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On New Type of Sets in Ideal Topological Spaces

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Abstract

In this paper, we introduce the notion of $I_{\tilde{g}\alpha}$-closed sets in ideal topological spaces and investigate some of their properties. Further, we introduce the concept of mildly $I_{\tilde{g}\alpha}$-closed sets and $I_{\tilde{g}\alpha}$ normal space.

1. Introduction and Preliminaries

Levine [7, 8] introduced the concept of generalized closed sets and semiclosed sets in topological spaces. The concept of $\tilde{g}\alpha$-closed sets were introduced by Devi et al. [2]. Dontchev et al. [4] introduced the notion of the generalized closed sets in ideal topological space (i.e. $\mathcal{I}$-g-closed sets) in 1999. In 2008, Navaneethakrishnan and Joseph have studied some characterizations of normal spaces via I_g open sets [10]. In this paper, we introduce the notion of $I_{\tilde{g}\alpha}$-closed sets in ideal topological spaces and investigate some of their properties. Further, we introduce the concept of mildly $I_{\tilde{g}\alpha}$-closed sets.

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An ideal $\mathcal{I}$ [5] on a topological space (X, τ) is a non-empty collection of subsets of X satisfies

(a) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and

(b) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \bigcup B \in \mathcal{I}$.

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $P(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : P(X) \to P(X)$, called a local function [5] of A with respect to τ and $\mathcal{I}$ is defined as follows: For $A \subset X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \bigcap A \neq \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. We will make use of the basic facts about the local functions [5, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the τ^*-topology, finer than τ is defined by $cl^*(A) = A \bigcup A^*(\mathcal{I}, \tau)$ [16]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called ideal space. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is τ^* closed [5] if $A^* \subset A$.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $cl(A)$ and $\text{int}(A)$ will, respectively, denote the closure and interior of A in (X, τ) and $\text{int}^*(A)$ will denote the interior of A in (X, τ^*).

Definition 1.1. A subset A of a space (X, τ) is called a

(a) *semi-open set* [8] if $A \subseteq cl(\text{int}(A))$ and a *semi-closed set* [8] if $\text{int}(cl(A)) \subseteq A$,

(b) α-*open set* [12] if $A \subseteq \text{int}(cl(\text{int}(A)))$ and an α-*closed set* [12] if $cl(\text{int}(cl(A))) \subseteq A$ and

(c) *regular open* [15] if $A = \text{int}(cl(A))$.

The semi-closure (resp. α-closure) of a subset A of a space (X, τ) is the intersection of all semi-closed (resp. α-closed) sets that contain A and is denoted by $scl(A)$ (resp. $\alpha cl(A)$).

Definition 1.2. A subset A of a topological space (X, τ) is called

(a) a *g-closed set* [7] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ),

(b) an αg*-closed set* [9] if $\alpha cl(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ),

(c) a $\hat{g}$*-closed set* [18, 20] if $cl(A) \subset U$ whenever $A \subset U$ and U is semi-open in (X, τ),

(d) a ${}^{*}g$*-closed set* [17] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in (X, τ),

(e) a ${}^{\#}gs$*-closed set* [19] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is ${}^{*}g$-open in (X, τ), and

(f) a $\tilde{g}\alpha$*-closed set* [2] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ${}^{\#}gs$-open set of (X, τ). The complement of an $\tilde{g}\alpha$-closed set is called $\tilde{g}\alpha$*-open*.

The set $\bigcap\{F \subset X : F \supseteq A, F \text{ is } \tilde{g}\alpha\text{-closed}\}$ is called $\tilde{g}\alpha$*-closure* of A and is denoted by $cl\tilde{g}\alpha(A)$.

Definition 1.3. A subset A of an ideal topological space (X, τ, I) is called

(a) an I_g *closed* [4] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open in (X, I, τ),

(b) an I_{rg} *closed* [11] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, I, τ),

(c) an $I_{\alpha gg}$ *closed* [13] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is αg-open in (X, I, τ),

(d) an *I-R closed* [1] if $A = cl^*(\text{int}(A))$ and

(e) a *pre-I-closed* [3] if $cl^*(\text{int}(A)) \subseteq A$.

Lemma 1.4 [14]**.** *Let (X, τ, I) be an ideal topological space $A \subseteq X$. If $A \subseteq A^*$, then $A^* = cl(A^*) = cl(A) = cl^*(A)$.*

Lemma 1.5 [5]. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then the following properties hold*:

(a) $A \subset B$ *implies* $A^* \subset B^*$,

(b) $A^* = cl(A^*) \subset cl(A)$,

(c) $(A^*)^* \subset A^*$,

(d) $(A \cup B)^* = A^* \cup B^*$.

2. Properties of $I_{\tilde{g}\alpha}$-closed Sets in Ideal Topological Spaces

Definition 2.1. A subset A of an ideal space (X, τ, I) is said to be $I_{\tilde{g}\alpha}$-*closed set* if $A^* \subseteq U$ whenever $A \subseteq U$ and U is $\tilde{g}\alpha$-open set.

