leftthreetimes(2m+1)-(1-\upsilon)\nu_{2m+1}]b_{2m+1}+\frac{1}{2}\Bigg[(1-\upsilon)\left((2-\upsilon)\nu_{m+1}-2\leftthreetimes(m+1)\right)\nu_{m+1}+\leftthreetimes(\leftthreetimes-1)(m+1)^2\Bigg]b_{m+1}^2=(1-\gamma)\upsilon_{2m}, \quad (3.8)$$

$$-[\leftthreetimes(m+1)-(1-\upsilon)\nu_{m+1}]b_{m+1}=(1-\gamma)y_m, \quad (3.9)$$

$$[\leftthreetimes(2m+1)-(1-\upsilon)\nu_{2m+1}][(m+1)b_{m+1}^2-b_{2m+1}]+\frac{1}{2}\Bigg[(1-\upsilon)((2-\upsilon)\nu_{m+1}-2\leftthreetimes(m+1))\nu_{m+1}+\leftthreetimes(\leftthreetimes-1)(m+1)^2\Bigg]b_{m+1}^2=(1-\gamma)y_{2m}. \quad (3.10)$$

From (3.7) and (3.9) we obtain

$$\upsilon_m=-y_m, \quad (3.11)$$

$$2[\leftthreetimes(m+1)-(1-\upsilon)\nu_{m+1}]^2b_{m+1}^2=(1-\gamma)^2(\upsilon_m^2+y_m^2). \quad (3.12)$$

Also from (3.8), (3.10) and (3.12) we have

$$\{(m+1)[\leftthreetimes(2m+1)-(1-\upsilon)\nu_{2m+1}]+(1-\upsilon)\nu_{m+1}((2-\upsilon)\nu_{m+1}-2\leftthreetimes(m+1))+\leftthreetimes(\leftthreetimes-1)(m+1)^2\}b_{m+1}^2=(1-\gamma)(\upsilon_{2m}+y_{2m}). \quad (3.13)$$

Therefore, we have

$$b_{m+1}^2=\frac{(1-\gamma)(\upsilon_{2m}+y_{2m})}{(m+1)[\leftthreetimes(2m+1)-(1-\upsilon)\nu_{2m+1}]+(1-\upsilon)\nu_{m+1}((2-\upsilon)\nu_{m+1}-2\leftthreetimes(m+1))+\leftthreetimes(\leftthreetimes-1)(m+1)^2}. \quad (3.14)$$

Thus, we obtain from relations (3.12) and (3.14) that

$$|b_{m+1}^2|\leq\frac{(1-\gamma)^2[|\upsilon_m^2|+|y_m^2|]}{2[\leftthreetimes(m+1)-(1-\upsilon)\nu_{m+1}]^2} \quad (3.15)$$

and

$$|b_{m+1}^2| \leq \frac{(1-\gamma)(|v_{2m}|+|y_{2m}|)}{(m+1)[\sphericalangle(2m+1)-(1-\upsilon)\nu_{2m+1}]+(1-\upsilon)\nu_{m+1}((2-\upsilon)\nu_{m+1} - 2\sphericalangle(m+1))+\sphericalangle(\sphericalangle-1)(m+1)^2}, \tag{3.16}$$

respectively. Applying Lemma 1.1 for the coefficients v_m, v_{2m}, y_m, y_{2m} we have the desired estimates on $|b_{m+1}|$ as given in (3.3). Next, in order to find the bound on $|b_{2m+1}|$, by subtracting (3.10) from (3.8), we obtain

$$2[\sphericalangle(2m+1)-(1-\upsilon)\nu_{2m+1}]b_{2m+1} - [(2m+1)\sphericalangle-(1-\upsilon)\nu_{2m+1}](m+1)b_{m+1}^2 = (1-\gamma)(v_{2m}-y_{2m}). \tag{3.17}$$

Then, in view of (3.11) and (3.12), and applying Lemma 1 for the coefficients v_m, v_{2m}, y_m, y_{2m}, we have

$$|b_{2m+1}| \leq \frac{2(1-\gamma)^2(m+1)}{[\sphericalangle(m+1)-(1-\upsilon)\nu_{m+1}]^2} + \frac{2(1-\gamma)}{(2m+1)\sphericalangle-(1-\upsilon)\nu_{2m+1}}. \tag{3.18}$$

This completes the proof of Theorem 3.2. □

4 Corollaries and Consequences

By setting $\upsilon = 0$ in Theorems 2.2 and 3.2, we have the following corollary.

