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On Computation of Highly Oscillatory Integrals with Bessel Kernel

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Abstract

In this paper, we introduce a new numerical scheme for approximation of highly oscillatory integrals having Bessel kernel. We transform the given integral to a special form having improper nonoscillatory Laguerre type and proper oscillatory integrals with Fourier kernels. Integrals with Laguerre weights over $[0, \infty)$ will be solved by Gauss-Laguerre quadrature and oscillatory integrals with Fourier kernel can be evaluated by meshless-Levin method. Some numerical examples are also discussed to check the efficiency of proposed method.

1. Introduction

In this paper, we are concern with highly oscillatory integrals of the form [1]

$$I[g] = \int_a^b g(x) J_\mu(\kappa x)\, dx, \tag{1}$$

where $g(x)$ is a smooth function, $J_\mu(\kappa x)$ is Bessel function of first kind of order μ and κ is parameter of frequency. For large value of κ the integral become highly oscillatory and cannot be approximate by usual quadrature rules. To handle this type of problems, we formulate special numerical schemes.

Highly oscillatory integrals are applicable in many areas of science and engineering

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technology like optics, astronomy, seismology image processing, electromagnetics, plasma transport and computerized tomography [3, 4, 7, 8, 11].

In [5], the author presented an efficient method for numerical approximation of oscillatory integrals with trigonometric and Bessel kernels and then extended this procedure in [6]. In [9], different approaches are presented for different types of oscillatory integrals, the method designed for Bessel type oscillatory integrals is based on Lagrange's identity. Highly oscillatory integrals are evaluated by convolution quadrature in [13]. To check effectiveness of proposed method some test problems are included. In [10], the oscillatory integrals over infinite positive domain $[0, \infty)$ are evaluated by "integration then summation with extrapolation" (ISE) method. [14], proposed Filon-type method and Clenshaw-Curtis-Filon-type method based on Fast Fourier transform and fast computation of modified moments for evaluation of highly oscillatory integrals containing Bessel functions. The proposed methods are high accurate for large frequencies, which is clear from numerical examples. In [15] the authors proposed meshless procedure for approximation of oscillatory integrals with Bessel kernel, the case of singularity is handled with multi-resolution quadrature based on Haar wavelet quadrature and hybrid function. [2], has transformed Bessel oscillatory integrals to special type integrals with Fourier kernel and integrals with Laguerre weights and then used Levin type method with Gauss-Laguerre quadrature for approximation.

In current work we extend [2], method with some modification. For transformation purpose the same approach can be used while for approximation we use meshless-Levin method based on Gauss-Laguerre quadrature.

Symbols Chart

Symbols	**Description**
κ	Frequency parameter
μ	Order of Bessel function
$Q^{GL}[g]$	Gauss-Laguerre quadrature
$Q_{mL}^{GL}[g]$	Meshless-Levin method based on Gauss-Laguerre quadrature
M	Nodal points of meshless-Levin method
N	Nodal points of Gauss-Laguerre quadrature

2. Transformation of Integrals

Bessel function of first kind can be determine as following [12]

$$J_\mu(x) = \frac{(x/2)^\mu}{\sqrt{\pi}\Gamma(\mu + 1/2)} \int_{-1}^{1} (1 - y^2)^{\mu - 1/2} e^{ixy} dy. \tag{2}$$

Substituting (2) into (1), we get

$$I[g] = \int_a^b g(x) \frac{(x/2)^\mu}{\sqrt{\pi}\Gamma(\mu + 1/2)} \int_{-1}^{1} (1 - y^2)^{\mu - 1/2} e^{ixy} dy\, dx. \tag{3}$$

