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Trifunction Bihemivariational Inequalities

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Abstract

In this paper, we consider a new class of hemivariational inequalities, which is called the trifunction bihemivariational inequality. We suggest and analyze some iterative methods for solving the trifunction bihemivariational inequality using the auxiliary principle technique. The convergence analysis of these iterative methods is also considered under some mild conditions. Several special cases are also considered. Results proved in this paper can be viewed as a refinement and improvement of the known results.

1 Introduction

Variational inequalities theory introduced in 1964 by Stampacchia [31] can be viewed as a novel and significant generalization of the variational principles. The origin of the variational principles can be traced back to Euler, Newton, Lagrange and Bernoulli's brothers. These variational principles have emerged as a powerful tool to investigate and study a wide class of unrelated problems arising in industrial, regional, physical, pure and applied sciences in a unified and general framework. Variational inequalities have been extended and generalized in several direction using novel and new techniques. Panagiotopoulos [28] introduced the hemivariational inequalities by using the concept of the generalized directional derivatives of nonconvex and nondifferentiable functions. This class has important

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applications in structural analysis and nonconvex optimization. It has been shown [7] that, if a nonsmooth and nonconvex superpotential of a structure is quasidifferentiable, then these problems can be studied in the general framework of hemivariational inequalities. The solution of the hemivariational inequalities gives the position of the state equilibrium of the structure. We would like to point out that the hemivariational inequalities include the problem of finding the difference of two monotone operators, which is itself an interesting problem, see [8, 28].

Noor and Oettli [16] introduced triequilibrium problems and have shown variational inequalities, fixed-point problems, Nash equilibrium problems and saddle-point problems can be studied in the framework of triequilibrium problems. Thus it is clear that hemivariational inequalities and equilibrium problems are different generalizations of variational inequalities. Noor and Noor [17] investigated the trifunction hemivariational inequalities, which can be viewed a significant extension of variational inequalities and hemivariational inequalities. We would like to emphasize that hemivariational inequality theory provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization.

Convexity theory is a branch of mathematical sciences with a wide range of applications in industry, physics, social, regional and engineering sciences. The general theory of the convexity started soon after the introduction of differential and integral calculus by Newton and Leibnitz, although some individual optimization problems had been investigated before that. It is worth mentioning that variational inequalities represent the optimality conditions for the differentiable convex functions on the convex sets. The convex sets and convex functions have been extended and generalized in several directions using innovative ideas to consider completed problems. See an excellent book by Cristescu and Lupsa [3]. Inspired by the research work going on in this field, Noor and Noor [21, 22, 23, 24] introduced and and considered a new class of nonconvex sets and nonconvex functions with respect to an arbitrary bifunction. This class of nonconvex set is called the biconvex set and the noncovex function is called biconvex function. functions is called the biconvex functions. Noor et al [19, 21, 22, 23, 24, 26, 27] have studied some basic properties of the biconvex functions. It have been shown that the biconvex functions have characterizations as the convex functions enjoy. In particular, it have been shown that the optimization conditions of the differentiable biconvex functions are characterized by a class of variational inequalities, called the bivariational inequalities, see [19, 21, 22, 23, 24, 26, 27] and references therein.

Variational inequalities and hemivariational inequalities have witnessed an explosive growth in theoretical advances, algorithmic developments and applications across almost all disciplines of engineering, pure and applied sciences. There are several methods for solving variational inequalities and bivariational inequalities. Due to the nature of the hemivariational inequalities, projection and resolvent methods can not be applied for solving hemivariational inequalities. In recent years, the auxiliary principle technique is being used to suggest and analyze some iterative methods for solving variational inequalities and equilibrium problems. Glowinski, Lions and Tremolieres [5] used this technique to study the existence problem for mixed variational inequalities, whereas Noor [8, 11, 12, 13, 14] and Zhu et al.[32] have used this approach to suggest and analyze some iterative methods for solving various classes of variational inequalities and equilibrium problems. In this paper, we again use the auxiliary principle technique to suggest several new iterative schemes for trifunction bihemivariational inequalities. We also prove that the convergence of these methods require either pseudomonotonicity or partially relaxed strongly monotonicity. These are weaker conditions than monotonicity. As a special case, we obtain new iterative schemes for solving bihemivariational inequalities, variational inequalities and optimization problem. The comparison of these methods with other methods is a subject of future research.

