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$f-$Biharmonic Curves in the Three-dimensional Para-Sasakian Space Forms

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Abstract

In this paper, we give some characterizations for proper $f-$biharmonic curves in the para-Bianchi-Cartan-Vranceanu space forms with 3-dimensional para-Sasakian structures.

1 Introduction

As a natural generalization of biharmonic curves, the concept of $f-$biharmonic curves was introduced by Lu in [4]. Since this paper, many authors studied f-biharmonic curves in several spaces: Ou considered $f-$biharmonic curves on a generic manifold and gave a characterization for them in $n-$dimensional space forms [6]. Guvenc and Ozgur studied $f-$biharmonic Legendre curves in Sasakian space forms [2]. Karaca and Ozgur investigated $f-$biharmonic curves in Sol spaces, Cartan Vranceanu three-dimensional spaces and homogenous contact three-manifolds [3]. Dua and Zhang examined $f-$biharmonic curves in Lorentz–Minkowski spaces [1].

On the other hand, in a very recent paper [5], Lee constructed the para-Bianchi-Cartan-Vranceanu model with 3-dimensional para-Sasakian structure and found the necessary and sufficient conditions for biharmonic Frenet curves.

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In this paper, we investigate $f-$biharmonic curves in this 3-dimensional para-Sasakian manifolds. We obtain some characterizations with respect to the special situations of curvature and torsion functions of these curves. Throughout the paper, all geometric objects (curves, manifolds, vector fields, functions etc.) are assumed to be smooth.

2 Preliminaries

2.1 Para-Sasakian manifolds

We recall fundamental ingredients of para-Sasakian manifolds from [5]. A $(2n+1)-$dimensional differentiable manifold M is said to be an almost paracontact manifold if it admits a (1,1)-tensor field φ, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi,\ \ \eta(\xi) = 1.$$

For an almost paracontact manifold M, we have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

If a $(2n+1)-$dimensional manifold M with almost paracontact structure (φ, ξ, η) admits a compatible pseudo-Riemannian metric such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

then we say M is an almost paracontact metric manifold with the paracontact metric structure (φ, ξ, η, g). Putting $Y = \xi$, we have

$$\eta(X) = g(X, \xi). \tag{2.2}$$

If the compatible pseudo-Riemannian metric g satisfies

$$d\eta(X, Y) = g(X, \varphi Y),$$

then η is a contact form on M, ξ the associated Reeb vector field, g an associated metric and $(M, \varphi, \xi, \eta, g)$ is called a paracontact metric manifold.

For a paracontact metric manifold M, an almost paracomplex structure J on $M \times \mathbb{R}$ is defined by

$$J(X, f\frac{d}{dt}) = (\varphi X + f\xi, \eta(X)\frac{d}{dt}),$$

where X is a vector field on M, t the coordinate of $\mathbb{R}$ and f a function of $M \times \mathbb{R}$. If the almost paracomplex structure J is integrable, then the paracontact metric manifold M is said to be normal or para-Sasakian.

Proposition 1. *[8] An almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is para-Sasakian if and only if*

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.3}$$

for any vector fields X, Y on M, where ∇ is Levi-Civita connection of g.

2.2 Frenet-Serret equations

Let $\gamma : I \to M$ be a unit speed curve in a three-dimensional Lorentzian manifold M such that γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$. The constant ε_1 is said to be the causal character of γ. A unit speed curve is called spacelike or timelike if its causal character is 1 or -1, respectively. A unit speed curve is called a Frenet curve if $g(\gamma'', \gamma'') \neq 0$. A Frenet curve has an orthonormal frame field $\{T = \gamma', N, B\}$ along γ. Then the Frenet-Serret equations are given by

$$\begin{aligned} \nabla_T T &= \varepsilon_2 \kappa N, \\ \nabla_T N &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau B, \\ \nabla_T B &= \varepsilon_2 \tau N, \end{aligned}$$

where $\kappa = \left\| \nabla_{\gamma'} \gamma' \right\|$ is the geodesic curvature and τ is the geodesic torsion of γ. The vector fields T, N and B are called tangent vector field, principal normal vector field and binormal vector field of γ, respectively.

