



The Topp-Leone Weibull Distribution: Its Properties and Application

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Abstract

This paper presents a new generalization of the Topp-Leone distribution called the Topp-Leone Weibull Distribution (TLWD). Some of the mathematical properties of the proposed distribution are derived, and the maximum likelihood estimation method is adopted in estimating the parameters of the proposed distribution. An application of the proposed distribution alongside with some well-known distributions belonging to the Topp-Leone generated family of distributions, to a real lifetime data set reveals that the proposed distribution exhibits more flexibility in modeling lifetime data based on some comparison criteria such as maximized log-likelihood, Akaike Information Criterion [$AIC = 2k - 2 \log(L)$], Kolmogorov-Smirnov test statistic ($K - S$) and Anderson Darling test statistic (A^*) and Crammer-Von Mises test statistic (W^*).

1. Introduction

Lifetime distributions are statistical models used for analyzing real life problems based on survival time. Many statistical distributions have been proposed to model lifetime data and the Topp-Leone distribution introduced by [14] is one of such distributions. In practice, most lifetime data sets encountered exhibits a bathtub hazard rate property and the one parameter Topp-Leone distribution happens to be the simplest distribution with such hazard rate property, but being a single parameter defined on a unit interval, its flexibility is limited in handling lifetime data sets.

Several generalizations of the distribution have been introduced to address the aforementioned drawback. These generalizations are found in the works of [1-5,9,10]. In this paper, we introduce a new generalization of the Topp-Leone distribution which serves as an alternative distribution among the Topp-Leone generated family of

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distributions. We shall call the proposed distribution, “Topp-Leone Weibull Distribution (TLWD)”.

[14] proposed a J-shaped univariate distribution with cumulative distribution function defined by

$$F(x) = \left(\frac{x}{b}\right)^\alpha \left(2 - \frac{x}{b}\right)^\alpha, \quad 0 \leq x \leq b < \infty, \quad 0 < \alpha < 1 \quad (1)$$

and the corresponding density function given by

$$f(x) = \frac{2\alpha}{b} \left(\frac{x}{b}\right)^{\alpha-1} \left(1 - \frac{x}{b}\right) \left(2 - \frac{x}{b}\right)^{\alpha-1}. \quad (2)$$

The survival function and the hazard rate function of Topp-Leone distribution are obtained using equations (1) and (2) as

$$S(x) = 1 - F(x) = 1 - \left(\frac{x}{b}\right)^\alpha \left(2 - \frac{x}{b}\right)^\alpha \quad (3)$$

and

$$H(x) = \frac{F(x)}{1-F(x)} = \frac{\frac{2\alpha}{b} \left(\frac{x}{b}\right)^{\alpha-1} \left(1 - \frac{x}{b}\right)}{\left\{\left(2 - \frac{x}{b}\right)^{1-\alpha} - \left(\frac{x}{b}\right)^\alpha \left(2 - \frac{x}{b}\right)\right\}}. \quad (4)$$

[11] improved on the mathematical properties of the Topp-Leone distribution by deriving higher moments for the distribution and a general usefulness of the distribution. Motivations for generalizing the Topp-Leone distribution arose after the work of [11]. One of such motivation is that, suppose a random variable x follow the Topp-Leone distribution then the random variable x can have either a finite support ($0 < x < b$) or an infinite support ($0 < x < b < \infty$).

Suppose we fix the parameter $b = 1$ in equations (1) and (2), then we obtain the cumulative distribution function and the density function of the one parameter Topp-Leone distribution defined on a unit interval respectively as

$$F(x) = x^\alpha (2 - x)^\alpha, \quad 0 < x < 1, \alpha > 0 \quad (5)$$

and

$$f(x) = 2\alpha x^{\alpha-1} (1 - x) (2 - x)^{\alpha-1}. \quad (6)$$

Figure 1 shows the graphical illustration of the density function and the hazard rate function of the Topp-Leone distribution for varying values of the shape parameter defined in the interval $0 < \alpha < 1$ and a fixed value of the scale parameter $b = 1$.

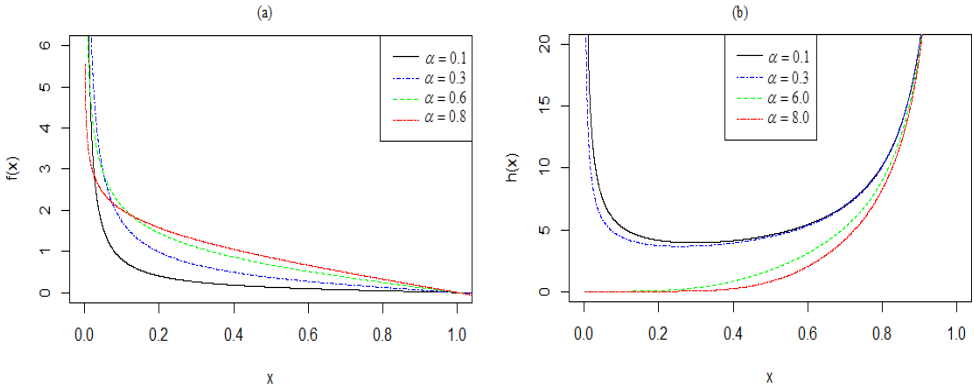


Figure 1: Density and hazard rate functions of the Topp-Leone distribution.

The plots in Figure 1(a), clearly shows that the density function of the TLD exhibits a reversed-J shape for different values of the shape parameter defined in the interval $0 < \alpha < 1$ and a fixed value of the scale parameter $b = 1$, while the plots in Figure 1(b) indicates that the hazard rate function of TLD exhibits a bathtub shape whenever $0 < \alpha < 1$ and a non-decreasing shape for $\alpha \geq 1$. The remaining sections of this paper are organized as follows: Section 2 presents some mathematical properties of the proposed distribution which include; the probability density function, cumulative distribution function, Hazard rate function, Survival function, Quantile function, Moments, Moment generating function, Renyi entropy, and distribution of ordered statistics. The model parameter estimation and simulation study on the maximum likelihood estimates of the proposed distribution are given in Section 3. Finally, in Section 4, we applied the proposed distribution to a real data set and compared its fit with the fit of some existing Topp-Leone generated family of distributions.