The analytic integral $\int_{-1}^{1} (1 - y^2)^{\mu - 1/2} e^{ixy} dy,$ can be transformed to the following form with variable u [2],

$$\int_{-1}^{1} (1 - y^2)^{\mu - 1/2} e^{i\kappa xy} dy = \frac{ie^{-i\kappa x}}{(\kappa x)^{2\mu}} \int_0^\infty (u^2 + 2i\kappa xu)^{\mu - 1/2} e^{-u} du$$

$$- \frac{ie^{-i\kappa x}}{(\kappa x)^{2\mu}} \int_0^\infty (u^2 - 2i\kappa xu)^{\mu - 1/2} e^{-u} du \tag{4}$$

$$= \frac{ie^{-i\kappa x}}{(\kappa x)^{2\mu}} \Psi_1(x) - \frac{ie^{i\kappa x}}{(\kappa x)^{2\mu}} \Psi_2(x), \tag{5}$$

where,

$$\Psi_1(x) = \int_0^\infty (u^2 + 2i\kappa xu)^{\mu - 1/2} e^{-u} du \tag{5.1}$$

and,

$$\Psi_2(x) = \int_0^\infty (u^2 - 2i\kappa xu)^{\mu - 1/2} e^{-u} du. \tag{5.2}$$

$\Psi_1(x)$ and $\Psi_2(x)$ are nonoscillatory integrals having Laguerre weights and can be evaluated by Gauss-Laguerre quadrature.

Using (5) in (3) we get,

$$I[g] = \frac{i}{2^\mu \kappa^\mu \sqrt{\pi}\Gamma(\mu + 1/2)} \left[\int_a^b \frac{g(x)}{x^\mu} e^{-i\kappa x} \Psi_1(x) dx - \int_a^b \frac{g(x)}{x^\mu} e^{i\kappa x} \Psi_2(x) dx \right] \tag{6}$$

(6) is a special type integral containing nonoscillatory improper Laguerre type integrals and proper oscillatory integrals with Fourier kernels. (6) can be written in more simplified as,

$$I[g] = \frac{i}{2^{\mu}\kappa^{\mu}\sqrt{\pi}\Gamma(\mu + 1/2)}[I_1(g, \kappa) - I_2(g, \kappa)], \tag{7}$$

where,

$$I_1(g, \kappa) = \int_a^b \frac{g(x)}{x^{\mu}} e^{-i\kappa x}\Psi_1(x)dx \tag{7.1}$$

and,

$$I_2(g, \kappa) = \int_a^b \frac{g(x)}{x^{\mu}} e^{i\kappa x}\Psi_2(x)dx. \tag{7.2}$$

From (7.1) and (7.2) we see that the proposed method fails at $x = 0$, so we choose the domain $0 < a < b$.

3. Meshless-Levin Method based on Gauss-Laguerre Quadrature

Integrals (5.1) and (5.2) contains Gauss-Laguerre weights, can be evaluated by Gauss-Laguerre quadrature as

$$\int_0^{\infty} g(x)e^x dx = \sum_{j=1}^{N} \Omega(x_j)g(x_j)dx + \frac{(N!)^2}{(2N)!} g^{(2N)}(\zeta),\ 0 < \zeta < \infty$$

$$= Q^{GL}[g] + E_{rr}.$$

The weights functions $\Omega(x_j)$ is given by [16]

$$\Omega(x_j) = \frac{x_j}{(x+1)^2[L_{N+1}(x_j)^2]}, \quad j = 1, 2, 3, \ldots, N.$$

The quadrature points of Gauss-Laguerre quadrature are the zeros of the following polynomial

$$L_N(x) = \sum_{j=0}^{N} \frac{(-1)^j x^j N!}{(N-j)!(j!)^2}.$$

After computing $\Psi_1(x)$ and $\Psi_2(x)$ by Gauss-Laguerre quadrature $Q^{GL}[g]$, (6) can be written as

$$I[g] = \frac{i}{2^{\mu}\kappa^{\mu}\sqrt{\pi}\Gamma(\mu + 1/2)}\left[\int_a^b G_1(x)e^{-i\kappa x}dx - \int_a^b G_2(x)e^{i\kappa x}dx\right], \tag{8}$$

where,

$$G_1(x) = \frac{g(x)}{x^{\mu}}\Psi_1(x) \tag{8.1}$$

and,

$$G_2(x) = \frac{g(x)}{x^{\mu}}\Psi_2(x). \tag{8.2}$$

Finally we got the highly oscillatory integral with Fourier kernels, which can be evaluated by Leven type method based on multiquadric radial basis functions (MQ RBF) as following.