2 Preliminaries and Basic Results

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$ respectively. Let K be a nonempty set in H.

We now recall some concepts of biconvex sets and biconvex functions, which are mainly due to Noor et al. [21, 22, 23, 24].

Definition 2.1. The set K_β in H is said to be *biconvex set* with respect to an arbitrary bifunction $\beta(\cdot - \cdot)$, if

$$u + \lambda\beta(v-u) \in K_\beta, \qquad \forall u, v \in K_\beta, \lambda \in [0,1].$$

The biconvex set K_β is also called β-connected set. If $\beta(v-u) = v-u$, then the biconvex set K_β is a convex set, but the converse is not true. For example, the set $K_\beta = R - (-\frac{1}{2}, \frac{1}{2})$ is an biconvex set with respect to β, where

$$\beta(v-u) = \begin{cases} v-u, & \text{for} \quad v > 0, u > 0 \quad \text{or} \quad v < 0, u < 0 \\ u-v, & \text{for} \quad v < 0, u > 0 \quad \text{or} \quad v < 0, u < 0. \end{cases}$$

It is clear that K_β is not a convex set.

Remark 2.1. We would like to emphasize that, if $u + \beta(v-u) = v, \quad \forall u, v \in K_\beta$, then $\beta(v-u) = v-u$. Consequently, the β-biconvex set reduces to the convex set K. Thus, $K_\beta \subset K$. This implies that every convex set is a biconvex set, but the converse is not true.

Definition 2.2. The function F on the biconvex set K_β is said to be *strongly biconvex*, if

$$\begin{aligned} F(u + \lambda\beta(v-u)) \quad &\leq \quad (1-\lambda)F(u) + \lambda F(v) \\ &\qquad -\nu\lambda(1-\lambda)\|\beta(v-u)\|^2, \quad \forall u, v \in K_\beta, \lambda \in [0,1]. \end{aligned}$$

Note that every convex function is a biconvex, but the converse is not true.

If $\lambda = \frac{1}{2}$, then the function F satisfies

$$F(\frac{2u + \beta(v-u)}{2}) \leq \frac{1}{2}\{F(u) + F(v)\} - \nu\frac{1}{4}\|\beta(v-u)\|^2, \quad \forall u, v \in K_\beta,$$

which is called Jensen biconvex function.

If $\nu = 0$, then Definition(2.2) reduces to

Definition 2.3. The function F on the biconvex set K_β is said to be *biconvex*, if

$$F(u+\lambda\beta(v-u)) \leq (1-\lambda)F(u) + \lambda F(v) \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

We now consider the biconvex function on the interval $I_\beta = [a, a+\beta(b-a)]$.

Definition 2.4. Let $I_\beta = [a, a+\beta(b-a)]$. Then F is a *biconvex function*, if and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a+\beta(b-a) \\ F(a) & F(x) & F(b) \end{vmatrix} \geq 0; \quad a \leq x \leq a+\beta(b-a).$$

One can easily show that the following are equivalent:

1. F is a biconvex function.
2. $F(x \leq F(a) + \frac{F(b)-F(a)}{\beta(b-a)}(x-a)$.
3. $\frac{F(x)-F(a)}{x-a} \leq \frac{F(b)-F(a)}{\beta(b-a)}$.
4. $\frac{F(a)}{(\beta(b-a)))(a-x)} + \frac{F(x)}{(x-a)-\beta(b-a))(a-x)} + \frac{F(b)}{\beta(b-a)(x-b)} \leq 0$,

where $x = a+\lambda\beta(b-a) \in [a, a+\beta(b-a)]$.

To derive the main results, we need the following assumption regarding the bifunction $\beta(\cdot - \cdot)$.