2. Mathematical Properties of the Proposed Distribution

2.1 The density and cumulative distribution functions of the proposed distribution

[13] introduced a new class of the Topp-Leone generated family of distributions with the cumulative distribution function defined as

$$F(x) = [G(x)]^\alpha [2 - G(x)]^\alpha, \tag{7}$$

and the corresponding density function given by

$$f(x) = 2\alpha g(x)(1 - G(x))[G(x)]^{\alpha-1}[2 - G(x)]^{\alpha-1}. \tag{8}$$

Taking $G(x)$ as the distribution function of the Exponentiated exponential distribution developed by [8], the authors obtained the cumulative distribution function of the Topp-Leone Generalized Exponential Distribution as

$$F(x) = [1 - e^{-\lambda x}]^{\beta\alpha} [2 - (1 - e^{-\lambda x})^{\beta}]^{\alpha}, \quad (9)$$

and the corresponding density function given by

$$f(x) = 2\alpha\beta\lambda e^{-\lambda x} [1 - e^{-\lambda x}]^{\beta\alpha-1} (1 - (1 - e^{-\lambda x})^{\beta}) \\ \times [2 - (1 - e^{-\lambda x})^{\beta}]^{\alpha-1}. \quad (10)$$

Using a similar approach of generating new distributions defined in equations (7) and (8), we assume the random variable X to follow the 2-parameter Weibull distribution with cumulative distribution function and probability density function respectively defined by

$$G(x) = 1 - e^{-\theta x^{\alpha}} \quad (11)$$

and

$$g(x) = \alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}. \quad (12)$$

Inserting equations (11) and (12) into (7) and (8), we define the cumulative distribution function and the probability distribution function of the Topp Leone Weibull distribution (TLWD) respectively as

$$F(x) = \{(1 - e^{-\theta x^{\alpha}})[(2 - (1 - e^{-\theta x^{\alpha}}))]^{\lambda}\}^{\lambda} \\ = \{1 - e^{-2\theta x^{\alpha}}\}^{\lambda} \quad (13)$$

and

$$f(x) = 2\lambda\alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}} [1 - (1 - e^{-\theta x^{\alpha}})] [(1 - e^{-\theta x^{\alpha}})]^{\lambda-1} [2 - (1 - e^{-\theta x^{\alpha}})]^{\lambda-1} \\ = 2\lambda\alpha\theta x^{\alpha-1} e^{-2\theta x^{\alpha}} (1 - e^{-2\theta x^{\alpha}})^{\lambda-1}. \quad (14)$$

The series representation of equation (14) can be obtain as

$$(1 - e^{-2\theta x^{\alpha}})^{\lambda-1} = \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j e^{-2\theta x^{\alpha}j} \\ f(x) = 2\lambda\alpha\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j x^{\alpha-1} e^{-2\theta x^{\alpha}(1+j)}. \quad (15)$$

The graphical plots of the probability density function are shown in Figure 2.

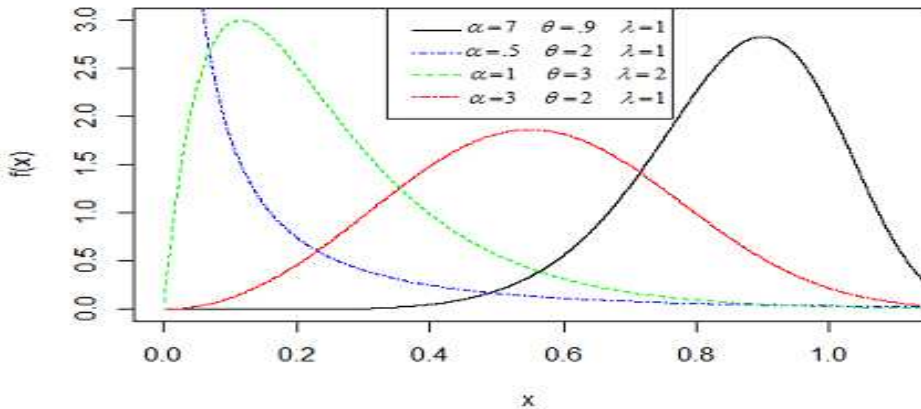


Figure 2: Density function of the TLWD for fixed value of the parameters.

The plots in Figure 2 clearly indicate that the density function of the TLWD exhibits a left-skewed, right-skewed, reversed J-shape and symmetric unimodal shapes.

2.2 Survival function and hazard function of the proposed distribution

Using equations (13) and (14), the survival function and the hazard rate function of the Topp Leone Weibull distribution are given by

$$\begin{aligned}
 S(x) &= 1 - F(x) \\
 &= 1 - \{1 - e^{-2\theta x^\alpha}\}^\lambda
 \end{aligned}
 \tag{16}$$

and

$$\begin{aligned}
 H(x) &= \frac{f(x)}{1-F(x)} \\
 &= \frac{2\lambda\alpha\theta x^{\alpha-1}e^{-2\theta x^\alpha}}{\{(1-e^{-2\theta x^\alpha})^{1-\lambda} - (1-e^{-2\theta x^\alpha})\}}.
 \end{aligned}
 \tag{17}$$

The graphical plots of the hazard rate function of the TLWD for different choice of the parameter value is shown in Figure 3.

