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New Families of Bi-Univalent Functions Governed by Gegenbauer Polynomials

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Abstract

The aim of this article is to initiating an exploration of the properties of bi-univalent functions related to Gegenbauer polynomials. To do so, we introduce a new families $\mathbb{T}_{\Sigma}(\gamma,\phi,\mu,,\eta,\theta,\lambda,t,\delta)$ and $\mathbb{S}_{\Sigma}(\sigma,\eta,\theta,\lambda,t,\delta)$ of holomorphic and bi-univalent functions. We derive estimates on the initial coefficients and solve the Fekete-Szegö problem of functions in these families.

1. Introduction

"In [20] Legendre studied orthogonal polynomials comprehensively. The importance of orthogonal polynomials for contemporary mathematics as well as for a wide range of their applications in physics and engineering, is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations as well as in mathematical statistics. Their applications in quantum mechanics, scattering theory, automatic control, signal analysis, and axially symmetric potential theory are also known [7,12]. In practical, the Gegenbauer polynomials is special case of orthogonal polynomials. They are representatively related with typically real functions T_R as discovered in [19]. Typically, real functions play an important role in the geometric function theory because of the relation $T_R = \overline{co}S_R$ and its role of estimating coefficient bounds, where S_R indicates the family of univalent functions in the unit disk with real coefficients and $\overline{co}S_R$ denotes the closed convex hull of S_R."

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On this subject in geometric function theory, the so-called Fekete-Szegö type inequalities (or problems) which estimate some upper bounds for $\left|a_3 - \mu a_2^2\right|$ for holomorphic univalent functions. Its origin was in the disproof by Fekete and Szegö [16] conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity.

Let $\mathcal{A}$ stand for the collection of functions f have the type:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \tag{1.1}$$

which are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Further, symbolized by S the subfamily of $\mathcal{A}$ consisting the functions that are univalent in U.

According to the "Koebe One-Quarter Theorem" [13] each function f from S has an inverse f^{-1}, which fulfills

$$f^{-1}\big(f(z)\big) = z, \qquad (z \in U)$$

and

$$f\big(f^{-1}(w)\big) = w, \qquad \left(|w| < r_0(f),\, r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is named bi-univalent in U if together f and f^{-1} are univalent in U. Let Σ indicate the family of bi-univalent functions in U satisfying (1.1). Beginning with Srivastava et al. pioneering work [36] on the subject, the large number of works associated with the subject have been presented (see, for example [1,2,4,5,8,9,10,11,14, 17,18,21,22,25,28,29,30,31,32,33,34,35,37,38,39,40,41]). We see that the set Σ is not empty. We see that the functions

$$\frac{z}{1-z},\ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \text{ and } -\log(1-z)$$

are in the set Σ with the corresponding inverse functions

$$\frac{w}{1+w},\ \frac{e^{2w}-1}{e^{2w}+1} \text{ and } \frac{e^{w}-1}{e^{w}},$$

respectively. But the functions

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}.$$

are not a member of the set Σ.

The problem to find the bound of $|a_n|, (n = 3,4, \ldots)$ of functions $f \in \Sigma$ is still an open problem.

The fundamental distributions like, the Pascal, the Binomial, the Poisson, the Logarithmic, the Borel have been partially considered in the "Geometric Function Theory" from a theoretical point of view (see for example [6,15,24,26,43]).

We say that the discrete random variable x have a beta negative binomial distribution, if it has the values $0,1,2,3,\ldots$ with the probabilities $\frac{\beta(\eta+\theta,\lambda)}{\beta(\eta,\lambda)}, \theta\frac{\beta(\eta+\theta,\lambda+1)}{\beta(\eta,\lambda)}, \frac{1}{2}\theta(\theta+1)\frac{\beta(\eta+\theta,\lambda+2)}{\beta(\eta,\lambda)}, \ldots$, respectively, where η, θ, λ are named the parameters.

$$\operatorname{Prob}(x=\tau) = \binom{\theta+\tau-1}{\tau}\frac{\beta(\eta+\theta,\lambda+\tau)}{\beta(\eta,\lambda)} = \frac{\Gamma(\theta+\tau)}{\tau!\ \Gamma(\theta)}\frac{\Gamma(\eta+\theta)\Gamma(\lambda+\tau)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+\tau)\Gamma(\eta)\Gamma(\lambda)}$$

$$= \frac{(\eta)_\theta(\theta)_\tau(\lambda)_\tau}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_\tau\,\tau!}, \quad \tau = 0,1,2,3,\ldots,$$

where $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1 & (n=0), \\ \alpha(\alpha+1)\ldots(\alpha+n-1) & (n \in \mathbb{N}). \end{cases}$$

Recently, Wanas and Al-Ziadi [42] studied the following power series whose coefficients are probabilities of the beta negative binomial distribution:

$$\mathfrak{X}_{\eta,\lambda}^{\theta}(z) = z + \sum_{n=2}^{\infty}\frac{(\eta)_\theta(\theta)_{n-1}(\lambda)_{n-1}}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_{n-1}(n-1)!}z^n, \quad z \in U,$$

where $\eta, \lambda, \theta > 0$. We see that, by making use of ratio test we deduce that the radius of convergence of the above power series is infinity.