The constants ε_2 and ε_3 are defined by $g(N, N) = \varepsilon_2$ and $g(B, B) = \varepsilon_3$, and called second causal character and third causal character of γ, respectively. The equation $\varepsilon_1 \varepsilon_2 = -\varepsilon_3$ holds.

A Frenet curve γ is a geodesic if and only if $\kappa = 0$.

Proposition 2. *Let $\{T, N, B\}$ are orthonormal frame field in a Lorentzian 3-manifold. Then,*

$$T \wedge_L N = \varepsilon_3 B,\ N \wedge_L B = \varepsilon_1 T,\ B \wedge_L T = \varepsilon_2 N.$$

2.3 $f-$Biharmonic maps

A map $\phi : (M_m, g) \to (N_n, h)$ between two pseudo-Riemannian manifolds is called harmonic if it is a critical point of the energy

$$E(\phi) = \frac{1}{2}\int_{\Omega} \|d\phi\|^2\, dv_g,$$

where Ω is a compact domain of M_m. The tension field $\tau(\phi)$ of ϕ is defined by

$$\tau(\phi) = tr(\nabla^{\phi} d\phi) = \sum_{i=1}^{m} \varepsilon_i (\nabla^{\phi}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)),$$

where ∇^{ϕ} and $\{e_i\}$ denote the induced connection by ϕ on the bundle $\phi^* TN_n$. A map ϕ is called harmonic if its tension field vanishes. The bienergy $E_2(\phi)$ of the map ϕ is defined by

$$E_2(\phi) = \frac{1}{2}\int_{\Omega} \|\tau(\phi)\|^2\, dv_g,$$

and ϕ is called biharmonic if it is a critical point of the bienergy, where Ω is a compact domain of M_m. Clearly, all harmonic maps are biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps. The bitension field $\tau_2(\phi)$ of ϕ is defined by

$$\tau_2(\phi) = \sum_{i=1}^{m} \varepsilon_i ((\nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla_{e_i} e_i}) \tau(\phi) - R^N(\tau(\phi), d\phi(e_i)) d\phi(e_i)), \tag{2.4}$$

where R^N denotes the curvature tensor of N_n. A map ϕ is called biharmonic if its bitension field vanishes.

A map ϕ is called $f-$harmonic with a function $f : M \to \mathbb{R}$, if it is a critical point of the energy

$$E_f(\phi) = \frac{1}{2}\int_{\Omega} f\, \|d\phi\|^2\, dv_g,$$

where Ω is a compact domain of M_m. The $f-$tension field $\tau_f(\phi)$ of ϕ is given by

$$\tau_f(\phi) = f\tau(\phi) + d\phi(gradf)$$

see [7]. The $f-$bitension field $\tau_{2,f}(\phi)$ of ϕ is defined by

$$\tau_{2,f}(\phi) = f\tau_2(\phi) + \Delta f\tau(\phi) + 2\nabla^{\phi}_{gradf}\tau(\phi). \tag{2.5}$$

A map ϕ is called $f-$biharmonic if its $f-$bitension field vanishes (see [1], [4]). Non-harmonic and non-biharmonic $f-$biharmonic curves are called proper $f-$biharmonic curves, and if f is constant, then an $f-$biharmonic curve turns into a biharmonic curve [4].

3 $f-$Biharmonic Curves in Para-Sasakian Space Forms

Lee introduced the concept of para-Bianchi-Cartan-Vranceanu model with 3-dimensional para-Sasakian structure in [5] as follows:

Consider the set

$$D = \{(x, y, z) \in \mathbb{R}^3 : 1 + \frac{c}{2}(x^2 + y^2) > 0\},$$

where c is a real number. Remark that if $c \geq 0$, then D is the whole $\mathbb{R}^3(x, y, z)$. On the region D, the contact form η is taken as

$$\eta = dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)}.$$

Then, the characteristic vector field of η is $\xi = \frac{\partial}{\partial z}$.