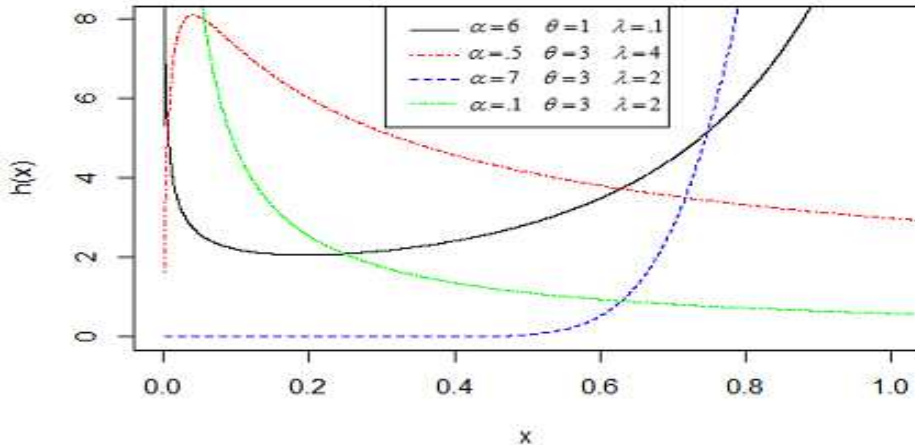


Figure 3: Hazard rate function of the TLWD for fixed value of the parameters.

2.3 Quantile function of the proposed distribution

The quantile function of a random variable x is obtained by solving the system equation of $F(x) = p$. Thus, given the cumulative distribution function $F(x)$ defined in equation (13), the p^{th} quantile function of the TLWD can be obtain as

$$\begin{aligned} \{1 - e^{-2\theta x^\alpha}\}^\lambda &= p \\ 1 - e^{-2\theta x^\alpha} &= p^{1/\lambda} \\ e^{-2\theta x^\alpha} &= 1 - p^{1/\lambda} \end{aligned}$$

taking the natural logarithm of both sides we have

$$\begin{aligned} -2\theta x^\alpha &= \log(1 - p^{1/\lambda}) \\ x^\alpha &= -\frac{1}{2\theta} \log(1 - p^{1/\lambda}) \\ x &= \left\{-\frac{1}{2\theta} \log(1 - p^{1/\lambda})\right\}^{1/\alpha}, \quad 0 < p < 1. \end{aligned} \quad (18)$$

The median of the TLWD can be obtained by substituting $p = \frac{1}{2}$ in equation (18) which yields,

$$median = \left\{-\frac{1}{2\theta} \log(1 - (0.5)^{1/\lambda})\right\}^{1/\alpha}. \quad (19)$$

2.4 Moments of the proposed distribution

Let X be a continuous random variable with density function $f(x)$, then the r^{th} moment about the origin of X is defined by

$$E[X^r] = \int_{-\infty}^{\infty} x^r f(x) dx \tag{20}$$

inserting the density function of the TLWD into equation (20), the r^{th} moment about the origin of the random variable X is defined by

$$E[X^r] = \int_0^{\infty} 2\lambda\alpha\theta x^{r+\alpha-1} e^{-2\theta x^\alpha} \{1 - e^{-2\theta x^\alpha}\}^{\lambda-1} dx \tag{21}$$

$$\{1 - e^{-2\theta x^\alpha}\}^{\lambda-1} = \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j e^{-2\theta x^\alpha j}$$

so that equation (21) now becomes,

$$E[X^r] = 2\lambda\alpha\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \int_0^{\infty} x^{r+\alpha-1} e^{-2\theta x^\alpha(1+j)} dx \tag{22}$$

evaluating the integral part of equation (22), we have

$$\int_0^{\infty} x^{r+\alpha-1} e^{-2\theta x^\alpha(1+j)} dx$$

let $y = 2\theta x^\alpha(1+j)$, which implies that, $x = \left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha}$

so that, $dx = \frac{1}{\alpha 2\theta(1+j)} \left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha-1} dy$

$$\int_0^{\infty} \left\{ \left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha} \right\}^{r+\alpha-1} e^{-y} \frac{1}{\alpha 2\theta(1+j)} \left[\frac{y}{2\theta(1+j)}\right]^{1/\alpha-1} dy$$

$$\int_0^{\infty} \frac{y^{r/\alpha+1-1/\alpha+1/\alpha-1}}{\alpha [2\theta(1+j)]^{r/\alpha+1-1/\alpha+1/\alpha}} e^{-y} dy$$

$$\int_0^{\infty} \frac{y^{r/\alpha} e^{-y}}{\alpha [2\theta(1+j)]^{r/\alpha+1}} dy = \frac{\Gamma(r/\alpha+1)}{\alpha [2\theta(1+j)]^{r/\alpha+1}} .$$

Thus, the r^{th} moment about the origin of the TLWD is given by

$$E[X^r] = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(r/\alpha+1)}{[2\theta(1+j)]^{r/\alpha+1}} . \tag{23}$$

The first four raw moments of the TLWD are obtained from equation (23) as

$$\mu'_1 = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(1/\alpha+1)}{[2\theta(1+j)]^{1/\alpha+1}},$$

$$\mu'_2 = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(2/\alpha+1)}{[2\theta(1+j)]^{2/\alpha+1}},$$

$$\mu'_3 = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(3/\alpha+1)}{[2\theta(1+j)]^{3/\alpha+1}},$$

$$\mu'_4 = 2\lambda\theta \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{\Gamma(4/\alpha+1)}{[2\theta(1+j)]^{4/\alpha+1}}.$$

Furthermore, using the first four raw moments defined above, the measures of skewness (S_k) and kurtosis (K_S) are obtained as

$$S_k = \frac{\mu'_3 - 3\mu'_2\mu + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}} \quad \text{and} \quad K_S = \frac{\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}.$$

Tables 1 and 2 show the theoretical moments of the TLWD for different values of the parameters.

Table 1: Theoretical moments of TLWD for fixed value of the parameter ($\alpha = 2$).

μ'_r	$(\theta = 2, \lambda = 3)$	$(\theta = 2, \lambda = 5)$	$(\theta = 4, \lambda = 3)$	$(\theta = 4, \lambda = 5)$
μ'_1	0.6452	0.7310	0.4562	0.5169
μ'_2	0.4583	0.5708	0.2292	0.2854
μ'_3	0.3542	0.4741	0.1252	0.1676
μ'_4	0.2951	0.4173	0.0738	0.1043
σ^2	0.0420	0.0364	0.0211	0.0182
S_k	0.4980	0.5143	0.4581	0.5078
K_S	3.2886	3.3859	3.5931	3.4248

Table 2: Theoretical moments of TLWD for fixed value of the parameter ($\alpha = 5$).