Theorem 2.2.

(a) *Every* $*$-*closed set is* $I_{\tilde{g}\alpha}$-*closed set.*

(b) *Every* $I_{\alpha gg}$-*closed set is* $I_{\tilde{g}\alpha}$-*closed set.*

(c) *Every* $I_{\tilde{g}\alpha}$-*closed set is* I_{rg}-*closed set.*

(d) *Every* $I_{\tilde{g}\alpha}$-*closed set is* I_g-*closed set.*

Proof.

(a) It is obvious.

(b) Let $A \subseteq U$ and U is $\tilde{g}\alpha$-open set and hence αg-open set. Since A is $I_{\alpha gg}$-closed, we have $A^* \subseteq U$. Therefore A is $I_{\tilde{g}\alpha}$-closed set.

(c) Let $A \subseteq U$ and U is regular open set and hence $\tilde{g}\alpha$-open set. Since A is $I_{\tilde{g}\alpha}$-closed, we have $A^* \subseteq U$. Therefore A is I_{rg}-closed set.

(d) Let $A \subseteq U$ and U is open set and hence $\tilde{g}\alpha$-open set. Since A is $I_{\tilde{g}\alpha}$-closed, we have $A^* \subseteq U$. Therefore A is I_g-closed set.

The converse of the above theorems need not be true by the following examples.

Example 2.3.

(a) Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c\}$, $\tau = \{X, \varnothing, \{a, b\}\}$ and $I = \{\varnothing\}$. Then $\{a, c\}$ is $I_{\tilde{g}\alpha}$-closed set but not $*$-closed.

(b) Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c\}$, $\tau = \{X, \varnothing, \{a\}, \{b, c\}\}$ and $I = \{\varnothing, \{c\}\}$. Then $\{b\}$ is $I_{\tilde{g}\alpha}$-closed set but not $I_{\alpha gg}$-closed.

(c) Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\varnothing\}$. Then $\{c\}$ is I_{rg}-closed set but not $I_{\tilde{g}\alpha}$-closed.

(d) Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c\}$, $\tau = \{X, \varnothing, \{a\}\}$ and $I = \{\varnothing, \{a\}\}$. Then $\{a, c\}$ is I_g-closed set but not $I_{\tilde{g}\alpha}$-closed.

Theorem 2.4. *The union of two $I_{\tilde{g}\alpha}$-closed sets is $I_{\tilde{g}\alpha}$-closed set.*

Proof. Let A and B are $I_{\tilde{g}\alpha}$-closed sets. Let U be an $I_{\tilde{g}\alpha}$-open set containing $A \cup B$. Since A and B are $I_{\tilde{g}\alpha}$-closed sets, $A^* \subseteq U$ and $B^* \subseteq U$. We have $(A \cup B)^* = A^* \cup B^*$, $(A \cup B)^* \subseteq U$. Therefore $A \cup B$ is $I_{\tilde{g}\alpha}$-closed set.

Remark 2.5. The intersection of two $I_{\tilde{g}\alpha}$-closed sets need not be $I_{\tilde{g}\alpha}$-closed.

Proof. It follows from the following example.

Example 2.6. Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\varnothing\}$. Then $A = \{a, c\}$ and $B = \{a, d\}$ are $I_{\tilde{g}\alpha}$-closed set but $A \cap B = \{a\}$ is not $I_{\tilde{g}\alpha}$-closed.

Theorem 2.7. *Let (X, τ, I) be an ideal topological space. For every $A \in I$, A is $I_{\tilde{g}\alpha}$-closed.*

Proof. Let $A \subseteq U$ and U is $\tilde{g}\alpha$-open set. Since $A^* = \varnothing$, $A^* \subseteq U$. Therefore A is $I_{\tilde{g}\alpha}$-closed.

Theorem 2.8. *If* (X, τ, I) *be an ideal topological space, then* A^* *is always* $I_{\tilde{g}\alpha}$*-closed for every subset A of X.*

Proof. Let $A^* \subseteq U$ and U is $\tilde{g}\alpha$-open set. Since $(A^*)^* \subseteq A^*$, we have $(A^*)^* \subseteq U$ implies $A^* \subseteq U$. Hence A^* is $I_{\tilde{g}\alpha}$-closed.

Theorem 2.9. *If* (X, τ, I) *be an ideal topological space, then every* $I_{\tilde{g}\alpha}$*-closed,* $\tilde{g}\alpha$*-open set is* $*$*-closed set.*

Proof. Since A is $I_{\tilde{g}\alpha}$-closed and $\tilde{g}\alpha$-open set. Then $A^* \subseteq A$, $A \subseteq A$ and A is $\tilde{g}\alpha$-open. Hence A is $*$-closed set.