Next, the Lorentzian metric is equipped as

$$g_c = \frac{-dx^2 + dy^2}{\{1 + \frac{c}{2}(x^2 + y^2)\}^2} + (dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)})^2.$$

The Lorentzian orthonormal frame field (e_1, e_2, e_3) on (D, g_c) is given by

$$e_1 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \{1 + \frac{c}{2}(x^2 + y^2)\}\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}.$$

Then the endomorphism field φ is given by

$$\varphi(e_1) = e_2,\ \varphi(e_2) = e_1,\ \varphi(e_3) = 0.$$

The Levi-Civita connection ∇ of (D, g_c) is described as

$$\begin{gathered}\nabla_{e_1}e_1 = -cye_2,\ \nabla_{e_1}e_2 = -cye_1 + e_3,\ \nabla_{e_1}e_3 = -e_2,\\ \nabla_{e_2}e_1 = -cxe_2 - e_3,\ \nabla_{e_2}e_2 = -cxe_1,\ \nabla_{e_2}e_3 = -e_1,\\ \nabla_{e_3}e_1 = -e_2, \nabla_{e_3}e_2 = -e_1, \nabla_{e_3}e_3 = 0.\end{gathered}$$

The contact form η on D fulfills

$$d\eta(X, Y) = g_c(X, \varphi Y),\ X, Y \in \chi(D).$$

Furthermore the structure $(g_c, \varphi, \xi, \eta)$ is para-Sasakian. The non-vanishing components of the curvature tensor R of (D, g_c) is given by

$$\begin{aligned}R(e_1, e_2)e_2 &= -\{3 + c^2(x^2 - y^2)\}e_1,\ R(e_1, e_3)e_3 = e_1,\\ R(e_2, e_1)e_1 &= \{3 + c^2(x^2 - y^2)\}e_2,\ R(e_2, e_3)e_3 = e_2,\\ R(e_3, e_1)e_1 &= -e_3,\ R(e_3, e_2)e_2 = e_3.\end{aligned}$$

For the sectional curvature K of (D, g_c), we have

$$K(e_2, e_3) = -1 = -K(e_3, e_1),$$

and

$$K(e_1, e_2) = R(e_1, e_2, e_1, e_2) = -\{3 + c^2(x^2 - y^2)\}.$$

So, (D, g_c) is of holomorphic sectional curvature $H = -\{3 + c^2(x^2 - y^2)\}$.

For the case $c = 0$, the holomorphic sectional curvature H equals -3, thus the space D becomes para-Sasakian space form. In the next, we will deal with the case $c = 0$.

Now, suppose that $\gamma : I \to (D, g_c)$ is a curve parametrized by arc-length and $\{T, N, B\}$ is an orthonormal frame field tangent to D along γ, where $T = T_1e_1 + T_2e_2 + T_3e_3$, $N = N_1e_1 + N_2e_2 + N_3e_3$ and $B = B_1e_1 + B_2e_2 + B_3e_3$.

The $f-$biharmonicity condition for curves on (D, g_c) is obtained in the following theorem.

Theorem 1. *Let $\gamma : I \to (D, g_c)$ be a curve parametrized by arc-length. Then γ is $f-$biharmonic if and only if the following relations are satisfied:*

$$\begin{aligned} 3\kappa\kappa' f + 2\kappa^2 f' = 0, \\ \kappa f'' + 2\kappa' f' + f[\kappa'' + \varepsilon_3\kappa^3 + \varepsilon_1\kappa\tau^2 + \kappa\varepsilon_2(\varepsilon_3 - 4\eta(B)^2)] = 0, \\ -2\kappa\tau f' - f(2\kappa'\tau + \kappa\tau') - 4\varepsilon_1\kappa f\eta(N)\eta(B) = 0. \end{aligned} \tag{3.1}$$

Proof. Let $\gamma = \gamma(s)$ be a curve parametrized by arc-length. We use formula (2.5). From [5], we have

$$\tau(\gamma) = \varepsilon_1\nabla_T T = -\varepsilon_3\kappa N, \tag{3.2}$$

$$\begin{aligned} R(T, N, T, N) = \varepsilon_3 - 4B_3^2, \\ R(T, N, T, B) = -4\varepsilon_1 N_3 B_3, \end{aligned} \tag{3.3}$$

$$\tau_2(\gamma) = 3\varepsilon_3\kappa\kappa' T + \varepsilon_2(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2))N + \varepsilon_1(2\kappa'\tau + \kappa\tau')B + \varepsilon_2\kappa R(T, N)T. \tag{3.4}$$