μ'_r	$(\theta = 2, \lambda = 3)$	$(\theta = 2, \lambda = 5)$	$(\theta = 4, \lambda = 3)$	$(\theta = 4, \lambda = 5)$
μ'_1	0.8288	0.8750	0.7215	0.7617
μ'_2	0.6986	0.7740	0.5294	0.5866
μ'_3	0.5982	0.6921	0.3946	0.4566
μ'_4	0.5199	0.6254	0.2986	0.3592
σ^2	0.0117	0.0084	0.0088	0.0064
S_k	-0.1406	0.2517	-0.1393	0.0333
K_S	3.3840	0.9577	4.4313	5.0395

From Tables 1 and 2, we observed that the TLWD can be right skewed ($S_k > 0$), left skewed ($S_k < 0$) and approximately symmetric ($S_k \approx 0$). Also, at some fixed values of the parameters, the distribution can be leptokurtic ($K_s > 3$), platykurtic ($K_s < 3$) as well as mesokurtic ($K_s \approx 3$). This claim was clearly shown in Figure 2.

2.5 Moment generating function of the proposed distribution

Let X be a continuous random variable with density function $f(x)$, then the moment generating function of X is defined by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \tag{24}$$

inserting the density function of the TLWD into equation (24), we obtain the moment generating function of the TLWD as

$$M_X(t) = \int_0^{\infty} e^{tx} 2\lambda\alpha\theta x^{\alpha-1} e^{-2\theta x^\alpha} \{1 - e^{-2\theta x^\alpha}\}^{\lambda-1} dx \tag{25}$$

using the series representation,

$$\begin{aligned} \{1 - e^{-2\theta x^\alpha}\}^{\lambda-1} &= \sum_{j=0}^{\infty} \binom{\lambda-1}{j} (-1)^j e^{-2\theta x^\alpha j} \\ e^{tx} &= \sum_{m=0}^{\infty} \frac{(tx)^m}{m!} \end{aligned}$$

equation (25) becomes,

$$M_X(t) = 2\lambda\alpha\theta \sum_{j=0}^{\infty} \sum_{M=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{t^M}{M!} \int_0^{\infty} x^{M+\alpha-1} e^{-2\theta x^\alpha(1+j)} dx \tag{26}$$

evaluating the integral part of equation (26), we have

$$\begin{aligned} &\int_0^{\infty} x^{M+\alpha-1} e^{-2\theta x^\alpha(1+j)} dx \\ &= \int_0^{\infty} \left\{ \left[\frac{y}{2\theta(1+j)} \right]^{1/\alpha} \right\}^{M+\alpha-1} e^{-y} \frac{1}{\alpha 2\theta(1+j)} \left[\frac{y}{2\theta(1+j)} \right]^{1/\alpha-1} dy \\ &= \int_0^{\infty} \frac{y^{M/\alpha+1-1/\alpha+1/\alpha-1}}{\alpha [2\theta(1+j)]^{r/\alpha+1-1/\alpha+1/\alpha}} e^{-y} dy \\ &= \int_0^{\infty} \frac{y^{M/\alpha} e^{-y}}{\alpha [2\theta(1+j)]^{M/\alpha+1}} dy \\ &\int_0^{\infty} x^{M+\alpha-1} e^{-2\theta x^\alpha(1+j)} dx = \frac{\Gamma(M/\alpha+1)}{\alpha [2\theta(1+j)]^{M/\alpha+1}} \end{aligned} \tag{27}$$

upon substituting equation (27) into equation (26), the moment generating function of the

TLWD is given by

$$M_X(t) = 2\lambda\theta \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \binom{\lambda-1}{j} (-1)^j \frac{t^m}{m!} \frac{\Gamma(m/\alpha+1)}{[2\theta(1+j)]^{m/\alpha+1}}. \quad (28)$$

2.6 Renyi entropy of the proposed distribution

[12] defined an entropy of a random variable X as a measure of variation of uncertainty associated with the random variable X . The Renyi entropy of X with density function $f(x)$, is defined by,

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log \int f^\gamma(x) dx, \quad \gamma > 0, \quad \gamma \neq 1 \quad (29)$$

inserting the density function of the TLWD into equation (29), we obtain the Renyi entropy of the random variable X following the TLWD as

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log \left[(2\lambda\alpha\theta)^\gamma \int_0^\infty x^{(\alpha-1)\gamma} e^{-2\theta x^\alpha} \{1 - e^{-2\theta x^\alpha}\}^{(\lambda-1)\gamma} dx \right] \quad (30)$$

using series representation,

$$\{1 - e^{-2\theta x^\alpha}\}^{\lambda\gamma-\gamma} = \sum_{j=0}^{\infty} \binom{\lambda\gamma-\gamma}{j} (-1)^j e^{-2\theta x^\alpha j} \quad (31)$$

substituting equation (31) into equation (30), we have

$$= \frac{1}{1-\gamma} \log \left[(2\lambda\alpha\theta)^\gamma \sum_{j=0}^{\infty} \binom{\lambda\gamma-\gamma}{j} (-1)^j \int_0^\infty x^{(\alpha-1)\gamma} e^{-2\theta x^\alpha(\gamma+j)} dx \right] \quad (32)$$

again, evaluating the integral part of equation (32),

$$\begin{aligned} & \int_0^\infty x^{(\alpha-1)\gamma} e^{-2\theta x^\alpha(\gamma+j)} dx \\ &= \int_0^\infty \left\{ \left[\frac{y}{2\theta(\gamma+j)} \right]^{1/\alpha} \right\}^{\alpha\gamma-\gamma} e^{-y} \frac{1}{\alpha 2\theta(\gamma+j)} \left[\frac{y}{2\theta(\gamma+j)} \right]^{1/\alpha-1} dy \\ &= \int_0^\infty \frac{y^{\gamma-\gamma/\alpha+1/\alpha-1}}{\alpha [2\theta(\gamma+j)]^{\gamma-\gamma/\alpha+1/\alpha}} e^{-y} dy \\ &= \int_0^\infty \frac{y^{\gamma+(1-\gamma)/\alpha-1} e^{-y}}{\alpha [2\theta(1+j)]^{\gamma+(1-\gamma)/\alpha}} dy \\ & \int_0^\infty x^{(\alpha-1)\gamma} e^{-2\theta x^\alpha(\gamma+j)} dx = \frac{\Gamma(\gamma+(1-\gamma)/\alpha)}{\alpha [2\theta(\gamma+j)]^{\gamma+(1-\gamma)/\alpha}} \quad (33) \end{aligned}$$