Theorem 2.10. *If* (X, τ, I) *be an ideal topological space and A is a subset of X, then the following are equivalent.*

(a) *A is* $I_{\tilde{g}\alpha}$*-closed.*

(b) $cl^*(A) \subseteq U$, $A \subseteq U$ *and U is* $\tilde{g}\alpha$*-open in X.*

(c) *For all* $x \in cl^*(A)$, $\tilde{g}\alpha cl\{x\} \cap A \neq \emptyset$.

(d) $cl^*(A) - A$ *contains no non-empty* $\tilde{g}\alpha$*-closed set.*

(e) $A^* - A$ *contains no non-empty* $\tilde{g}\alpha$*-closed set.*

Proof. (a) $\Rightarrow$ (b) If A is $I_{\tilde{g}\alpha}$-closed, then $A^* \subseteq U$, $A \subseteq U$ and U is $\tilde{g}\alpha$-open in X and so $cl^*(A) = A \cup A^* \subseteq U$, $A \subseteq U$ and U is $\tilde{g}\alpha$-open in X.

(b) $\Rightarrow$ (c) Suppose $x \in cl^*(A)$. If $\tilde{g}\alpha cl\{x\} \cap A = \emptyset$, then $A \subseteq X - \tilde{g}\alpha cl\{x\}$. By (b) $cl^*(A) \subseteq X - \tilde{g}\alpha\{x\}$, a contradiction.

(c) $\Rightarrow$ (d) Suppose $F \subseteq cl^*(A) - A$, F is $\tilde{g}\alpha$-closed and $x \in F$. Since $F \subseteq X - A$ and F is $\tilde{g}\alpha$-closed, then $A \subseteq X - F$ and F is $\tilde{g}\alpha$-closed, $\tilde{g}\alpha cl\{x\} \cap A = \emptyset$. Since $x \in cl^*(A)$, by (c) $\tilde{g}\alpha cl\{x\} \cap A \neq \emptyset$. Therefore $cl^*(A) - A$ contains no non-empty $\tilde{g}\alpha$-closed set.

(d) $\Rightarrow$ (e) Since $cl^*(A)-A=(A\cup A^*)-A=(A\cup A^*)\cap A^c=(A\cap A^c)\cup(A^*\cap A^c)$ $= A^*\cap A^c = A^* - A$. Therefore $A^* - A$ contains no non-empty $\tilde{g}\alpha$-closed set.

(e) $\Rightarrow$ (a) Let $A \subseteq U$ and U is $\tilde{g}\alpha$-closed set. Therefore $X - U \subseteq X - A$ and $A^* \cap (X-U) \subseteq A^* \cap (X - A) = A^* - A$. Since A^* is always closed set, so $A^* \cap (X-U)$ is $\tilde{g}\alpha$-closed set contained in $A^* - A$. Therefore, $A^* \cap (X-U) = \varnothing$ and hence $A^* \subseteq U$ which implies A is $I_{\tilde{g}\alpha}$-closed.

Theorem 2.11. *If (X, τ, I) be an ideal topological space and A be an $I_{\tilde{g}\alpha}$-closed, then the following are equivalent.*

(a) *A is a $*$-closed set.*

(b) *$cl^*(A) - A$ is a $\tilde{g}\alpha$-closed set.*

(c) *$A^* - A$ is a $\tilde{g}\alpha$-closed set.*

Proof. (a) $\Rightarrow$ (b) If A is $*$-closed, then $A^* \subseteq A$ and so $cl^*(A) - A = (A \cup A^*) - A = \varnothing$. Hence $cl^*(A) - A$ is $\tilde{g}\alpha$-closed.

(b) $\Rightarrow$ (c) Since $cl^*(A)-A=(A\cup A^*)-A=(A\cup A^*)\cap A^c=(A\cap A^c)\cup(A^*\cap A^c)$ $= A^*\cap A^c = A^* - A$ and so $A^* - A$ is $\tilde{g}\alpha$-closed.

(c) $\Rightarrow$ (a) If $A^* - A$ is a $\tilde{g}\alpha$-closed set and A is $I_{\tilde{g}\alpha}$-closed set, by Theorem 2.10. $A^* - A = \varnothing$ and so A is $*$-closed.

Theorem 2.12. *If (X, τ, I) be an ideal topological space and A is a subset of X. Then A is $I_{\tilde{g}\alpha}$-closed if and only if $A = F - N$, where F is $*$-closed and N contains no non-empty $\tilde{g}\alpha$-closed set.*

Proof. If A is $I_{\tilde{g}\alpha}$-closed, then by Theorem 2.10, $N = A^* - A$ contains no nonempty $\tilde{g}\alpha$-closed set. If $F = cl^*(A)$, then F is $*$-closed such that $F - N = (A\cup A^*) - (A^* - A) = (A \cup A^*) \cap (A^* \cap A^c)^c = (A\cup A^*)\cap(A\cup(A^*)^c) = A\cup(A^* \cap (A^*)^c) = A$.