Moreover, from [1], we have

$$\begin{aligned} \nabla^{\gamma}_{grad f}\tau(\gamma) = f'\nabla_T(\nabla_T T) = \varepsilon_2 f'[\kappa' N + \kappa(-\varepsilon_1\kappa T - \varepsilon_3\tau B)], \\ \Delta f\tau(\gamma) = f''\nabla_T T = f''\varepsilon_2\kappa N. \end{aligned} \tag{3.5}$$

Therefore, combining the equations (3.2), (3.4) and (3.5), we obtain

$$\begin{aligned} \tau_{2,f}(\gamma) \;=\; & 3\varepsilon_3\kappa\kappa' f T + \varepsilon_2 f(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2))N + \varepsilon_1 f(2\kappa'\tau + \kappa\tau')B \\ & + \varepsilon_2 f\kappa R(T, N)T + \varepsilon_2\kappa f'' N + 2\varepsilon_2 f'[\kappa' N + \kappa(-\varepsilon_1\kappa T - \varepsilon_3\tau B)]. \end{aligned} \tag{3.6}$$

If we take inner product of equation (3.6) with T, N and B, respectively and use the equations (3.3), we get (3.1). □

Proposition 3. *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If κ is a non-zero constant, then γ is biharmonic.*

Proof. Under the assumption κ is a non-zero constant, from the first equation in (3.1), obviously we get $f' = 0$. So, γ is a biharmonic curve. □

Proposition 4. *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If τ is a non-zero constant and $\eta(N)\eta(B) = 0$, then γ is biharmonic.*

Proof. Under the assumption τ is a non-zero constant and $\eta(N)\eta(B) = 0$, using the first and third equations in (3.1), we get

$$\frac{\kappa'}{\kappa} = -\frac{2f'}{3f} \tag{3.7}$$

and

$$\tau(\frac{\kappa'}{\kappa} + \frac{f'}{f}) = 0. \tag{3.8}$$

Putting equation (3.7) in (3.8) shows that f is constant, therefore γ is a biharmonic curve. □

Proposition 5. *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If τ is a non-zero constant, then $f = e^{\int -\frac{6\varepsilon_1 \eta(N)\eta(B)}{\tau}}$.*

Proof. Under the assumption τ is a non-zero constant, if we use the first and third equations in (3.1), we obtain

$$\frac{\kappa'}{\kappa} = -\frac{2f'}{3f} \tag{3.9}$$

and

$$-2\kappa\tau f' - 2f\kappa'\tau - 4\varepsilon_1 \kappa f \eta(N)\eta(B) = 0. \tag{3.10}$$

Setting equation (3.9) in (3.10), we get the result. □

Proposition 6. *Let $\gamma : I \to (D, g_c)$ be a non-geodesic curve parametrized by arc-length and suppose that $\tau = 0$. In this case, γ is $f-$biharmonic if and only if the following equations are valid:*

$$f^2\kappa^3 = c_1^2, \tag{3.11}$$

$$(f\kappa)'' = -f\kappa(\varepsilon_3\kappa^2 + \varepsilon_2(\varepsilon_3 - 4\eta(B)^2)), \tag{3.12}$$

$$\eta(N)\eta(B) = 0, \tag{3.13}$$

where $c_1 \in \mathbb{R}$.

Proof. Under the assumption $\tau = 0$, if we use equations in (3.1) by integrating first equation, we deduce the results. □

Proposition 7. *Let $\gamma : I \to (D, g_c)$ be a non-geodesic curve parametrized by arc-length and suppose that τ and κ are non-constants. In this case, γ is $f-$biharmonic if and only if the following equations are valid:*

$$f^2\kappa^3 = c_1^2, \tag{3.14}$$

$$(f\kappa)'' = -f\kappa(\varepsilon_3\kappa^2 + \varepsilon_1\tau^2 + \varepsilon_2(\varepsilon_3 - 4\eta(B)^2)), \tag{3.15}$$

$$f^2\kappa^2\tau = e^{\int -\frac{4\varepsilon_1\eta(N)\eta(B)}{\tau}}, \tag{3.16}$$

where $c_1 \in \mathbb{R}$.