upon substituting equation (33) into equation (32), the Renyi entropy of the random

variable X following the TLWD is given by

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log \left[(2\lambda\alpha\theta)^\gamma \sum_{j=0}^{\infty} \binom{\lambda\gamma - \gamma}{j} (-1)^j \frac{\Gamma(\gamma + (1-\gamma)/\alpha)}{\alpha[2\theta(\gamma+j)]^{\gamma + (1-\gamma)/\alpha}} \right]. \tag{34}$$

2.7 The distribution of the ordered statistics of the proposed distribution

Suppose that $Y_{1:n} < Y_{2:n} < \dots < Y_{n:n}$ is the order statistics of a random sample generated from TLWD, then the probability density function of the k^{th} order statistics, say $X = Y_{n:n}$ is given by

$$h_i(x) = \frac{n!}{(n-k)!(k-1)!} [F(x)]^{k-1} [1 - F(x)]^{n-k} f(x) \tag{35}$$

substituting the cumulative distribution function and the density function of TLWD defined in equations (13) and (14), into equation (35), we have

$$h_i(x) = \frac{2\lambda\alpha\theta n!}{(n-k)!(k-1)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n-k}{i} \binom{\lambda-1}{j} (-1)^{i+j} e^{-2\theta x^\alpha(1+j)} \times x^{\alpha-1} \left[\{1 - e^{-2\theta x^\alpha}\}^\lambda \right]^{i+k-1} \tag{36}$$

using the series representation,

$$\{1 - e^{-2\theta x^\alpha}\}^{\lambda(i+k-1)} = \sum_{m=0}^{\infty} \binom{\lambda(i+k-1)}{m} (-1)^m e^{-2\theta x^\alpha m}$$

so that equation (36) becomes,

$$h_i(x) = \frac{2\lambda\alpha\theta n!}{(n-k)!(k-1)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \binom{n-k}{i} \binom{\lambda-1}{j} \binom{\lambda(i+k-1)}{m} \times (-1)^{i+j+m} x^{\alpha-1} e^{-2\theta x^\alpha(1+j+m)} \tag{37}$$

Hence, the s^{th} moment of the k^{th} order statistics from the TLWD is defined by

$$E(X_k^s) = \int_0^\infty x^s h_i(x) dx \tag{38}$$

$$= \frac{2\lambda\alpha\theta n!}{(n-k)!(k-1)!} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \binom{n-k}{i} \binom{\lambda-1}{j} \binom{\lambda(i+k-1)}{m} \times (-1)^{i+j+m} \int_0^\infty x^{s+\alpha-1} e^{-2\theta x^\alpha(1+j+m)} \tag{39}$$

evaluating the integral part of (39),

$$\int_0^\infty x^{s+\alpha-1} e^{-2\theta x^\alpha(1+j+m)}$$

let $y = 2\theta x^\alpha(1 + j + m)$, which implies that, $x = \left[\frac{y}{2\theta(1+j+m)}\right]^{1/\alpha}$

so that, $dx = \frac{1}{\alpha 2\theta(1+j+m)} \left[\frac{y}{2\theta(1+j+m)}\right]^{1/\alpha - 1} dy$

$$\int_0^\infty \left\{ \left[\frac{y}{2\theta(1+j+m)}\right]^{1/\alpha} \right\}^{s+\alpha-1} e^{-y} \frac{1}{\alpha 2\theta(1+j+m)} \left[\frac{y}{2\theta(1+j+m)}\right]^{1/\alpha - 1} dy$$

$$\int_0^\infty \frac{y^{s/\alpha+1-1/\alpha+1/\alpha-1}}{\alpha [2\theta(1+j+m)]^{s/\alpha+1-1/\alpha+1/\alpha}} e^{-y} dy$$

$$\int_0^\infty \frac{y^{s/\alpha} e^{-y}}{\alpha [2\theta(1+j+m)]^{s/\alpha+1}} dy = \frac{\Gamma(s/\alpha+1)}{\alpha [2\theta(1+j+m)]^{s/\alpha+1}}$$

equation (39) now becomes

$$E(X_k^s) = \frac{2\lambda\theta n!}{(n-k)!(k-1)!} \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{n-k}{i} \binom{\lambda-1}{j} \binom{\lambda(i+k-1)}{m}$$

$$\times (-1)^{i+j+m} \frac{\Gamma(s/\alpha+1)}{[2\theta(1+j+m)]^{s/\alpha+1}}. \quad (40)$$

3.0 Parameter Estimation of the Proposed Distribution

3.1 Maximum likelihood estimation

Let (x_1, x_2, \dots, x_n) be random samples from the TLWD, then the log-likelihood function of the TLWD is defined by

$$\ell(x, \phi) = \sum_{i=1}^n \log \left[2\lambda\alpha\theta x^{\alpha-1} e^{-2\theta x^\alpha} \{1 - e^{-2\theta x^\alpha}\}^{\lambda-1} \right], \quad \phi = (\alpha, \theta, \lambda)^T$$

$$= n \log(2\lambda\alpha\theta) + (\alpha - 1) \sum_{i=1}^n \log(x_i) - 2\theta \sum_{i=1}^n x_i^\alpha$$

$$+ (\lambda - 1) \sum_{i=1}^n \log(1 - e^{-2\theta x_i^\alpha}). \quad (41)$$

On differentiating the log-likelihood function with respect to the parameters, we obtain the score function as,