Now, we consider the linear operator $\mathfrak{B}_{\eta,\lambda}^{\theta} : \mathcal{A} \longrightarrow \mathcal{A}$ which is defined as follows:

$$\mathfrak{B}_{\eta,\lambda}^{\theta}f(z) = \mathfrak{X}_{\eta,\lambda}^{\theta}(z) * f(z) = z + \sum_{n=2}^{\infty}\frac{(\eta)_\theta(\theta)_{n-1}(\lambda)_{n-1}}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_{n-1}(n-1)!}a_n z^n, \quad z \in U,$$

where "$*$" indicate the convolution of two series.

"For the functions f and g be holomorphic in U. We say that the function f is said to be subordinate to g, if there exists a Schwarz function w holomorphic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)(z \in U)$. It is well known that (see [23]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$."

Recently, Amourah [3] studied the generating function of Gegenbauer polynomials $H_\delta(z,t)$ that are given by the following recurrence relation:

$$H_\delta(z,t) = \frac{1}{(1-2tz+z^2)^\delta},$$

where $\delta \in \mathbb{R} \setminus \{0\}$, $t \in [-1,1]$ and $z \in U$. For fixed t, the function H_δ is holomorphic in U, so it may be expanded in a Taylor-Maclaurin series as note that if $t = \cos\beta$, where $\beta \in (-\frac{\pi}{3}, \frac{\pi}{3})$, then

$$H_\delta(z,t) = \frac{1}{(1-2tz+z^2)^\delta} = \sum_{n=0}^{\infty} \mathcal{G}_n^\delta(t) z^n,$$

where $\mathcal{G}_n^\delta(t)$ is Gegenbauer polynomial of degree n.

Clearly, H_δ generates nothing when $\delta = 0$. Thus, the generating function of the Gegenbauer polynomial is set to be

$$H_0(z,t) = 1 - \log(1-2tz+z^2) = \sum_{n=0}^{\infty} \mathcal{G}_n^0(t) z^n.$$

Furthermore, it is worth to mention that a normalization of δ to be greater than $-\frac{1}{2}$ is desirable [12,27]. Also, Gegenbauer polynomials can be introduced by the following recurrence relations:

$$\mathcal{G}_n^\delta(t) = \frac{1}{2}\left[2t(n+\delta-1)\mathcal{G}_{n-1}^\delta(t) - (n+2\delta-2)\mathcal{G}_{n-1}^\delta(t)\right],$$

with the initial values

$$\mathcal{G}_0^\delta(t) = 1, \ \ \mathcal{G}_1^\delta(t) = 2\delta t \ \text{ and } \ \mathcal{G}_2^\delta(t) = 2\delta(\delta+1)t^2 - \delta\,. \tag{1.3}$$

Remark 1.1. Choosing the special values of δ, the Gegenbauer polynomial $\mathcal{G}_n^\delta(t)$ reduces to the following well-known polynomials:

1) Taking $\delta = 1$, we have the Chebyshev Polynomials.

2) Taking $\delta = \frac{1}{2}$, we obtain the Legendre Polynomials.

2. Main Results

This section start with defining the families $\mathbb{T}_\Sigma(\gamma, \phi, \mu,, \eta, \theta, \lambda, t, \delta)$ and $\mathbb{S}_\Sigma(\sigma, \eta, \theta, \lambda, t, \delta)$ as follows:

Definition 2.1. For $0 \le \gamma \le 1, 0 \le \phi \le 1, 0 \le \mu \le 1$, $t \in \left(\frac{1}{2}, 1\right]$ and δ is a nonzero real constant, a function $f \in \Sigma$ is called in the family $\mathbb{T}_\Sigma(\gamma, \phi, \mu, \eta, \theta, \lambda, t, \delta)$ if it fulfills the subordinations:

$$\left(\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)}\right)^{\gamma} \left[(1-\mu)\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)} + \mu\left(1 + \frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'}\right)\right]^{\phi} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$\left(\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)}\right)^{\gamma} \left[(1-\mu)\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)} + \mu\left(1 + \frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'}\right)\right]^{\phi} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

Definition 2.2. For $0 \le \sigma \le 1$, $t \in \left(\frac{1}{2}, 1\right]$ and δ is a nonzero real constant, a function $f \in \Sigma$ is called in the family $\mathbb{S}_\Sigma(\sigma, \eta, \theta, \lambda, t, \delta)$ if it fulfills the subordinations:

$$\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)' + (2\sigma+1)z^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'' + \sigma z^3\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)'''}{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)' + \sigma z^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} f(z)\right)''} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)' + (2\sigma+1)w^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'' + \sigma w^3\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)'''}{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)' + \sigma w^2\left(\mathfrak{B}_{\eta,\lambda}^{\theta} g(w)\right)''} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