Conversely suppose $A = F - N$, where F is $*$-closed and N contains no nonempty $\tilde{g}\alpha$-closed set. Let U be a $\tilde{g}\alpha$-open set such that $A \subseteq U$. Then $F - N \subseteq U$ implies $F \cap (X - U) \subseteq N$. Now $A \subseteq F$ and $F^* \subseteq F$, then $A^* \subseteq F^*$ and so $A^* \cap (X - U) \subseteq F^* \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. By hypothesis, since $A^* \cap (X - U)$ is $\tilde{g}\alpha$-closed, $A^* \cap (X - U) = \varnothing$ and so $A^* \subseteq U$. Hence A is $I_{\tilde{g}\alpha}$-closed.

Theorem 2.13. *If (X, τ, I) be an ideal topological space. If A and B are subset of X such that $A \subseteq B \subseteq cl^*(A)$ and A is $I_{\tilde{g}\alpha}$-closed, then B is $\tilde{g}\alpha$-closed.*

Proof. Since A is $I_{\tilde{g}\alpha}$-closed, by Theorem 2.10(d) $cl^*(A) - A$ contains no non-empty $\tilde{g}\alpha$-closed set. Since $cl^*(B) - B \subseteq cl^*(A) - A$ and so $cl^*(B) - B$ contains no non-empty $\tilde{g}\alpha$-closed set. Hence B is $I_{\tilde{g}\alpha}$-closed set.

Theorem 2.14. *If (X, τ, I) be an ideal topological space and A is a subset of X. Then A is $I_{\tilde{g}\alpha}$-open if and only if $F \subseteq \operatorname{int}^*(A)$ whenever F is $\tilde{g}\alpha$-closed and $F \subseteq A$.*

Proof. Suppose A is $I_{\tilde{g}\alpha}$-open. If F is $\tilde{g}\alpha$-closed and $F \subseteq A$, then $X - A \subseteq X - F$ and so $cl^*(X - A) \subseteq X - F$ by Theorem 2.10. Therefore, $F \subseteq X - cl^*(X - A) = \operatorname{int}^*(A)$.

Conversely suppose the condition holds. Let U be a $\tilde{g}\alpha$-open set such that $X - A \subseteq U$. Then $X - U \subseteq A$ and so $X - U \subseteq \operatorname{int}^*(A)$ implies $cl^*(X - A) \subseteq U$, by Theorem 2.10, $X - A$ is $I_{\tilde{g}\alpha}$-closed. Hence A is $I_{\tilde{g}\alpha}$-open set.

Theorem 2.15. *Let (X, τ, I) be an ideal topological space and A is a subset of X. If A is $I_{\tilde{g}\alpha}$-open and $\operatorname{int}^*(A) \subseteq B \subseteq A$ then B is $I_{\tilde{g}\alpha}$-open.*

Proof. Since A is $I_{\tilde{g}\alpha}$-open, $X - A$ is $I_{\tilde{g}\alpha}$-closed. By Theorem 2.10, $cl^*(X - A) - (X - A)$ contains no non-empty $\tilde{g}\alpha$-closed set. Since $\operatorname{int}^*(A) \subseteq \operatorname{int}^*(B)$ which implies $cl^*(X - B) \subseteq cl^*(X - A)$ and so $cl^*(X - B) - (X - B) \subseteq cl^*(X - A) - (X - A)$. Hence B is $I_{\tilde{g}\alpha}$-open.

Theorem 2.16. *If* (X, τ, I) *be an ideal topological space and A is a subset of X, then the following are equivalent.*

(a) *A is* $I_{\tilde{g}\alpha}$*-closed.*

(b) $A \cup (X - A^*)$ *is* $I_{\tilde{g}\alpha}$*-closed.*

(c) $A^* - A$ *is* $I_{\tilde{g}\alpha}$*-open.*

Proof. (a) $\Rightarrow$ (b) Suppose A is $I_{\tilde{g}\alpha}$-closed. If U is any $\tilde{g}\alpha$-open set such that $A \cup (X - A^*) \subseteq U$, then $X - U \subseteq X - (A \cup (X - A^*)) = X \cap (A \cup (A^*)^c)^c = A^* \cap A^c = A^* - A$. Since A is $\tilde{g}\alpha$-closed, by Theorem 2.10(e), it follows that $X - U = \varnothing$ and so $X = U$. Therefore $A \cup (X - A^*) \subseteq U$ which implies $A \cup (X - A^*) \subseteq X$ and so $(A \cup (X - A^*))^* \subseteq X^* \subseteq X = U$. Hence $A \cup (X - A^*)$ is $I_{\tilde{g}\alpha}$-closed.