Let the approximate solution $\bar{y}(x) = \sum_{j=0}^{M} \alpha_j \phi(x)$ satisfies the following ODE [17]

$$\mathbf{D}[\theta(x)] = G(x), \quad x \in [a, b] \text{ and } x \neq 0. \tag{9}$$

The differential operator **D** can be defined as

$$\mathbf{D}[\theta(x)] = \mathbf{D}'(x) + i\kappa\mathbf{D}(x).$$

For weights α_j, applying interpolation condition on (9) as under

$$\mathbf{D}[\bar{y}(x)] = G(x_j), \quad j = 0, 1, 2, ..., M \tag{10}$$

(10) gives a system of linear equations, which can be written in matrix form as following

$$\mathbf{A}\alpha = \mathbf{G} \tag{11}$$

where **A** is square matrix while α and **G** are column vectors.

$\phi(x)$ can be defined as

$$\phi(x) = \sqrt{r_j(x)^2 + \varepsilon^2}, \qquad r_j(x) = \| x - x_j^c \|_2,$$

where, ε is shape parameter of multiquadric radial basis function (MQ RBF).

Solution of linear equations system (11) for α gives the approximate solution $\bar{y}(x)$, and hence the value of integral (1) can be determined as

$$Q_{mL}^{GL}[g] = \bar{y}(b)e^{i\kappa b} - \bar{y}(a)e^{i\kappa a}.$$

4. Error Analysis

Theorem. Let the integral (6) has no stationary point in $[a, b]$. Then the error bound of proposed method is given by

$$| E_{rr} | = | I[g] - Q_{mL}^{GL}[g] | \leq \rho_1 \frac{(b - x_1)^{M-1}}{\kappa^{3+\mu}},$$

where, ρ_1 is real constant free of κ and x.

Proof. The integrals (5.1) and (5.2) can be computed by $Q^{GL}[g]$ as following,

$$\Psi(x) = \int_0^\infty g(x, u)e^{-u}du$$

$$= \sum_{j=1}^{N} \Omega(x_j)g(x, u_j) + \frac{(N!)^2}{(2N)!}g^{(2N)}(x, \zeta), \; 0 < \zeta < \infty, \tag{12}$$

where, $g(x, u) = (u^2 + 2i\kappa xu)^{\mu - \frac{1}{2}}$ or $g(x, u) = (u^2 - 2i\kappa xu)^{\mu - \frac{1}{2}}$.

So (6) can be written as following,

$$I[g] = \frac{i}{2^\mu \kappa^\mu \sqrt{\pi}\Gamma(\mu + 1/2)}\left[\left\{\int_a^b \frac{g(x)}{x^\mu} Q^{GL}[g]e^{-i\kappa x}dx - \int_a^b \frac{g(x)}{x^\mu} Q^{GL}[g]e^{i\kappa x}dx\right\}\right.$$

$$\left. + \frac{(N!)^2}{(2N)!}\left\{\int_a^b \frac{g(x)}{x^\mu} g^{(2N)}(x, \zeta)e^{-i\kappa x}dx - \int_a^b \frac{g(x)}{x^\mu} g^{(2N)}(x, \zeta)e^{i\kappa x}dx\right\}\right]$$

$$\leq \frac{i}{2^\mu \kappa^\mu \sqrt{\pi}\Gamma(\mu + 1/2)} \frac{(N!)^2}{(2N)!}\left[\int_a^b \frac{g(x)}{x^\mu} g^{(2N)}(x, \zeta)e^{-i\kappa x}dx\right.$$

$$\left. - \int_a^b \frac{g(x)}{x^\mu} g^{(2N)}(x, \zeta)e^{i\kappa x}dx\right]$$

$$= \rho\left[\int_a^b F(x)e^{-i\kappa x}dx - \int_a^b F(x)e^{i\kappa x}dx\right], \tag{13}$$

where, $F(x) = \dfrac{g(x)}{x^{\mu}} g^{(2N)}(x, \zeta)$ and $\rho = \dfrac{i}{2^{\mu} k^{\mu} \sqrt{\pi}\Gamma(\mu + 1/2)} \dfrac{(N!)^2}{(2N)!}$.