In particular, if we choose $\phi = 0$ and $\gamma = 1$ in Definition 2.1, the family $\mathbb{T}_\Sigma(\gamma, \phi, \mu,, \eta, \theta, \lambda, t, \delta)$ reduces to the family $\mathfrak{T}_\Sigma(\eta, \theta, \lambda, t, \delta)$ of bi-starlike functions which fulfills the conditions:

$$\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

If we choose $\sigma = 0$ in Definition 2.2, the family $\mathbb{S}_{\Sigma}(\sigma,\eta,\theta,\lambda,t,\delta)$ reduces to the family $\mathfrak{H}_{\Sigma}(\eta,\theta,\lambda,t,\delta)$ of bi-convex functions which fulfills the conditions:

$$1+\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$1+\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

Theorem 2.1. *For $0 \le \gamma \le 1, 0 \le \phi \le 1, 0 \le \mu \le 1,\ t \in (\frac{1}{2}, 1]$ and δ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathbb{T}_{\Sigma}(\gamma,\phi,\mu,\eta,\theta,\lambda,t,\delta)$. Then*

$$|a_2| \le \frac{2|\delta|t\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\sqrt{2|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -2\left[\begin{array}{c}\delta(\delta+1)\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}\right|}}$$

and

$$|a_3| \le \frac{4\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\delta^2t^2}{\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}$$

$$+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)|\delta|t}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)},$$

where

$$\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)=\frac{2(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\lambda+2)}{\Gamma(\eta+\theta+\lambda+2)}$$

$$+\frac{\theta\Gamma(\eta+\theta)\Gamma^2(\lambda+1)\Gamma(\eta+\lambda)\big[\gamma(\gamma-1)+\phi(\mu+1)\big(2\gamma+(\phi-1)(\mu+1)\big)-2\big(\gamma+\phi(3\mu+1)\big)\big]}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}. \quad (2.1)$$

Proof. Let $f\in\mathbb{T}_{\Sigma}(\gamma,\phi,\mu,\eta,\theta,\lambda,t,\delta)$. Then there exists two holomorphic functions $u,v\colon U\longrightarrow U$ given by

$$u(z)=u_1z+u_2z^2+u_3z^3+\cdots \qquad (z\in U) \qquad (2.2)$$

and

$$v(w)=v_1w+v_2w^2+v_3w^3+\cdots \qquad (w\in U), \qquad (2.3)$$

with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, $z,w\in U$ such that

$$\left(\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)}\right)^{\gamma}\left[(1-\mu)\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)}+\mu\left(1+\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}\right)\right]^{\phi}$$

$$=1+\mathcal{G}_1^{\delta}(t)u(z)+\mathcal{G}_2^{\delta}(t)u^2(z)+\cdots \qquad (2.4)$$

and

$$\left(\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)}\right)^{\gamma}\left[(1-\mu)\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)}+\mu\left(1+\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}\right)\right]^{\phi}$$

$$=1+\mathcal{G}_1^{\delta}(t)v(w)+\mathcal{G}_2^{\delta}(t)v^2(w)+\cdots\ . \qquad (2.5)$$

Combining (2.2), (2.3), (2.4) and (2.5), we obtain

$$\left(\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)}\right)^{\gamma}\left[(1-\mu)\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)}+\mu\left(1+\frac{z\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}f(z)\right)'}\right)\right]^{\phi}$$

$$=1+\mathcal{G}_1^{\delta}(t)u_1z+\left[\mathcal{G}_1^{\delta}(t)u_2+\mathcal{G}_2^{\delta}(t)u_1^2\right]z^2+\cdots \qquad (2.6)$$

and

$$\left(\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)}\right)^{\gamma}\left[(1-\mu)\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}{\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)}+\mu\left(1+\frac{w\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)''}{\left(\mathfrak{B}_{\eta,\lambda}^{\theta}g(w)\right)'}\right)\right]^{\phi}$$

$$=1+\mathcal{G}_1^{\delta}(t)v_1w+\left[\mathcal{G}_1^{\delta}(t)v_2+\mathcal{G}_2^{\delta}(t)v_1^2\right]w^2+\cdots\ . \qquad (2.7)$$

It is quite well-known that if $|u(z)|<1$ and $|v(w)|<1$, $z,w\in U$, then

$$|u_i|\le 1 \quad \text{and} \quad |v_i|\le 1 \quad \text{for all } i\in\mathbb{N}. \qquad (2.8)$$

Equating the coefficients in (2.6) and (2.7), we deduce that

$$\frac{\theta\big(\gamma+\phi(\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2=\mathcal{G}_1^{\delta}(t)u_1, \qquad (2.9)$$

$$\frac{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}a_3$$

$$+\frac{\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\big[\gamma(\gamma-1)+\phi(\mu+1)\big(2\gamma+(\phi-1)(\mu+1)\big)-2\big(\gamma+\phi(3\mu+1)\big)\big]}{2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2$$

$$=\mathcal{G}_1^{\delta}(t)u_2+\mathcal{G}_2^{\delta}(t)u_1^2, \qquad (2.10)$$

$$-\frac{\theta\big(\gamma+\phi(\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2=\mathcal{G}_1^{\delta}(t)v_1 \qquad (2.11)$$

and

$$\frac{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(2a_2^2-a_3)$$

$$+\frac{\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\big[\gamma(\gamma-1)+\phi(\mu+1)\big(2\gamma+(\phi-1)(\mu+1)\big)-2\big(\gamma+\phi(3\mu+1)\big)\big]}{2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2$$

$$=\mathcal{G}_1^{\delta}(t)v_2+\mathcal{G}_2^{\delta}(t)v_1^2. \qquad (2.12)$$

From (2.9) and (2.11), we conclude that

$$u_1=-v_1 \qquad (2.13)$$

and

$$\frac{2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2=\left(\mathcal{G}_1^{\delta}(t)\right)^2(u_1^2+v_1^2). \qquad (2.14)$$