(b) $\Rightarrow$ (a) Suppose $A \cup (X - A^*)$ is $I_{\tilde{g}\alpha}$-closed. If F is any $\tilde{g}\alpha$-closed-set such that $F \subseteq A^* - A$, then $F \subseteq A^*$ and F does not contained in A which implies $(A - A^*) \subseteq X - F$ and $A \subseteq X - F$. Therefore $A \cup (X - A^*) \subseteq A \cup (X - F) = X - F$ and $X - F$ is $\tilde{g}\alpha$-open. Since $(A \cup (X - A^*))^* \subseteq X - F$ which implies $A^* \cup (X - A^*)^* \subseteq X - F$ and so $A^* \subseteq X - F$ which implies $F \subseteq X - A^*$. Since $F \subseteq A^*$, it follows that $F = \varnothing$. Hence A is $I_{\tilde{g}\alpha}$-closed.

(b) $\Leftrightarrow$ (c) Since $X - (A^* - A) = X \cap (A^* \cap A^c)^c = X \cap (A^{*c} \cup A) = (X \cap (A^*)^c) \cup (X \cap A) = A \cup (X - A^*)$.

Theorem 2.17. *Let* (X, τ, I) *be an ideal topological space. Then every subset X is* $I_{\tilde{g}\alpha}$*-closed if and only if every* $\tilde{g}\alpha$*-open set is* $*$*-closed.*

Proof. Suppose every subset of X is $I_{\tilde{g}\alpha}$-closed. If $U \subseteq X$ is $\tilde{g}\alpha$-open, then U is $I_{\tilde{g}\alpha}$-closed and so $U^* \subseteq U$. Hence U is $*$-closed.

Conversely suppose that every $\tilde{g}\alpha$-open set is $*$-closed. If U is $\tilde{g}\alpha$-open such that $A \subseteq U \subseteq X$, then $A^* \subseteq U^* \subseteq U$ and so A is $I_{\tilde{g}\alpha}$-closed.

Theorem 2.18. *Let* (X, τ, I) *be an ideal topological space. Then either* $\{x\}$ *is* $\tilde{g}\alpha$-*closed or* $\{x\}^c$ *is* $I_{\tilde{g}\alpha}$-*closed for every* $x \in X$.

Proof. Suppose $\{x\}$ is not $\tilde{g}\alpha$-closed, then $\{x\}^c$ is not $\tilde{g}\alpha$-open and the only $\tilde{g}\alpha$-open set containing $\{x\}^c$ is X and hence $(\{x\}^c)^* \subseteq X$. Thus $\{x\}^c$ is $I_{\tilde{g}\alpha}$-closed.

Definition 2.19. An ideal topological space (X, τ, I), is said to be an $I_{\tilde{g}\alpha}$ normal space if every pair of disjoint closed subsets A and B of X, there exist disjoint $I_{\tilde{g}\alpha}$ open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 2.20. *Let* (X, τ, I) *be an ideal space. Then the following are equivalent*:

(i) X is $I_{\tilde{g}\alpha}$ *normal.*

(ii) *For every closed set A and an open set V containing A there exist an* $I_{\tilde{g}\alpha}$ *open set U such that* $A \subset U \subset cl^*(U) \subset V$.

Proof. (i) $\Rightarrow$ (ii) Let A be a closed set and V be an open set containing A. Then A and $X - V$ are disjoint closed set and so there exist disjoint $I_{\tilde{g}\alpha}$ open sets U and W such that $A \subset U$ and $X - V \subset W$. Now $U \cap W = \varnothing$ implies that $U \cap \operatorname{int}^*(W) = \varnothing$ which implies that $U \subset X - \operatorname{int}^*(W) = \varnothing$ and so $cl^*(U) \subset X - \operatorname{int}^*(W)$. Again, $X - V \subset W$ implies that $X - W \subset V$, where V is open which implies that $cl^*(X - W) \subset V$ and so $X - \operatorname{int}^*(W) \subset V$. Thus $A \subset U \subset cl^*(U) \subset X - \operatorname{int}^*(W)$ $\subset V$. Therefore $A \subset U \subset cl^*(U) \subset V$, where U is $I_{\tilde{g}\alpha}$ open.

(ii) $\Rightarrow$ (i) Let A and B be two disjoint closed subsets of X, by hypothesis, there exists an $I_{\tilde{g}\alpha}$ open set U such that $A \subset U \subset cl^*(U) \subset X - B$. Now $cl^*(U) \subset X - B$ implies that $B \subset X - cl^*(U)$. If $X - cl^*(U) = W$, then W is an $I_{\tilde{g}\alpha}$ open. Hence U and W are the required disjoint $I_{\tilde{g}\alpha}$ open sets containing A and B, respectively. Therefore (X, τ, I) is $I_{\tilde{g}\alpha}$ normal.

3. Mildly $I_{\tilde{g}\alpha}$-closed Sets in Ideal Topological Spaces

Definition 3.1. A subset A of an ideal space (X, τ, I) is said to be *mildly* $I_{\tilde{g}\alpha}$-*closed set* if $(\text{int}(A))^* \subseteq U$ whenever $A \subseteq U$ and U is $\tilde{g}\alpha$-open set.