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log(x_i) - 2\theta \sum_{i=1}^n x_i^\alpha \log(x_i) + 2\theta(\lambda - 1) \sum_{i=1}^n \frac{x_i^\alpha \log(x_i) e^{-2\theta x_i^\alpha}}{\{1 - e^{-2\theta x_i^\alpha}\}}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \log(1 - e^{-2\theta x_i^\alpha})$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - 2 \sum_{i=1}^n x_i^\alpha + 2(\lambda - 1) \sum_{i=1}^n \frac{x_i^\alpha e^{-2\theta x_i^\alpha}}{\{1 - e^{-2\theta x_i^\alpha}\}}.$$

The maximum likelihood estimator $\hat{\phi}$ of ϕ can be obtained by solving the system of equation $\frac{\partial \ell}{\partial \phi} = 0$. This equation cannot be solved using direct method since the system of equation is non-linear. However, an iterative method of solution such as the Newton Raphson method can be used. The Newton Raphson iterative scheme has the form:

$$\hat{\phi} = \phi_k - H^{-1}(\phi_k)U(\phi_k), \phi = (\alpha, \theta, \lambda)^T \tag{42}$$

where $U(\phi_k)$ is the score function and $H(\phi_k)$ is the Hessian matrix, which is the second derivative of the log-likelihood function. The Hessian matrix is defined by

$$H(\phi_k) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \alpha \partial \alpha} & \frac{\partial^2 \ell}{\partial \alpha \partial \theta} & \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \theta \partial \alpha} & \frac{\partial^2 \ell}{\partial \theta \partial \theta} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda \partial \lambda} \end{bmatrix}$$

where,

$$\frac{\partial^2 \ell}{\partial \alpha \partial \alpha} = -\frac{n}{\alpha^2} - 2\theta \sum_{i=1}^n x_i^\alpha [\log(x_i)]^2 + 2\theta(\lambda - 1) \sum_{i=1}^n \frac{x_i^\alpha [\log(x_i)]^2 e^{-2\theta x_i^\alpha} (1 - 2\theta x_i^\alpha - e^{-2\theta x_i^\alpha})}{\{1 - e^{-2\theta x_i^\alpha}\}^2}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \theta} = \frac{\partial^2 \ell}{\partial \theta \partial \alpha} = -2 \sum_{i=1}^n x_i^\alpha \log(x_i) + 2(\lambda - 1) \sum_{i=1}^n \frac{x_i^\alpha \log(x_i) e^{-2\theta x_i^\alpha} (1 - 2\theta x_i^\alpha - e^{-2\theta x_i^\alpha})}{\{1 - e^{-2\theta x_i^\alpha}\}^2}$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial \theta} = 2 \sum_{i=1}^n \frac{x_i^\alpha e^{-2\theta x_i^\alpha}}{\{1 - e^{-2\theta x_i^\alpha}\}}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} = 2\theta \sum_{i=1}^n \frac{x_i^\alpha \log(x_i) e^{-2\theta x_i^\alpha}}{\{1 - e^{-2\theta x_i^\alpha}\}}$$

$$\frac{\partial^2 \ell}{\partial \lambda \partial \lambda} = -\frac{n}{\lambda^2}$$

$$\frac{\partial^2 \ell}{\partial \theta \partial \theta} = -\frac{n}{\theta^2} - 4(\lambda - 1) \sum_{i=1}^n \frac{x_i^{2\alpha} e^{-2\theta x_i^\alpha}}{\{1 - e^{-2\theta x_i^\alpha}\}^2}$$

3.2 Interval estimate

The asymptotic confidence intervals (CIs) for the parameters of TLWD $(\alpha, \theta, \lambda)$ are

obtained according to the asymptotic distribution of the maximum likelihood estimates of the parameters.

Suppose $\hat{\phi} = (\hat{\alpha}, \hat{\theta}, \hat{\lambda})$ is MLE of ϕ , then the estimators are approximately bi-variate normal with mean $(\alpha, \theta, \lambda)$ and the Fisher information matrix is given by:

$$I(\phi_k) = -E(H(\phi_k)). \quad (43)$$

The approximate $(1 - \delta)100$ CIs for the parameters α, θ and λ respectively, are

$$\hat{\alpha} \pm Z_{\delta/2} \sqrt{\text{var}(\hat{\alpha})}, \hat{\theta} \pm Z_{\delta/2} \sqrt{\text{var}(\hat{\theta})} \text{ and } \hat{\lambda} \pm Z_{\delta/2} \sqrt{\text{var}(\hat{\lambda})}$$

where $\text{var}(\hat{\alpha})$, $\text{var}(\hat{\theta})$ and $\text{var}(\hat{\lambda})$ are the variance of α, θ and λ which are given by the diagonal elements of the variance-covariance matrix $I^{-1}(\phi_k)$ and $Z_{\delta/2}$ is the upper $(\delta/2)$ percentile of the standard normal distribution.

3.3 Simulation Study

In this section, we investigate the asymptotic behaviour of the maximum likelihood estimates of the parameters of the Topp-Leone Weibull distribution (TLWD) through a simulation study. A Monte Carlo simulation study is repeated 1000 times for different sample sizes $n = 50, 75, 100, 200$ and parameter values $(\alpha = 1.0, \theta = 0.3, \lambda = 0.5)$, $(\alpha = 1.0, \theta = 0.3, \lambda = 0.5)$ and $(\alpha = 1.0, \theta = 0.3, \lambda = 0.5)$. [6] suggested an algorithm for the simulation study employing the following steps.

1. Choose the value N (i.e. number of Monte Carlo simulation);
2. choose the values $\phi_0 = (\alpha_0, \theta_0, \lambda_0)$ corresponding to the parameters of the TLWD $(\alpha, \theta, \lambda)$;
3. generate a sample of size n from TLWD;
4. compute the maximum likelihood estimates $\hat{\phi}_0$ of ϕ_0 ;
5. repeat steps (3-4), N -times;
6. compute the:

(i) Average Bias = $\frac{1}{N} \sum_{i=1}^N (\hat{\phi}_i - \phi_0)$,

(ii) Mean Square Error (MSE) = $\frac{1}{N} \sum_{i=1}^N (\hat{\phi}_i - \phi_0)^2$ and

(iii) Coverage Probability of the 95% confidence interval of the estimates $\hat{\phi}_i$ given by

$$CP(\phi_0) = \frac{1}{N} \sum_{i=1}^N I \left(\hat{\phi}_i - Z_{\delta/2} \sqrt{\text{var}(\hat{\phi})} < \phi_0 < \hat{\phi}_i + Z_{\delta/2} \sqrt{\text{var}(\hat{\phi})} \right)$$

where $I(\cdot)$ is an indicator function and $\sqrt{\text{var}(\hat{\phi})}$ is the standard error of the estimate ϕ_i .