Adding (2.10) to (2.12), yields

$$\frac{\theta\Gamma(\eta+\theta)\Gamma(\eta+\lambda)}{\Gamma(\eta)\Gamma(\lambda)}\Omega(\gamma,\phi,\mu)a_2^2 = \mathcal{G}_1^\delta(t)(u_2+v_2)+\mathcal{G}_2^\delta(t)(u_1^2+v_1^2),$$

where $\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)$ is given by (2.1). Consequently, we have

$$\left[\frac{\theta\Gamma(\eta+\theta)\Gamma(\eta+\lambda)}{\Gamma(\eta)\Gamma(\lambda)}\Omega(\gamma,\phi,\mu)-\frac{2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\mathcal{G}_2^\delta(t)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2}\right]a_2^2$$

$$=\mathcal{G}_1^\delta(t)(u_2+v_2). \tag{2.15}$$

Further computations using (1.3), (2.8) and (2.15), we obtain

$$|a_2|\leq\frac{2|\delta|t\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\sqrt{2|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\end{array}\right|}}.$$

Next, if we subtract (2.12) from (2.10), we deduce that

$$\frac{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(a_3-a_2^2)$$

$$=\mathcal{G}_1^\delta(t)(u_2-v_2)+\mathcal{G}_2^\delta(t)(u_1^2-v_1^2). \tag{2.16}$$

In view of (2.13) and (2.14), we get from (2.16)

$$a_3=\frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2}{2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}(u_1^2+v_1^2)$$
$$+\frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)\mathcal{G}_1^\delta(t)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}(u_2-v_2).$$

Thus applying (1.3), we obtain

$$|a_3|\leq\frac{4\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\delta^2t^2}{\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}$$
$$+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)|\delta|t}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}.$$

Putting $\phi = 0$ and $\gamma = 1$ in Theorem 2.1, we demonstrate the next outcome:

Corollary 2.1. *For* $t \in \left(\frac{1}{2}, 1\right]$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathfrak{T}_\Sigma(\eta, \theta, \gimel, t, \delta)$. *Then*

$$|a_2| \le \frac{2|\delta| t \Gamma(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\sqrt{2|\delta|t}}{\sqrt{\left|\begin{matrix} \delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -2\left[\begin{matrix} \delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\mathfrak{S}(\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel) \end{matrix}\right] t^2 \end{matrix}\right|}}$$

and

$$|a_3| \le \frac{4\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\delta^2 t^2}{\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)} + \frac{2\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)|\delta|t}{\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)},$$

where

$$\mathfrak{S}(\eta,\theta,\gimel) = \frac{2(\theta+1)\Gamma(\gimel+2)}{\Gamma(\eta+\theta+\gimel+2)} + \frac{-2\theta\Gamma(\eta+\theta)\Gamma^2(\gimel+1)\Gamma(\eta+\gimel)}{\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)}.$$

Theorem 2.2. *For* $0 \le \sigma \le 1$, $t \in \left(\frac{1}{2}, 1\right]$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathbb{S}_\Sigma(\sigma, \eta, \theta, \gimel, t, \delta)$. *Then*

$$|a_2| \le \frac{\Gamma(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)|\delta| t\sqrt{|\delta|t}}{\sqrt{\left|\begin{matrix} \delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\left[\begin{matrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel) \end{matrix}\right] t^2 \end{matrix}\right|}}$$

and

$$|a_3| \le \frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\delta^2 t^2}{\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)} + \frac{2\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)|\delta|t}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)},$$

where

$$\Upsilon(\sigma,\eta,\theta,\gimel) = \frac{3(\theta+1)(2\sigma+1)\Gamma(\gimel+2)}{\Gamma(\eta+\theta+\gimel+2)}$$

$$-\frac{4\theta(\sigma+1)^2\Gamma(\eta+\theta)\Gamma^2(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}. \tag{2.17}$$