Condition M. The bifunction $\beta(,-,)$ is said to satisfy the following assumptions:

$$\begin{aligned} (i). \quad \beta(\gamma\beta(v-u)) &= \gamma\beta(v-u), \quad \forall u, v \in K_\beta, \quad \gamma \in R^n. \\ (ii). \quad \beta(v-u-\gamma\beta(v-u)) &= (1-\gamma)\beta(v-u), \quad \forall u, v \in K_\beta. \end{aligned}$$

Remark 2.2. Let $\beta(\cdot - \cdot) : K_\beta \times K_\beta \to H$ satisfy the assumption

$$\beta(v-u) = \beta(v-z) + \beta(z-u), \quad \forall u, v, z \in K_\beta.$$

One can easily show that $\beta(v-u)=0 \quad \Leftrightarrow \quad u=v, \quad \forall u,v\in K_\beta$. Consequently $\beta(v-u)=0$, for $v=u\in K_\beta$. Also $\beta(v-u)+\beta(u-v)=0, \quad \forall u,v,z\in K_\beta$. This implies that the bifunction $\beta(.-.)$ is skew symmetric.

Let $f: H \longrightarrow R$ be a locally Lipschitz continuous function. Let Ω be an open bounded subset of R^n. First of all, we recall the following concepts and results from nonsmooth analysis [2].

Definition 2.5. Let f be locally Lipschitz continuous at a given point $x\in H$ and v be any other vector in H. The *Clarke's generalized bidirectional derivative* of f at x in the direction $\beta(v-u)$, denoted by $f^0(x,\beta(v-u))$, is defined as

$$f^0(x,\beta(v-u)) = \lim_{t\to 0^+}\sup_{h\to 0}\frac{f(x+h+t\beta(v-u))-f(x+h)}{t}.$$

If $\beta(v-u)=v$, then Definition (2.5) reduces to the following concepts which are mainly due to Clarke [2].

Definition 2.6. *[2]* Let f be locally Lipschitz continuous at a given point $x\in H$ and v be any other vector in H. The *Clarke's generalized bidirectional derivative* of f at x in the direction v, denoted by $f^0(x,v)$, is defined as

$$f^0(x,v) = \lim_{t\to 0^+}\sup_{h\to 0}\frac{f(x+h+tv)-f(x+h)}{t}.$$

The generalized gradient of f at x, denoted $\partial f(x)$, is defined to be subdifferential of the function $f^0(x;v)$ at 0. That is

$$\partial f(x) = \{w\in H : \langle w,v\rangle \le f^0(x;v), \quad \forall v\in H.\}.$$

If f is convex on K and locally Lipschitz continuous at $x\in K$, then $\partial f(x)$ coincides with the subdifferential $f'(x)$ of f at x in the sense of convex analysis , and $f^0(x;v)$ coincides with the directional derivative $f'(x;v)$ for each $v\in H$, that is, $f^0(x;v)=\langle f'(x),v\rangle, \quad \forall v\in H$.

For a given nonlinear trifunction $F(.,.,.): K_\beta\times K_\beta\times K_\beta \longrightarrow H$ and a nonlinear continuous operator $T: K_\beta \longrightarrow H$, consider the problem of finding $u\in K_\beta$ such that

$$F(u, Tu, \beta(v-u)) + \int_{\Omega} f^0(u; \beta(v-u))d\Omega \geq 0, \quad \forall v \in K_{\beta}, \tag{2.1}$$

which is called the trifunction bihemivariational inequality.

Here $f^0(u;\beta(v-u)) := f^0(x,u;\beta(v-u)) := f^0(x,u(x);\beta(v(x)-u(x)))$ denotes the generalized bidirectional derivative of the function $f(x,.)$ at $u(x)$ in the direction $v(x)-u(x)$.

We now discuss some special cases of the trifunction bihemivariational inequalities (2.1).