The coverage probability computes the proportion of times the confidence interval contains the true value of the parameter ϕ_0 .

Table 3: Monte Carlo simulation results for average bias of the MLE.

Parameter	N	Average Bias (α)	Average Bias (θ)	Average Bias (λ)
$\alpha = 1.0$	50	0.2970	0.0314	0.0994
$\theta = 0.3$	75	0.1357	0.0293	0.0724
$\lambda = 0.5$	100	0.1364	0.0082	0.0362
	200	0.0507	0.0068	0.0157
$\alpha = 1.5$	50	0.4154	0.0341	0.1622
$\theta = 0.4$	75	0.2859	0.0141	0.0740
$\lambda = 0.6$	100	0.1714	0.0140	0.0540
	200	0.0721	0.0074	0.0242
$\alpha = 2.0$	50	0.5309	0.0403	0.2115
$\theta = 0.5$	75	0.2740	0.0087	0.0883
$\lambda = 0.8$	100	0.2340	-0.0114	0.0319
	200	0.0735	0.0104	0.0513

Table 4: Monte Carlo simulation results for mean square error (MSE) of the MLE.

Parameter	N	MSE (α)	MSE (θ)	MSE(λ)
$\alpha = 1.0$	50	0.6165	0.0727	0.2892
$\theta = 0.3$	75	0.2463	0.0422	0.1285
$\lambda = 0.5$	100	0.2105	0.0398	0.0916
	200	0.0598	0.0146	0.0271

$\alpha = 1.5$	50	1.5398	0.0995	0.4345
$\theta = 0.4$	75	0.8736	0.0511	0.1777
$\lambda = 0.6$	100	0.3915	0.0392	0.3917
	200	0.1208	0.0197	0.0477
$\alpha = 2.0$	50	3.4213	0.0975	0.6521
$\theta = 0.5$	75	0.9752	0.0568	0.2400
$\lambda = 0.8$	100	0.5449	0.0408	0.1602
	200	0.2143	0.0260	0.1023

Table 5: Monte Carlo simulation results for coverage probability of the 95% CIs of the MLE.

<i>Parameter</i>	<i>n</i>	<i>CP (α)</i>	<i>CP (θ)</i>	<i>CP (λ)</i>
$\alpha = 1.0$	50	0.8200	0.8133	0.8333
$\theta = 0.3$	75	0.8800	0.8933	0.8933
$\lambda = 0.5$	100	0.9200	0.8867	0.8967
	200	0.9300	0.9400	0.9467
$\alpha = 1.5$	50	0.8667	0.8000	0.8167
$\theta = 0.4$	75	0.9200	0.8967	0.8867
$\lambda = 0.6$	100	0.9433	0.9233	0.9033
	200	0.9500	0.9200	0.9133
$\alpha = 2.0$	50	0.9167	0.8900	0.8700
$\theta = 0.5$	75	0.9567	0.9133	0.8900
$\lambda = 0.8$	100	0.9700	0.9067	0.8800
	200	0.9300	0.9167	0.9233

Tables 3, 4 and 5 present the Monte Carlo simulation results for the average bias, mean square error and coverage probability of the 95% confidence interval of the parameter estimates of the TLWD at different choice of parameter values. Clearly from Table 3, we observe that the parameter α is positively biased, parameter λ is positively biased, while parameter θ could either be negatively or positively biased. Also from Table 4, as the sample size n increases, the values of the mean square error of the parameter estimates decreases (tends to zero). Finally, from Table 5, we observe that the coverage probabilities of the CIs are quite close to the nominal level of 95%.

4. Application of the Proposed Distribution

In this section, we fit the proposed distribution to a real data set alongside with some well-known lifetime distributions from the Topp-Leone Generated family of distributions with density functions given by;

(i) Topp-Leone Inverse Weibull Distribution (TLIWD)

$$f(x) = \frac{2\alpha\theta\lambda}{x^{\lambda+1}} e^{-\theta/x^\lambda} \left(1 - e^{-\theta/x^\lambda}\right) \left[1 - \left(1 - e^{-\theta/x^\lambda}\right)^2\right]^{\alpha-1};$$

(ii) Topp-Leone Bur XII Distribution (TLBXIID)

$$f(x) = 2\alpha\theta\lambda x^{\theta-1} (1+x^\theta)^{-(2\alpha+1)} \left[1 - (1+x^\theta)^{-2\alpha}\right]^{\lambda-1};$$

(iii) Topp-Leone Exponential Distribution (TLED)

$$f(x) = 2\alpha\lambda e^{-2\lambda x} (1 - e^{-2\lambda x})^{\alpha-1};$$

(iv) Topp-Leone Linear Exponential Distribution (TLLED)

$$f(x) = 2\alpha(\theta + \lambda x) e^{-2\left(\theta x + \frac{\lambda x^2}{2}\right)} \left[1 - e^{-2\left(\theta x + \frac{\lambda x^2}{2}\right)}\right]^{\alpha-1};$$

(v) Topp-Leone Nadarajah-Haghighi Distribution (TLNHD)

$$f(x) = 2\alpha\theta\lambda(1 + \theta x)^{\lambda-1} e^{2(1-(1+\theta x)^\lambda)} \left[1 - e^{2(1-(1+\theta x)^\lambda)}\right]^{\alpha-1}.$$

The dataset consists of the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm reported in [7]. Table 6 presents the dataset.

Table 6: Tensile strength, measured in GPa, of 69 carbon fibers.