Proof. Let $f \in \mathbb{S}_\Sigma(\sigma,\eta,\theta,\lambda,t,\delta)$. Then there exist two holomorphic functions $u, v : U \longrightarrow U$

$$\frac{z\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)' + (2\sigma+1)z^2\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)'' + \sigma z^3\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)'''}{z\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)' + \sigma z^2\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)''}$$

$$= 1 + \mathcal{G}^{\delta}_1(t)u(z) + \mathcal{G}^{\delta}_2(t)u^2(z) + \cdots \tag{2.18}$$

and

$$\frac{w\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)' + (2\sigma+1)w^2\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)'' + \sigma w^3\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)'''}{w\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)' + \sigma w^2\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)''}$$

$$= 1 + \mathcal{G}^{\delta}_1(t)v(w) + \mathcal{G}^{\delta}_2(t)v^2(w) + \cdots . \tag{2.19}$$

where u and v have the forms (2.2) and (2.3). Combining (2.18) and (2.19), yield

$$\frac{z\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)' + (2\sigma+1)z^2\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)'' + \sigma z^3\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)'''}{z\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)' + \sigma z^2\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)''}$$

$$= 1 + \mathcal{G}^{\delta}_1(t)u_1 z + \left[\mathcal{G}^{\delta}_1(t)u_2 + \mathcal{G}^{\delta}_2(t)u_1^2\right]z^2 + \cdots \tag{2.20}$$

and

$$\frac{w\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)' + (2\sigma+1)w^2\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)'' + \sigma w^3\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)'''}{w\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)' + \sigma w^2\left(\mathfrak{B}^{\theta}_{\eta,\lambda}g(w)\right)''}$$

$$= 1 + \mathcal{G}^{\delta}_1(t)v_1 w + \left[\mathcal{G}^{\delta}_1(t)v_2 + \mathcal{G}^{\delta}_2(t)v_1^2\right]w^2 + \cdots . \tag{2.21}$$

Equating the coefficients in (2.20) and (2.21), we deduce that

$$\frac{2\theta(\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2 = \mathcal{G}^{\delta}_1(t)u_1, \tag{2.22}$$

$$\frac{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}a_3$$

$$-\frac{4\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2=\mathcal{G}_1^\delta(t)u_2+\mathcal{G}_2^\delta(t)u_1^2, \tag{2.23}$$

$$-\frac{2\theta(\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}a_2=\mathcal{G}_1^\delta(t)v_1 \tag{2.24}$$

and

$$\frac{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(2a_2^2-a_3)$$
$$-\frac{4\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2=\mathcal{G}_1^\delta(t)v_2+\mathcal{G}_2^\delta(t)v_1^2. \tag{2.25}$$

In view of (2.22) and (2.24), we have

$$u_1=-v_1 \tag{2.26}$$

and

$$\frac{8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}a_2^2=\left(\mathcal{G}_1^\delta(t)\right)^2(u_1^2+v_1^2). \tag{2.27}$$

If we add (2.23) to (2.25), we conclude that

$$\frac{2\theta\Gamma(\eta+\theta)\Gamma(\eta+\lambda)}{\Gamma(\eta)\Gamma(\lambda)}\Upsilon(\sigma,\eta,\theta,\lambda)a_2^2=\mathcal{G}_1^\delta(t)(u_2+v_2)+\mathcal{G}_2^\delta(t)(u_1^2+v_1^2), \tag{2.28}$$

where $\Upsilon(\sigma,\eta,\theta,\lambda)$ is given by (2.17).

By substitute the value of $u_1^2+v_1^2$ from (2.27) in (2.28), yields

$$\left[\frac{2\theta\Gamma(\eta+\theta)\Gamma(\eta+\lambda)}{\Gamma(\eta)\Gamma(\lambda)}\Upsilon(\sigma,\eta,\theta,\lambda)-\frac{8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\mathcal{G}_2^\delta(t)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2}\right]a_2^2$$
$$=\mathcal{G}_1^\delta(t)(u_2+v_2),$$

or equivalently

$$a_2^2=\frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^3(u_2+v_2)}{\begin{array}{c}2\theta\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2\\-8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\mathcal{G}_2^\delta(t)\end{array}}, \tag{2.29}$$

Further computations using (1.3), (2.7) and (2.29), we obtain

$$|a_2| \leq \frac{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)|\delta|t\sqrt{|\delta|t}}{\sqrt{\left|\begin{matrix}\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \\ \left[\begin{matrix}2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{matrix}\right]t^2\end{matrix}\right|}}.$$

Next, if we subtract (2.25) from (2.23), we deduce that

$$\frac{6\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}(a_3-a_2^2)$$
$$= \mathcal{G}_1^\delta(t)(u_2-v_2)+\mathcal{G}_2^\delta(t)(u_1^2-v_1^2). \tag{2.30}$$

In view of (2.26) and (2.27), we get from (2.30)

$$a_3 = \frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^\delta(t)\right)^2}{8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}(u_1^2+v_1^2)$$
$$+\frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)\mathcal{G}_1^\delta(t)}{6\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}(u_2-v_2).$$

Thus applying (1.3), we obtain

$$|a_3| \leq \frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\delta^2t^2}{\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}$$
$$+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)|\delta|t}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}.$$

Putting $\sigma = 0$ in Theorem 2.2, we demonstrate the next outcome:

Corollary 2.2. *For $t \in \left(\frac{1}{2}, 1\right]$ and δ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathfrak{H}_\Sigma(\eta,\theta,\lambda,t,\delta)$. Then*

$$|a_2| \leq \frac{\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)|\delta|t\sqrt{|\delta|t}}{\sqrt{\left|\begin{matrix}\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\left[\begin{matrix}2\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\mathfrak{A}(\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{matrix}\right]t^2\end{matrix}\right|}}$$

and

$$|a_3| \leq \frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\delta^2t^2}{\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}+\frac{2\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)|\delta|t}{3\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)},$$

where

$$\mathfrak{A}(\eta,\theta,\lambda)=\frac{3(\theta+1)\Gamma(\lambda+2)}{\Gamma(\eta+\theta+\lambda+2)}-\frac{4\theta\Gamma(\eta+\theta)\Gamma^2(\lambda+1)\Gamma(\eta+\lambda)}{\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)}.$$

Next theorems, show "Fekete-Szegö problem" of the families $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\lambda,t,\delta)$ and $\mathbb{S}_\Sigma(\sigma,\eta,\theta,\lambda,t,\delta)$.

Theorem 2.3. *For* $0\le\gamma\le 1, 0\le\phi\le 1, 0\le\mu\le 1,\ t\in\left(\frac{1}{2},1\right],\ \xi\in\mathbb{R}$ *and* δ *is a nonzero real constant, let* $f\in\mathcal{A}$ *be in the family* $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\lambda,t,\delta)$. *Then*

$$|a_3-\xi a_2^2|$$

$$\le\begin{cases}\dfrac{2t|\delta|\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}; \\ \quad for\ \ |\xi-1|\le\dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}\times \\ \quad\times\left|\dfrac{\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)-2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2}{2\delta^2t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}\right| \\ \dfrac{8t^3|\delta^3|\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)|\xi-1|}{\left|\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)-2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2\right|}; \\ \quad for\ \ |\xi-1|\ge\dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}\times \\ \quad\times\left|\dfrac{\delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)-2\left[\begin{array}{c}\delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{array}\right]t^2}{2\delta^2t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}\right|\end{cases}.$$

Proof. In the light of (2.15) and (2.16), we deduce that

$$a_3 - \xi a_2^2 = (1-\xi)a_2^2 + \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)\mathcal{G}_1^{\delta}(t)(u_2-v_2)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}$$

$$= (1-\xi)\frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\left(\mathcal{G}_1^{\delta}(t)\right)^3(u_2+v_2)}{\theta\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma(\eta)\Gamma(\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\left(\mathcal{G}_1^{\delta}(t)\right)^2 - 2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\mathcal{G}_2^{\delta}(t)}$$
$$+ \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)\mathcal{G}_1^{\delta}(t)(u_2-v_2)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}$$

$$= \mathcal{G}_1^{\delta}(t)\left[\left(\psi(\xi) + \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\right)u_2 \right.$$
$$\left. + \left(\psi(\xi) - \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\right)v_2\right],$$

where

$$\psi(\xi) = \frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\left(\mathcal{G}_1^{\delta}(t)\right)^2(1-\xi)}{\theta\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma(\eta)\Gamma(\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\left(\mathcal{G}_1^{\delta}(t)\right)^2 - 2\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\mathcal{G}_2^{\delta}(t)}.$$

According to (1.3), we deduce that

$$\left|a_3 - \xi a_2^2\right| \le \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}, \\ 0 \le |\psi(\xi)| \le \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \\ 4t|\delta||\psi(\xi)|, \\ |\psi(\xi)| \ge \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \end{cases}.$$

After some computations, we obtain

$$|a_3-\xi a_2^2|$$

$$\leq \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}; \\ \quad \text{for } |\xi-1| \leq \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \times \\ \quad \times \left| \dfrac{\begin{array}{c} \delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -2\left[\begin{array}{c} \delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel) \end{array}\right] t^2 \end{array}}{2\delta^2 t^2\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)} \right| \\[3ex] \dfrac{8t^3|\delta^3|\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)|\xi-1|}{\left| \begin{array}{c} \delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -2\left[\begin{array}{c} \delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel) \end{array}\right] t^2 \end{array} \right|}; \\ \quad \text{for } |\xi-1| \geq \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{2\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \times \\ \quad \times \left| \dfrac{\begin{array}{c} \delta\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -2\left[\begin{array}{c} \delta(\delta+1)\theta^2\big(\gamma+\phi(\mu+1)\big)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Omega(\gamma,\phi,\mu,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel) \end{array}\right] t^2 \end{array}}{2\delta^2 t^2\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)} \right| \end{cases}.$$

Putting $\xi = 1$ in Theorem 2.3, we demonstrate the next outcome:

Corollary 2.3. *For* $0 \leq \gamma \leq 1, 0 \leq \phi \leq 1, 0 \leq \mu \leq 1,\ t \in \left(\frac{1}{2}, 1\right]$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\gimel,t,\delta)$. *Then*

$$\left|a_3-a_2^2\right| \leq \frac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{\theta(\theta+1)\big(\gamma+\phi(2\mu+1)\big)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}.$$

Putting $\phi = 0$ and $\gamma = 1$ in Theorem 2.3, we demonstrate the next outcome:

Corollary 2.4. *For $t \in \left(\frac{1}{2}, 1\right]$, $\xi \in \mathbb{R}$ and δ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathfrak{T}_\Sigma(\eta, \theta, \daleth, t, \delta)$. Then*

$$\left|a_3 - \xi a_2^2\right| \le \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\daleth+2)\Gamma(\eta)\Gamma(\daleth)}{\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\daleth+2)\Gamma(\eta+\daleth)}; \\ \quad \text{for } |\xi - 1| \le \dfrac{\Gamma(\eta+\theta+\daleth+2)\Gamma(\eta)\Gamma(\daleth)}{2\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\daleth+2)\Gamma(\eta+\daleth)} \times \\ \quad \times \left| \dfrac{\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\daleth+1)\Gamma^2(\eta+\daleth) - 2\begin{bmatrix} \delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\daleth+1)\Gamma^2(\eta+\daleth) \\ -\theta\delta^2\mathfrak{S}(\eta,\theta,\daleth)\Gamma(\eta+\theta)\Gamma(\eta+\daleth)\Gamma^2(\eta+\theta+\daleth+1)\Gamma(\eta)\Gamma(\daleth) \end{bmatrix} t^2}{2\delta^2 t^2 \Gamma^2(\eta+\theta+\daleth+1)\Gamma^2(\eta)\Gamma^2(\daleth)} \right| \\ \dfrac{8t^3|\delta^3|\Gamma^2(\eta+\theta+\daleth+1)\Gamma^2(\eta)\Gamma^2(\daleth)|\xi-1|}{\left| \delta\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\daleth+1)\Gamma^2(\eta+\daleth) - 2\begin{bmatrix} \delta(\delta+1)\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\daleth+1)\Gamma^2(\eta+\daleth) \\ -\theta\delta^2\mathfrak{S}(\eta,\theta,\daleth)\Gamma(\eta+\theta)\Gamma(\eta+\daleth)\Gamma^2(\eta+\theta+\daleth+1)\Gamma(\eta)\Gamma(\daleth) \end{bmatrix} t^2 \right|}; \\ \quad \text{for } |\xi - 1| \ge \dfrac{\Gamma(\eta+\theta+\daleth+2)\Gamma(\eta)\Gamma(\daleth)}{2\theta(\theta+1)\left(\gamma+\phi(2\mu+1)\right)\Gamma(\eta+\theta)\Gamma(\daleth+2)\Gamma(\eta+\daleth)} \times \\ \quad \times \left| \dfrac{\delta\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\daleth+1)\Gamma^2(\eta+\daleth) - 2\begin{bmatrix} \delta(\delta+1)\theta^2\left(\gamma+\phi(\mu+1)\right)^2\Gamma^2(\eta+\theta)\Gamma^2(\daleth+1)\Gamma^2(\eta+\daleth) \\ -\theta\delta^2\mathfrak{S}(\eta,\theta,\daleth)\Gamma(\eta+\theta)\Gamma(\eta+\daleth)\Gamma^2(\eta+\theta+\daleth+1)\Gamma(\eta)\Gamma(\daleth) \end{bmatrix} t^2}{2\delta^2 t^2 \Gamma^2(\eta+\theta+\daleth+1)\Gamma^2(\eta)\Gamma^2(\daleth)} \right| \end{cases}.$$

Theorem 2.4. *For $0 \le \sigma \le 1$, $t \in \left(\frac{1}{2}, 1\right]$, $\xi \in \mathbb{R}$ and δ is a nonzero real constant, let $f \in \mathcal{A}$ be in the family $\mathbb{S}_\Sigma(\sigma, \eta, \theta, \daleth, t, \delta)$. Then*

$$
\left|a_3-\xi a_2^2\right| \leq \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}; \\ \quad \text{for } |\xi-1| \leq \dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)} \times \\ \quad \times \left| \dfrac{\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \begin{bmatrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{bmatrix} t^2}{\delta^2 t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} \right| \\ \\ \dfrac{2t^3|\delta^3|\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)|\xi-1|}{\left| \delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \begin{bmatrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{bmatrix} t^2 \right|} \\ \quad \text{for } |\xi-1| \geq \dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)} \times \\ \quad \times \left| \dfrac{\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) - \begin{bmatrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda) \end{bmatrix} t^2}{\delta^2 t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} \right| \end{cases}.
$$

Proof. In view of (2.29) and (2.30), we deduce that

$$
a_3-\xi a_2^2=(1-\xi)a_2^2+\frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)\mathcal{G}_1^{\delta}(t)(u_2-v_2)}{6\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}
$$

$$
=(1-\xi)\frac{\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)\left(\mathcal{G}_1^{\delta}(t)\right)^3(u_2+v_2)}{\begin{array}{c}2\theta\Upsilon(\sigma,\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\left(\mathcal{G}_1^{\delta}(t)\right)^2 \\ -8\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\mathcal{G}_2^{\delta}(t)\end{array}}
$$

$$
+\frac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)\mathcal{G}_1^{\delta}(t)(u_2-v_2)}{6\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}
$$

$$= \frac{\mathcal{G}_1^\delta(t)}{2}\left[\left(\varphi(\xi) + \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\right)u_2 \right.$$
$$\left. + \left(\varphi(\xi) - \frac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}\right)v_2\right],$$

where

$$\varphi(\xi) = \frac{\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)\left(\mathcal{G}_1^\delta(t)\right)^2(1-\xi)}{\begin{array}{c}\theta\Upsilon(\sigma,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\left(\mathcal{G}_1^\delta(t)\right)^2 \\ -4\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel)\mathcal{G}_2^\delta(t)\end{array}}.$$