Since the oscillatory function of (13) is linear, therefore the error bound of proposed method which is used for approximation of (13) with M collocation points $x_1 < x_2 < \cdots < x_M = b$ is given by [18],

$$| E_{rr} | = | I[g] - Q_{mL}^{GL}[g]| \leq \left| \rho \frac{(b - x_1)^{M-1}}{\kappa^3} \right|,$$

$$= \rho_1 \frac{(b - x_1)^{M-1}}{\kappa^{3+\mu}},$$

where, $\rho_1 = \left| \dfrac{i}{2^{\mu}\sqrt{\pi}\Gamma(\mu + 1/2)} \dfrac{(N!)^2}{(2N)!} \right|$, independent of κ and x.

This completes the proof.

5. Numerical Examples

In this section some numerical examples are discussed. For exact solution Maple 16 has been used while numerical computation is done by Matlab 2015a by hp core-i5 laptop with 2.5 GHz processor and 4 GB of RAM.

Example 1.

$$I_1[g] = \int_1^2 \frac{1}{x^2 + 1} J_0(\kappa x)dx.$$

$I_1[g]$ is a highly oscillatory integral which is clear from Figure 1. The oscillation increases as value of κ increases. The proposed method is implemented for evaluation of integral and results are compared with exact solution. Comparison of exact and approximate solution is shown in Figure 2 from which it is clear that the approximate solution is very close to exact solution. The results for higher frequencies are given in term of absolute error in Table 1, which shows that the proposed method gives high accuracy for higher frequencies. Figure 3 shows the asymptotic order of convergence of proposed method for $I_1[g]$.

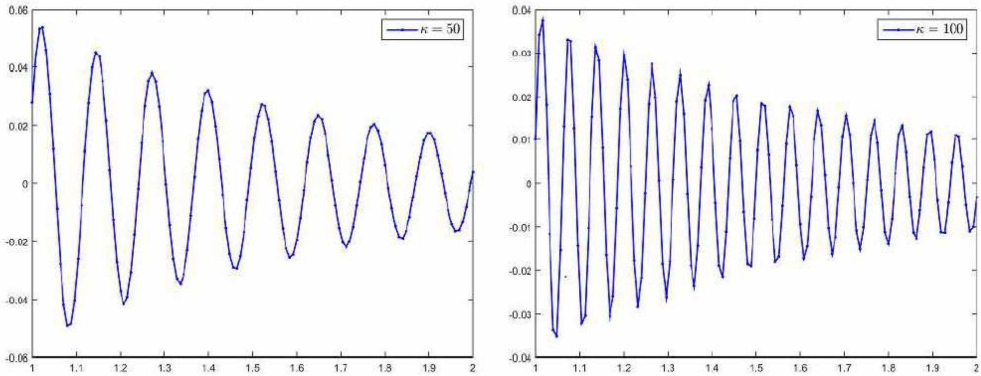

Figure 1. Oscillatory behavior of $I_1[g]$ for, $\kappa = 50$ (left) and $\kappa = 100$ (right).

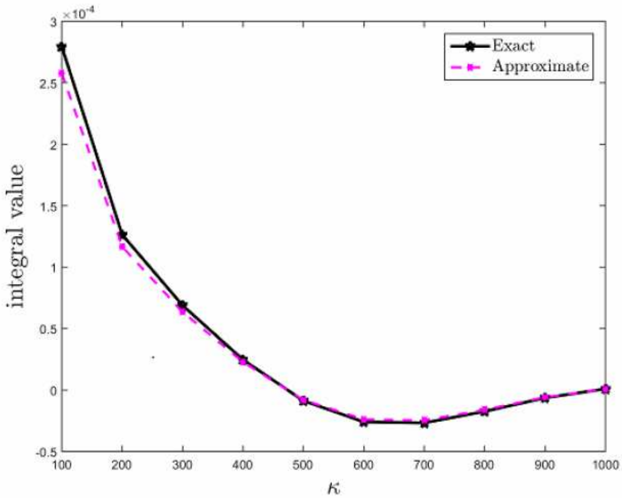

Figure 2. Comparison of exact and approximate solutions of $I_1[g]$ for different values of κ.