1.312	1.314	1.479	1.552	1.700	1.803	1.861
1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.027	2.055	2.063	2.098	2.14	2.179	2.224
2.240	2.253	2.270	2.272	2.274	2.301	2.301
2.359	2.382	2.382	2.426	2.434	2.435	2.478
2.490	2.511	2.514	2.535	2.554	2.566	2.57
2.586	2.629	2.633	2.642	2.648	2.684	2.697
2.726	2.770	2.773	2.800	2.809	2.818	2.821
2.848	2.88	2.954	3.012	3.067	3.084	3.090
3.096	3.128	3.233	3.433	3.585	3.585	

The comparison criteria considered in this paper includes, the estimates of the parameters of the distribution, $\text{Log} - \text{lik}$, Akaike Information Criterion $AIC = 2k - 2\log(L)$, Kolmogorov-Smirnov Statistic ($K - S$) and Anderson Darling test statistic (A^*) and Crammer-von Mises test statistic (W^*). Where, k is the number of estimated parameters and L is the value of the log-likelihood function evaluated at the parameter estimates.

Table 7: Summary statistics for the tensile strength of 69 carbon fibers dataset.

<i>Models</i>	<i>Estimates</i>	<i>Log-Lik</i>	<i>AIC</i>	<i>K-S</i> (<i>p-value</i>)	<i>W*</i> (<i>p-value</i>)	<i>A*</i> (<i>p-value</i>)
TLWD	$\alpha = 3.8590$	- 48.8598	103.7195	0.0395 (0.9999)	0.0154 (0.9996)	0.1435 (0.9990)
	$\theta = 0.0194$					
	$\lambda = 2.0312$					
TLBXIID	$\alpha = 1.4510$	- 59.9967	125.9933	0.1216 (0.2591)	0.3164 (0.1215)	2.1117 (0.0799)
	$\theta = 1.8205$					
	$\lambda = 110.9177$					
TLIWD	$\alpha = 0.5469$	- 58.0304	122.0608	0.1176 (0.2960)	0.2617 (0.1741)	1.7344 (0.1294)
	$\theta = 34.8899$					
	$\lambda = 3.4115$					
TLED	$\alpha = 88.1467$	- 54.6201	113.2403	0.0950 (0.5617)	0.1601 (0.3608)	1.1210 (0.2994)
	$\lambda = 1.0186$					
TLLED	$\alpha = 6.9393$	- 50.5881	107.1762	0.0660 (0.9240)	0.0627 (0.7985)	0.4406 (0.8072)
	$\theta = -0.0826$					
	$\lambda = 0.4742$					
TLNHD	$\alpha = 21.4659$	- 50.4692	106.9383	0.0652 (0.9311)	0.0604 (0.8129)	0.4235 (0.8246)
	$\theta = 0.0243$					
	$\lambda = 17.6504$					

Discussion of Result

The superiority of a model over another can be characterized by the model having the maximized loglikelihood and the least value in terms of AIC , $K-S$ Statistic, Anderson

Darling test statistic (A^*) and Crammer-von Mises test statistic (W^*). Table 7 clearly reveal that the proposed distribution (TPGLD) having the maximized loglikelihood and the least values in terms of AIC , $K-S$ Statistic, Anderson Darling test statistic (A^*) and Crammer-von Mises test statistic (W^*), demonstrates superiority over the Topp Leone Inverse Weibull Distribution (TLIWD), Topp Leone Bur XII Distribution (TLBXIID), Topp Leone Exponential Distribution (TLED), Topp Leone Linear Exponential Distribution (TLLED) and Topp Leone Nadarajah-Haghighi Distribution (TLNHD) in modeling the lifetime dataset under study. This claim was further supported by graphical illustration of the density and cumulative distribution fit of the distributions for the real lifetime dataset as displayed in Figure 4.

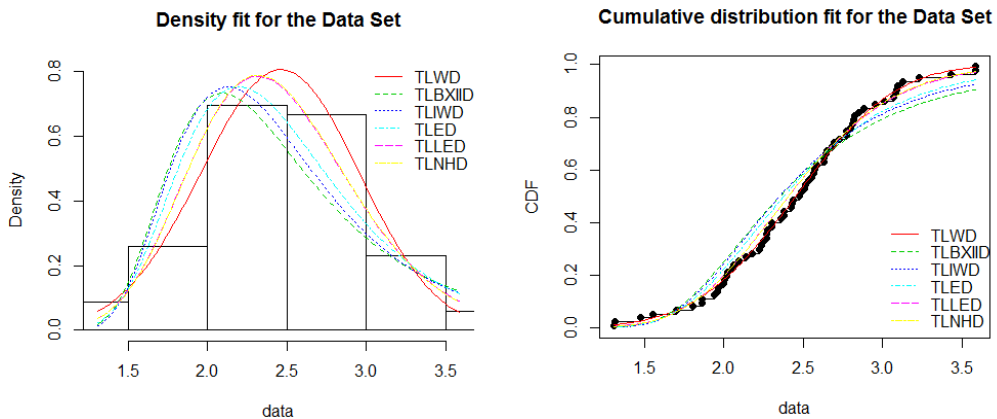


Figure 4: Density and cumulative distribution fit of the distributions for the dataset.

Conclusion

In this paper, we proposed a new member of the Topp-Leone generated family of distributions called the Topp-Leone Weibull Distribution (TLWD). The mathematical properties of the proposed distribution such as; the density function, cumulative distribution function, survival function, hazard rate function, moments and related measure, moment generating function, quantile function, Renyi entropy and the distribution of ordered statistics were derived. The method of maximum likelihood estimation was used in estimating the parameters of the proposed distribution.

Finally, an application of the proposed distribution to a real lifetime data set alongside with the Topp-Leone Bur XII distribution, Topp-Leone Inverse Weibull distribution, Topp-Leone Exponential distribution, Topp-Leone Linear Exponential distribution, and Topp-Leone Nadarajah-Haghighi distribution, reveals that the proposed

Topp-Leone Weibull distribution, demonstrates superiority and offers more flexibility in modeling the lifetime data set under study.

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