Corollary 4.1. *Let p given by (1.3) be in the class $\Lambda_{\Sigma_m}^{\sphericalangle}(\sigma,\phi)$, $0<\phi\leq 1$. Then*

$$|b_{m+1}| \leq \min\left\{ \frac{2\phi}{\sphericalangle(m+1)-\nu_{m+1}}, \frac{2\phi}{\sqrt{(m+1)[\phi\sphericalangle(2m+1)-\sphericalangle^2(m+1)(\phi-1)-(\phi\nu_{2m+1}+2\sphericalangle\nu_{m+1})]+(1-\phi)\nu_{m+1}^2}} \right\}$$

and

$$|b_{2m+1}| \leq \frac{2\phi^2(m+1)}{[\sphericalangle(m+1)-\nu_{m+1}]^2} + \frac{2\phi}{(2m+1)\sphericalangle-\nu_{2m+1}},$$

where $\nu_{m+1} = 1+\sigma+\cdots+\sigma^m$, $\nu_{2m+1} = 1+\sigma+\cdots+\sigma^{2m}$.

Corollary 4.2. *Let p given by (1.3) be in the class $\Lambda_{\Sigma_m}^{\reflectbox{λ}}(\sigma,\gamma)$, $0 \le \gamma < 1$. Then*

$$|b_{m+1}| \le \min\left\{ \frac{2(1-\gamma)}{\reflectbox{λ}(m+1)-\nu_{m+1}}, \frac{2\sqrt{1-\gamma}}{\sqrt{(m+1)[\reflectbox{λ}(2m+1)-\nu_{2m+1}] + (2\nu_{m+1} - 2\reflectbox{λ}(m+1))\nu_{m+1} + \reflectbox{λ}(\reflectbox{λ}-1)(m+1)^2}} \right\}$$

and

$$|b_{2m+1}| \le \frac{2(1-\gamma)^2(m+1)}{[\reflectbox{λ}(m+1)-\nu_{m+1}]^2} + \frac{2(1-\gamma)}{(2m+1)\reflectbox{λ} - \nu_{2m+1}},$$

where $\nu_{m+1} = 1+\sigma+\cdots+\sigma^m$, $\nu_{2m+1} = 1+\sigma+\cdots+\sigma^{2m}$.

By setting $m = 1$ (1-fold) in Corollaries 4.1 and 4.2, we have the Corollaries 4.3 and 4.4 as follows:

Corollary 4.3. *Let f given by (1.1) be in the class $\Lambda_{\Sigma}^{\reflectbox{λ}}(\sigma,\phi)$, $0 < \phi \le 1$. Then*

$$|a_2| \le \min\left\{ \frac{2\phi}{2\reflectbox{λ} - (1+\sigma)}, \frac{2\phi}{\sqrt{(2\reflectbox{λ}-(1+\sigma))^2 + \{4\reflectbox{λ}^2 + 2\reflectbox{λ} + 2(1+\sigma)^2 - 4\reflectbox{λ}(1+\sigma) - 2(1+\sigma+\sigma^2) - (2\reflectbox{λ}-(1+\sigma))^2\}\phi}} \right\}$$

and

$$|a_3| \le \frac{4\phi^2}{[2\reflectbox{λ}-(1+\sigma)]^2} + \frac{2\phi}{3\reflectbox{λ} - (1+\sigma+\sigma^2)}.$$

Corollary 4.4. *Let f given by (1.1) be in the class $\Lambda_{\Sigma}^{\reflectbox{λ}}(\sigma,\gamma)$, $0 \le \gamma < 1$. Then*

$$|a_2| \le \min\left\{ \frac{2(1-\gamma)}{2\reflectbox{λ} - (1+\sigma)}, \frac{2\sqrt{1-\gamma}}{\sqrt{(4\reflectbox{λ}^2 + 2\reflectbox{λ} - 4\reflectbox{λ}(1+\sigma) + 2(1+\sigma)^2 - 2(1+\sigma+\sigma^2))}} \right\}$$

and

$$|a_3| \le \frac{4(1-\gamma)^2}{[2\reflectbox{λ}-(1+\sigma)]^2} + \frac{2(1-\gamma)}{3\reflectbox{λ} - (1+\sigma+\sigma^2)}.$$

Remark 4.5. The above estimates $|a_2|$ and $|a_3|$ show that Corollaries 4.3 and 4.4 is an improvement of the estimate obtain by Lokesh and Keerthi ([11], Corollaries 3.1 and 3.2).

Corollary 4.6. *[11] Let f given by* (1.1) *be in the class $\Lambda_{\Sigma}^{\leftthreetimes}(\sigma,\phi)$, $0<\phi\leq 1$. Then*

$$|a_2| \leq \frac{2\phi}{\sqrt{\begin{array}{r}(2\leftthreetimes-(1+\sigma))^2+\{4\leftthreetimes^2+2\leftthreetimes+2(1+\sigma)^2-4\leftthreetimes(1+\sigma)\\ -2(1+\sigma+\sigma^2)-(2\leftthreetimes-(1+\sigma))^2\}\phi\end{array}}}$$

and

$$|a_3| \leq \frac{4\phi^2}{[2\leftthreetimes-(1+\sigma)]^2}+\frac{2\phi}{3\leftthreetimes-(1+\sigma+\sigma^2)}.$$

Corollary 4.7. *[11] Let f given by* (1.1) *be in the class $\Lambda_{\Sigma}^{\leftthreetimes}(\sigma,\gamma)$, $0\leq\gamma<1$. Then*

$$|a_2| \leq \frac{2\sqrt{1-\gamma}}{\sqrt{(4\leftthreetimes^2+2\leftthreetimes-4\leftthreetimes(1+\sigma)+2(1+\sigma)^2-2(1+\sigma+\sigma^2))}}$$

and

$$|a_3| \leq \frac{4(1-\gamma)^2}{[2\leftthreetimes-(1+\sigma)]^2}+\frac{2(1-\gamma)}{3\leftthreetimes-(1+\sigma+\sigma^2)}.$$

By setting $\leftthreetimes=0$ in Theorems 2.2 and 3.2, we have the following corollary.