Theorem 3.2. (a) *Every* $I_{\tilde{g}\alpha}$*-closed set is mildly* $I_{\tilde{g}\alpha}$*-closed set.*

(b) *Every pre-I-closed set is mildly* $I_{\tilde{g}\alpha}$*-closed set.*

Proof. (a) Let $A \subseteq U$ and U is $\tilde{g}\alpha$-open set. Since A is $I_{\tilde{g}\alpha}$-closed set, $A^* \subseteq U$ which implies $(\text{int}(A))^* \subseteq U$. Therefore A is mildly $I_{\tilde{g}\alpha}$-closed set.

(b) Let $A \subseteq U$ and U is $\tilde{g}\alpha$-open set. Since A is pre-I-closed set, $cl^*(\text{int}(A)) \subseteq A \subseteq U$. Therefore A is mildly $I_{\tilde{g}\alpha}$-closed set.

The converse of Theorem 3.2 need not be true by the following examples.

Example 3.3. (a) Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $I = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $\{c\}$ is mildly $I_{\tilde{g}\alpha}$-closed set but not $I_{\tilde{g}\alpha}$-closed.

(b) Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$ and $I = \{\varnothing\}$. Then $\{c, d\}$ is mildly $I_{\tilde{g}\alpha}$-closed set but not $\text{pre}^* I$-closed.

Remark 3.4. The union of two mildly $I_{\tilde{g}\alpha}$-closed set in an ideal topological space need not be a mildly $I_{\tilde{g}\alpha}$-closed set.

Proof. It follows from the following example.

Example 3.5. Let (X, τ, I) be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $I = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $\{a\}$ and $\{b, c\}$ are mildly $I_{\tilde{g}\alpha}$-closed set but their union $\{a, b, c\}$ is not mildly $I_{\tilde{g}\alpha}$-closed.

Theorem 3.6. *Let* (X, τ, I) *be an ideal topological space and A is a subset of X. The following properties are equivalent*

(i) *A is a mildly* $I_{\tilde{g}\alpha}$*-closed set*

(ii) $cl^*(\text{int}(A)) \subseteq U$ *whenever* $A \subseteq U$ *and U is a* $\tilde{g}\alpha$*-open set in X.*

Proof. (i) $\Rightarrow$ (ii) Let A is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I). Suppose that $A \subseteq U$ and U is a $\tilde{g}\alpha$-open set in X. We have $(\text{int}(A))^* \subseteq U$. Since $\text{int}(A) \subseteq A \subseteq U$, then $(\text{int}(A))^* \cup (\text{int}(A)) \subseteq U \Rightarrow cl^*(\text{int}(A)) \subseteq U$.

(ii) $\Rightarrow$ (i) Let $cl^*(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is a $\tilde{g}\alpha$-open set in X. Since $(\text{int}(A))^* \cup (\text{int}(A)) \subseteq U$, then $(\text{int}(A))^* \subseteq U$, $A \subseteq U$ and U is a $\tilde{g}\alpha$-open set in X. Therefore A is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I).

Theorem 3.7. *Let* (X, τ, I) *be an ideal topological space and A is a subset of X. If A is a* $\tilde{g}\alpha$*-open set and mildly* $I_{\tilde{g}\alpha}$*-closed set, then pre-I closed.*

Proof. Let A be a $\tilde{g}\alpha$-open set and mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I). Then $(\text{int}(A))^* \subseteq A$, $A \subseteq A$, A is $\tilde{g}\alpha$-open set, by Theorem 3.6, $cl^*(\text{int}(A)) \subseteq A$, $A \subseteq A$, A is $\tilde{g}\alpha$-open set. Thus A is a pre-*I* closed set in (X, τ, I).

Theorem 3.8. *Let* (X, τ, I) *be an ideal topological space and A is a subset of X. If A is a mildly* $I_{\tilde{g}\alpha}$*-closed set, then* $(\text{int}\, A)^* - A$ *contains no any nonempty* $\tilde{g}\alpha$*-closed set.*

Proof. Let A be a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I). Suppose that U is $\tilde{g}\alpha$-closed set such that $U \subseteq (\text{int}(A))^* - A$. Since A is a mildly $I_{\tilde{g}\alpha}$-closed set, $X - U$ is $\tilde{g}\alpha$-open set and $A \subseteq X - U$, then $(\text{int}(A))^* \subseteq X - U$. We have $U \subseteq X - (\text{int}(A))^*$. Hence $U \subseteq (\text{int}(A))^* \cap (X - (\text{int}(A))^*) = \varnothing$. Thus $(\text{int}(A))^* - A$ contains no any nonempty $\tilde{g}\alpha$-closed set.