Table 1. Absolute error of $I_1[g]$ for higher frequencies.

κ	10^0	10^1	10^2	10^3	10^4	10^5
Absolute error	1.33×10^{-3}	2.65×10^{-4}	2.16×10^{-5}	7.21×10^{-8}	1.56×10^{-8}	9.24×10^{-10}

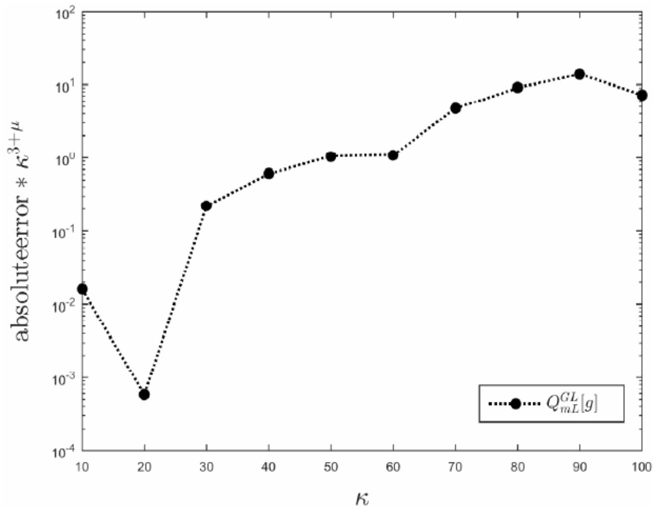

Figure 3. Absolute error scaled by $\kappa^{3+\mu}$, for $I_1[g]$.

Example 2.

$$I_2[g] = \int_1^2 \frac{\cos(x)}{1+x^3} J_2(\kappa x) dx.$$

The oscillatory integral $I_2[g]$ is approximated by the proposed method. Comparison of exact and approximate solutions is shown in Figure 5 for different values of κ. Results for higher frequencies are calculated in term of absolute error at different values of N and M. Table 2 shows that at fixed $N = 40$ the absolute error for higher frequencies decreases as M increases. Similarly Table 3 shows that at fixed $M = 15$ the absolute error for higher frequencies decreases as N increases. From Figure 5, we see that plot of exact and approximate solution overlap each other, which proves the efficiency of method. Figure 4 shows that the oscillations increase with increasing κ. The asymptotic order of convergence is shown in Figure 6 for this example.

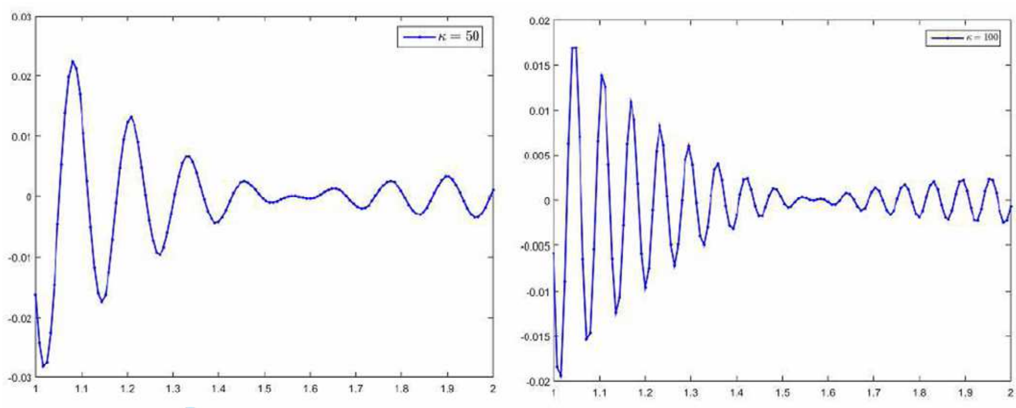

Figure 4. Oscillatory behavior of $I_2[g]$ for, $\kappa = 50$ (left) and $\kappa = 100$ (right).

Table 2. Absolute error of $I_2[g]$, for higher frequencies at different values of M and fixed $N = 40$.