Proof. Under the assumption τ and κ are non-constants, if we use equations in (3.1) by integrating first and third equations, we obtain (3.14), (3.15) and (3.16). □

From the last two propositions, we can give the following theorem.

Theorem 2. *An arc-length parametrized curve $\gamma : I \to (D, g_c)$ is proper $f-$biharmonic if and only if one of the following situations is true:*

(i) $\tau = 0$, $f = c_1\kappa^{-3/2}$ and the curvature κ solves the equation below:

$$3(\kappa')^2 - 2\kappa\kappa'' = -4\kappa^2[\varepsilon_3\kappa^2 + \varepsilon_2(\varepsilon_3 - 4\eta(B)^2)].$$

(ii) $\tau \neq 0$, $\frac{\tau}{\kappa} = \frac{e^{\int -\frac{4\varepsilon_1\eta(N)\eta(B)}{\tau}}}{c_1^2}$, $f = c_1\kappa^{-3/2}$ and the curvature κ solves the equation below:

$$3(\kappa')^2 - 2\kappa\kappa'' = -4\kappa^2[\varepsilon_3\kappa^2(1 - \varepsilon_2\frac{e^{\int -\frac{8\varepsilon_1\eta(N)\eta(B)}{\tau}}}{c_1^4}) + \varepsilon_2(\varepsilon_3 - 4\eta(B)^2)].$$

Proof. (i) The first equation of (3.1) gives

$$f = c_1\kappa^{-3/2}. \tag{3.17}$$

By replacing the above equation into (3.12), we obtain the result.

(ii) From the first equation of (3.1), we have

$$f = c_1\kappa^{-3/2}. \tag{3.18}$$

Setting the above equation in (3.16), we get

$$\frac{\tau}{\kappa} = \frac{e^{\int -\frac{4\varepsilon_1 \eta(N)\eta(B)}{\tau}}}{c_1^2}. \tag{3.19}$$

And finally putting equations (3.18) and (3.19) in (3.15), we obtain the result. □

Consequently, we can express the following corollary.

Corollary 1. *An arc-length parametrized $f-$biharmonic curve $\gamma : I \to (D, g_c)$ with constant geodesic curvature is biharmonic.*

References

[1] L. Du and J. Zhang, Classification of f-biharmonic curves in Lorentz–Minkowski space, *J. Math.* 2020 (2020), Article ID 7529284, 8 pp. https://doi.org/10.1155/2020/7529284

[2] Ş. Güvenç and C. Özgür, On the characterizations of f-biharmonic Legendre curves in Sasakian space forms, *Filomat* 31 (2017), 639-648. https://doi.org/10.2298/FIL1703639G

[3] F. Karaca and C. Özgür, On f-biharmonic curves, *Int. Electron. J. Geom.* 11(2) (2018), 18-27. https://doi.org/10.36890/iejg.545115

[4] W-J. Lu, On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds,*Sci. China Math.* 58 (2015), 1483-1498. https://doi.org/10.1007/s11425-015-4997-1

[5] J.E. Lee, Slant curves and biharmonic Frenet curves in 3-dimensional para-Sasakian manifolds *Balkan J. Geo. App.* 26(1) (2021), 21-33.

[6] Y-L. Ou, On f-biharmonic maps and f-biharmonic submanifolds, *Pacific J. Math.* 271 (2014), 461-477. https://doi.org/10.2140/pjm.2014.271.461

[7] S. Ouakkas, R. Nasri and M. Djaa, On the f-harmonic and f-biharmonic maps, *JP J. Geo. Top.* 10 (2010), 11-27.

[8] J. Welyczko, Slant curves in 3-dimensional normal almost paracontact metric manifolds, *Mediterr. J. Math.* 11 (2014), 965-978.
https://doi.org/10.1007/s00009-013-0361-2