Theorem 3.9. *Let (X, τ, I) be an ideal topological space and A is a subset of U. If A is a mildly $I_{\tilde{g}\alpha}$-closed set, then $cl^*(\text{int}(A)) - A$ contains no any nonempty $\tilde{g}\alpha$-closed set.*

Proof. Suppose U is a $\tilde{g}\alpha$-closed set such that $U \subseteq cl^*(\text{int}(A)) - A$ by Theorem 3.8. It follows from the fact that $cl^*(\text{int}(A)) - A = (\text{int}(A))^* \cup (\text{int}(A)) - A.$

Theorem 3.10. *Let (X, τ, I) be an ideal topological space and A is a subset of X. If A is mildly $I_{\tilde{g}\alpha}$-closed set, then $\text{int}(A) = H - K$, where H is I-R-closed and K contains no any non-empty $\tilde{g}\alpha$-closed set.*

Proof. Let A is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I). Take $K = (\text{int}(A))^* - A$. Then by Theorem 3.8., K contains no any nonempty $\tilde{g}\alpha$-closed set. Take $H = cl^*(\text{int}(A))$. Then $H = cl^*(\text{int}(H))$. Moreover we have

$$H - K = cl^*(\text{int}(A)) - ((\text{int}(A))^* - A) = \text{int}(A) \cup (\text{int}(A))^* - ((\text{int}(A))^* - A)$$

$$= \text{int}(A) \cup (\text{int}(A))^* \cap (X - ((\text{int}(A))^* - A)) = \text{int}(A).$$

Theorem 3.11. *Let (X, τ, I) be an ideal topological space. The following properties are equivalent.*

(i) *A pre-I closed for each mildly $I_{\tilde{g}\alpha}$-closed set A in (X, τ, I).*

(ii) *Each singleton $\{x\}$ of X is a $\tilde{g}\alpha$-closed set or $\{x\}$ is pre-I open.*

Proof. (i) $\Rightarrow$ (ii) Let A be pre-I closed for each mildly $I_{\tilde{g}\alpha}$-closed set A in (X, τ, I) and $x \in X$. We have $cl^*(\text{int}(A)) \subseteq A$ for each mildly $I_{\tilde{g}\alpha}$-closed set A in (X, τ, I). Assume that $\{x\}$ is not a $\tilde{g}\alpha$-closed set. It follows that X is the only $\tilde{g}\alpha$-open set containing $X - \{x\}$. Then $X - \{x\}$ is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I). Thus $cl^*(\text{int}(X - \{x\})) \subseteq X - \{x\}$ and hence $\{x\} \subseteq \text{int}^*(cl(\{x\}))$. Consequently $\{x\}$ is pre$^*I$ open.

(ii) $\Rightarrow$ (i) Let A be a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I). Let $x \in cl^*(\text{int}(A))$.

Suppose that $\{x\}$ is pre-I-open. We have $\{x\} \subseteq \text{int}^*(cl\{x\})$. Since $x \in cl^*(\text{int}(A))$, then $\text{int}^*(cl\{x\}) \cap \text{int}(A) \neq \emptyset$. It follows that $(cl\{x\}) \cap \text{int}(A) \neq \emptyset$. We have $(cl\{x\}) \cap \text{int}(A) \neq \emptyset$ and then $(cl\{x\}) \cap \text{int}(A) \neq \emptyset$. Hence $x \in \text{int}(A)$. Thus, we have $x \in A$. Suppose that $\{x\}$ is a $\tilde{g}\alpha$-closed set. By Theorem 3.9, $cl^*(\text{int}(A)) - A$ does not contain $\{x\}$. Since $x \in cl^*(\text{int}(A))$, we have $x \in A$. Thus, $cl^*(\text{int}(A)) \subseteq A$ and hence A is pre-I-closed.

Theorem 3.12. *Let (X, τ, I) be an ideal topological space and A is a subset of X. Assume that A is a mildly $I_{\tilde{g}\alpha}$-closed set. The following properties are equivalent.*

(i) *A is pre-I-closed.*

(ii) $cl^*(\text{int}(A)) - A$ *is a* $\tilde{g}\alpha$*-closed set.*

(iii) $(\text{int}(A))^* - A$ *is a* $\tilde{g}\alpha$*-closed set.*

Proof. (i) $\Rightarrow$ (ii) Let A be pre-I-closed. We have $cl^*(\text{int}(A)) \subseteq A$. Then $cl^*(\text{int}(A)) - A = \emptyset$. Thus $cl^*(\text{int}(A)) - A$ is a $\tilde{g}\alpha$-closed set.

(ii) $\Rightarrow$ (i) Let $cl^*(\text{int}(A)) - A$ be a $\tilde{g}\alpha$-closed set. Since A is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I), then by Theorem 3.9 $cl^*(\text{int}(A)) - A = \emptyset$. Hence $cl^*(\text{int}(A)) \subseteq A$. Thus, A is pre-I-closed.

(ii) $\Rightarrow$ (iii) It follows easily from that $cl^*(\text{int}(A)) - A = (\text{int}(A))^* - A$.