Corollary 4.8. *Let p given by* (1.3) *be in the class $\Lambda_{\Sigma_m}(\sigma,\phi,\upsilon)$, $0<\phi\leq 1$. Then*

$$|b_{m+1}| \leq \min\left\{\frac{2\phi}{(m+1)-(1-\upsilon)\nu_{m+1}},\ \frac{2\phi}{\sqrt{\begin{array}{r}(m+1)\{1+m(1-\phi)-(1-\upsilon)[\phi\nu_{2m+1}-2\nu_{m+1}]\}\\ +(1-\upsilon)[1-\upsilon+\phi]\nu_{2m+1}^2\end{array}}}\right\}$$

and

$$|b_{2m+1}| \leq \frac{2\phi^2(m+1)}{[(m+1)-(1-\upsilon)\nu_{m+1}]^2}+\frac{2\phi}{[(2m+1)-(1-\upsilon)\nu_{2m+1}]},$$

where $\nu_{m+1}=1+\sigma+\cdots+\sigma^m,\ \ \nu_{2m+1}=1+\sigma+\cdots+\sigma^{2m}.$

Corollary 4.9. *Let p given by* (1.3) *be in the class* $\Lambda^{\lessdot}_{\Sigma_m}(\sigma,\gamma,\upsilon)$, $0\le\gamma<1$. *Then*

$$|b_{m+1}|\le \min\left\{\frac{2(1-\gamma)}{(m+1)-(1-\upsilon)\nu_{m+1}},\ \frac{2\sqrt{1-\gamma}}{\sqrt{(m+1)[(2m+1)-(1-\upsilon)\nu_{2m+1}]+(1-\upsilon)((2-\upsilon)\nu_{m+1}-2(m+1))\nu_{m+1}}}\right\}$$

and

$$|b_{2m+1}|\le\frac{2(1-\gamma)^2(m+1)}{[(m+1)-(1-\upsilon)\nu_{m+1}]^2}+\frac{2(1-\gamma)}{(2m+1)-(1-\upsilon)\nu_{2m+1}},$$

where $\nu_{m+1}=1+\sigma+\cdots+\sigma^m$, $\nu_{2m+1}=1+\sigma+\cdots+\sigma^{2m}$.

Remark 4.10. The above estimates $|b_{m+1}|$ and $|b_{2m+1}|$ show that Corollaries 4.8 and 4.9 is an improvement of the estimate obtain by Senthil and Keerthi ([18], Theorem 6 and 7).

Corollary 4.11. *[18] Let p given by* (1.3) *be in the class* $\Lambda_{\Sigma_m}(\sigma,\phi,\upsilon)$, $0<\phi\le 1$. *Then*

$$|b_{m+1}|\le\frac{2\phi}{\sqrt{(m+1)\{1+m(1-\phi)-(1-\upsilon)[\phi\nu_{2m+1}-2\nu_{m+1}]\}+(1-\upsilon)[1-\upsilon+\phi]\nu_{2m+1}^2}}$$

and

$$|b_{2m+1}|\le\frac{2\phi^2(m+1)}{[(m+1)-(1-\upsilon)\nu_{m+1}]^2}+\frac{2\phi}{[(2m+1)-(1-\upsilon)\nu_{2m+1}]},$$

where $\nu_{m+1}=1+\sigma+\cdots+\sigma^m$, $\nu_{2m+1}=1+\sigma+\cdots+\sigma^{2m}$.

Corollary 4.12. *[18] Let p given by* (1.3) *be in the class* $\Lambda^{\lessdot}_{\Sigma_m}(\sigma,\gamma,\upsilon)$, $0\le\gamma<1$. *Then*

$$|b_{m+1}|\le\frac{2\sqrt{1-\gamma}}{\sqrt{(m+1)[(2m+1)-(1-\upsilon)\nu_{2m+1}]+(1-\upsilon)((2-\upsilon)\nu_{m+1}-2(m+1))\nu_{m+1}}}$$

and

$$|b_{2m+1}|\le\frac{2(1-\gamma)^2(m+1)}{[(m+1)-(1-\upsilon)\nu_{m+1}]^2}+\frac{2(1-\gamma)}{(2m+1)-(1-\upsilon)\nu_{2m+1}},$$

where $\nu_{m+1}=1+\sigma+\cdots+\sigma^m$, $\nu_{2m+1}=1+\sigma+\cdots+\sigma^{2m}$.

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