According to (1.3), we deduce that

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \dfrac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}, \\ \quad 0 \leq |\varphi(\xi)| \leq \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \\ 2t|\delta||\varphi(\xi)|, \\ \quad |\varphi(\xi)| \geq \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \end{cases}.$$

After some computations, we obtain

$$
|a_3 - \xi a_2^2| \le \begin{cases}
\dfrac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}; \\
\quad \text{for } |\xi-1| \le \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \times \\
\quad \times \left| \dfrac{\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) - \left[\begin{matrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{matrix}\right] t^2}{\delta^2 t^2\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)} \right| \\[2ex]
\dfrac{2t^3|\delta^3|\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)|\xi-1|}{\left| \delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) - \left[\begin{matrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{matrix}\right] t^2 \right|} \\
\quad \text{for } |\xi-1| \ge \dfrac{\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)} \times \\
\quad \times \left| \dfrac{\delta\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) - \left[\begin{matrix} 2\delta(\delta+1)\theta^2(\sigma+1)^2\Gamma^2(\eta+\theta)\Gamma^2(\gimel+1)\Gamma^2(\eta+\gimel) \\ -\theta\delta^2\Upsilon(\sigma,\eta,\theta,\gimel)\Gamma(\eta+\theta)\Gamma(\eta+\gimel)\Gamma^2(\eta+\theta+\gimel+1)\Gamma(\eta)\Gamma(\gimel)\end{matrix}\right] t^2}{\delta^2 t^2\Gamma^2(\eta+\theta+\gimel+1)\Gamma^2(\eta)\Gamma^2(\gimel)} \right|
\end{cases}.
$$

Putting $\xi = 1$ in Theorem 2.4, we demonstrate the next outcome:

Corollary 2.5. *For* $0 \le \sigma \le 1$, $t \in \left(\frac{1}{2}, 1\right]$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathbb{S}_\Sigma(\sigma, \eta, \theta, \gimel, t, \delta)$. *Then*

$$|a_3 - a_2^2| \le \frac{2t|\delta|\Gamma(\eta+\theta+\gimel+2)\Gamma(\eta)\Gamma(\gimel)}{3\theta(\theta+1)(2\sigma+1)\Gamma(\eta+\theta)\Gamma(\gimel+2)\Gamma(\eta+\gimel)}.$$

Putting $\sigma = 0$ in Theorem 2.4, we demonstrate the next outcome:

Corollary 2.6. *For* $t \in \left(\frac{1}{2}, 1\right]$, $\xi \in \mathbb{R}$ *and* δ *is a nonzero real constant, let* $f \in \mathcal{A}$ *be in the family* $\mathfrak{H}_\Sigma(\eta, \theta, \gimel, t, \delta)$. *Then*

$$
\left|a_3-\xi a_2^2\right|
\leq\begin{cases}
\dfrac{2t|\delta|\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}; \\
\quad \text{for } |\xi-1|\leq\dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}\times \\
\quad \times\left|\dfrac{\begin{matrix}\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\begin{bmatrix}2\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\mathfrak{A}(\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{bmatrix}t^2\end{matrix}}{\delta^2t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}\right| \\[2ex]
\dfrac{2t^3|\delta^3|\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)|\xi-1|}{\left|\begin{matrix}\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\begin{bmatrix}2\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\mathfrak{A}(\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{bmatrix}t^2\end{matrix}\right|} \\
\quad \text{for } |\xi-1|\geq\dfrac{\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)}{3\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}\times \\
\quad \times\left|\dfrac{\begin{matrix}\delta\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\begin{bmatrix}2\delta(\delta+1)\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)\\ -\theta\delta^2\mathfrak{A}(\eta,\theta,\lambda)\Gamma(\eta+\theta)\Gamma(\eta+\lambda)\Gamma^2(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)\end{bmatrix}t^2\end{matrix}}{\delta^2t^2\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}\right|
\end{cases}.
$$

Conclusion

The primary objective was to create the families $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\lambda,t,\delta)$ and $\mathbb{S}_\Sigma(\sigma,\eta,\theta,\lambda,t,\delta)$ of bi-univalent functions which governed by Gegenbauer polynomials. We generated Taylor coefficient inequalities for functions in the families $\mathbb{T}_\Sigma(\gamma,\phi,\mu,,\eta,\theta,\lambda,t,\delta)$ and $\mathbb{S}_\Sigma(\sigma,\eta,\theta,\lambda,t,\delta)$ and viewed the famous Fekete-Szegö problem.

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