Theorem 3.13. *Let (X, τ, I) be an ideal topological space and A a subset of X be a mildly $I_{\tilde{g}\alpha}$-closed set. Then $A \cup (X - (\text{int}(A))^*)$ is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I).*

Proof. Let A be a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I). Suppose U is a $\tilde{g}\alpha$-open set such that $A \cup (X - (\text{int}(A))^*) \subseteq U$. We have $X - U \subseteq X - (A \cup (X - (\text{int}(A))^*)) = (X - A) \cap (\text{int}(A))^* = (\text{int}(A))^* - A$. Since $X - U$ is a $\tilde{g}\alpha$-closed set and A is a mildly $I_{\tilde{g}\alpha}$-closed set, it follows from Theorem 3.8 that $X - U = \emptyset$. Hence $X = U$.

Thus X is the only $\tilde{g}\alpha$-open set containing $A \cup (X - \text{int}(A))^*$. Hence $A \cup (X - \text{int}(A))^*$ is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I).

Theorem 3.14. *Let (X, τ, I) be an ideal topological space and A a subset of X be a mildly $I_{\tilde{g}\alpha}$-closed set. Then $(\text{int}(A))^* - A$ is a mildly $I_{\tilde{g}\alpha}$-open set in (X, τ, I).*

Proof. Since $X - (\text{int}(A)^* - A) = A \cup X - (\text{int}(A))^*$, it is follows from Theorem 3.13 that $(\text{int}(A))^* - A$ is a mildly $I_{\tilde{g}\alpha}$-open set in (X, τ, I).

Theorem 3.15. *Let (X, τ, I) be an ideal topological space and A a subset of X be a mildly $I_{\tilde{g}\alpha}$-closed set. Then the following properties are equivalent.*

(i) *A is $*$-closed and open set.*

(ii) *A is I-R closed and open set.*

(iii) *A is a mildly $\tilde{g}\alpha$-closed and open set.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): Obvious. (iii) $\Rightarrow$ (i) Since A is mildly $I_{\tilde{g}\alpha}$-closed and open set, then $cl^*(\text{int}(A)) \subseteq A$ and so $A = cl^*(\text{int}(A))$. Then A is *I-R* closed and hence it is $*$-closed.

Theorem 3.16. *Let (X, τ, I) be an ideal topological space and A a subset of X be a mildly $I_{\tilde{g}\alpha}$-closed set. Then the following properties are equivalent.*

(i) *Each subset of (X, τ, I) is a mildly $I_{\tilde{g}\alpha}$-closed set.*

(ii) *A is pre-I-closed for each $\tilde{g}\alpha$-open set A in X.*

Proof. (i) $\Rightarrow$ (ii) Suppose that each subset of $(X\tau, I)$ is a mildly $I_{\tilde{g}\alpha}$-closed set. Let A be a $\tilde{g}\alpha$-open set. Since A is mildly $I_{\tilde{g}\alpha}$-closed set, then we have $cl^*(\text{int}(A)) \subseteq A$. Thus A is pre-I-closed.

(ii) $\Rightarrow$ (i) Let A be a subset of $(X\tau, I)$ and U be a $\tilde{g}\alpha$-open set such that $A \subseteq U$. We have $cl^*(\text{int}(A)) \subseteq cl^*(\text{int}(U)) \subseteq U$. Thus A is mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I).

Theorem 3.17. *Let (X, τ, I) be an ideal topological space and A be a subset of X. If A is mildly $I_{\tilde{g}\alpha}$-closed set and $A \subseteq U \subseteq cl^*(\text{int}(A))$, then U is mildly $I_{\tilde{g}\alpha}$-closed set.*

Proof. Let $U \subseteq K$ and K be a $\tilde{g}\alpha$-open set in X. Since $A \subseteq K$ and A is mildly $I_{\tilde{g}\alpha}$-closed set, $cl^*(\text{int}(A)) \subseteq K$. Since $U \subseteq cl^*(\text{int}(A))$, $cl^*(\text{int}(U)) \subseteq cl^*(\text{int}(A)) \subseteq K$. Thus $cl^*(\text{int}(U)) \subseteq K$ and hence U is a mildly $I_{\tilde{g}\alpha}$-closed set.

Theorem 3.18. *Let (X, τ, I) be an ideal topological space and A be a subset of X. If A is mildly $I_{\tilde{g}\alpha}$-closed and open set, then $cl^*(A)$ is a mildly $I_{\tilde{g}\alpha}$-closed set.*

Proof. Let A be mildly $I_{\tilde{g}\alpha}$-closed and open set in $(X\tau, I)$. We have $A \subseteq cl^*(A) = cl^*(\text{int}(A))$. Hence by Theorem 3.17, $cl^*(A)$ is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I).

Theorem 3.19. *Let (X, τ, I) be an ideal topological space and A be a subset of X. If A is nowhere dense set, then A is a mildly $I_{\tilde{g}\alpha}$-closed set.*

Proof. Let A be a nowhere dense set in X. Since $\text{int}(A) \subseteq \text{int}(cl(A))$, $\text{int}(A) = \varnothing$. Hence $cl^*(\text{int}(A)) = \varnothing$. Thus, A is a mildly $I_{\tilde{g}\alpha}$-closed set in (X, τ, I).

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