κ	$M = 5$	$M = 10$	$M = 15$
10^0	4.37×10^{-3}	3.55×10^{-4}	2.71×10^{-14}
10^1	5.15×10^{-5}	2.87×10^{-6}	4.14×10^{-7}
10^2	2.00×10^{-7}	1.28×10^{-8}	5.10×10^{-9}
10^3	3.14×10^{-9}	4.32×10^{-10}	5.52×10^{-11}
10^4	1.06×10^{-11}	2.27×10^{-12}	8.18×10^{-13}
10^5	5.15×10^{-14}	3.14×10^{-14}	2.79×10^{-14}

Table 3. Absolute error of $I_2[g]$, for higher frequencies at different values of N and fixed $M = 15$.

κ	$N = 20$	$N = 30$	$N = 40$
10^0	2.82×10^{-5}	2.73×10^{-5}	2.71×10^{-14}
10^1	2.97×10^{-7}	3.88×10^{-7}	4.14×10^{-7}
10^2	1.89×10^{-9}	3.54×10^{-9}	5.10×10^{-9}
10^3	4.38×10^{-12}	4.18×10^{-11}	5.52×10^{-11}
10^4	3.62×10^{-12}	1.44×10^{-12}	8.18×10^{-13}
10^5	1.57×10^{-13}	5.69×10^{-14}	2.79×10^{-14}

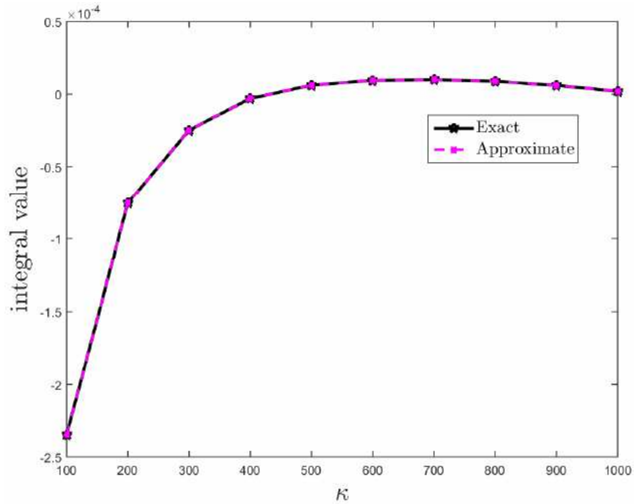

Figure 5. Comparison of exact and approximate solutions of $I_2[g]$, for different values of κ.

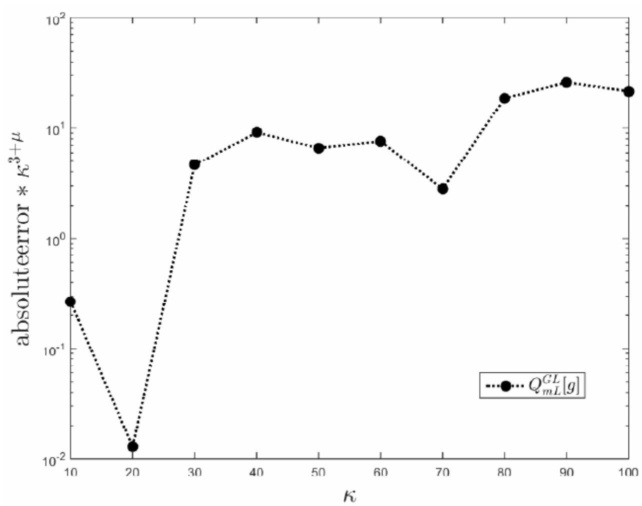

Figure 6. Absolute error scaled by $\kappa^{3+\mu}$, for $I_2[g]$.

6. Conclusion

Numerical method $Q_{mL}^{GL}[g]$ proposed for integrals with Bessel oscillatory kernels. With the help of numerical examples it is proved that the main advantage of the proposed method is high accuracy for higher frequencies. The accuracy also improves with increase in values of M as well as N. The asymptotic order of convergence of proposed method is $O(\kappa^{-(3+\mu)})$.

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