(I). If $F(u,Tu,\beta(v-u)) = W(u,\beta(v-u))$, where $B(.,.)$ is a continuous bifunction, then problem (2.1) is equivalent to finding $u \in K_\beta$ such that

$$W(u, \beta(v-u)) + \int_{\Omega} f^0(u; \beta(v-u))d\Omega \geq 0, \quad \forall v \in K_{\beta}, \tag{2.2}$$

which is called the bifunction bihemivariational inequality and appears to be a new one.

(II). If $F(u,Tu,\beta(v-u)) = \langle Au, \beta(v-u)\rangle$, where A is a nonlinear operator, then problem (2.1) is equivalent to finding $u \in K_\beta$ such that

$$\langle Au, \beta(v-u)\rangle + \int_{\Omega} f^0(u; \beta(v-u))d\Omega \geq 0, \quad \forall v \in K_{\beta}, \tag{2.3}$$

which is known as the bihemivariational inequality.

(III). If $F(u,Tu,\beta(v-u)) = \langle Au, v-u\rangle$, where A is a nonlinear operator, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Au, v-u\rangle + \int_{\Omega} f^0(u; v-u)d\Omega \geq 0, \quad \forall v \in K, \tag{2.4}$$

which is known as the hemivariational inequality introduced and studied by Panagiotopoulos [28, 29] in order to formulate variational principles connected to energy functions which are neither convex nor smooth. It is has been shown that the technique of hemivariational inequalities is very efficient to describe the behaviour of complex structure arising in engineering and industrial sciences.

(IV). If f is a differentiable convex function, then problem (2.1) is equivalent to finding $u \in K_\beta$ such that

$$F(u, Tu, \beta(v-u)) + \langle f'(u), \beta(v-u)\rangle \geq 0, \quad \forall v \in K_\beta, \tag{2.5}$$

which is known as the mildly nonlinear trifunction bihemivariational inequality and appear to be a new one.

(V). If $f = 0$, then problem (2.1) is equivalent to finding $u \in K_\beta$ such that

$$F(u, Tu, \beta(v-u)) \quad \geq 0, \quad \forall v \in K_\beta, \tag{2.6}$$

which is called the trifunction bivariational inequality.

In brief, for suitable and appropriate choice of the trifunction, one can obtain several classes of bihemivariational and bivariational inequalities. This clear shows that the problem (2.1) is more general and flexible and includes the previous ones as special cases.

Definition 2.7. The trifunction $F(.,.,.)$ and the operator T is said to be:
(a) *jointly bimonotone,* if

$$F(u, Tu, \beta(v-u)) + F(v, Tv, \beta(u-v)) \leq 0, \quad \forall u, v \in K_\beta.$$

(b) *jointly pseudo-bimonotone* with respect to $\int_\Omega f^0(u; \beta(v-u))d\Omega$, if

$$\begin{aligned} & F(u, Tu, \beta(v-u)) + \int_\Omega f^0(u; \beta(v-u))d\Omega \geq 0 \\ & \Longrightarrow \\ & -F(v, Tv, \beta(u-v)) - \int_\Omega f^0(u; \beta(v-u))d\Omega \geq 0, \quad \forall u, v \in K_\beta. \end{aligned}$$

(c) *partially relaxed strongly jointly bimonotone,* if there exists a constant $\gamma > 0$ such that

$$F(u, Tu, \beta(v-u)) + F(v, Tv, \beta(z-v)) \leq \gamma \|\beta(u-z)\|^2, \quad \forall u, v, z \in K_\beta.$$

Note that for $z = u$ partially relaxed strongly jointly bimonotonicity reduces to jointly bimonotonicity. This shows that partially relaxed strongly jointly bimonotonicity implies jointly bimonotonicity, but the converse is not true.

Definition 2.8. The function $\int_\Omega f^0(u;\beta(v-u))d\Omega$ is said to be *partially relaxed strongly bimonotone*, if there exists a constant $\alpha > 0$ such that

$$\int_\Omega f^0(u;\beta(v-u))d\Omega + \int_\Omega f^0(z;\beta(u-v))d\Omega \leq \alpha\|\beta(z-v)\|^2, \quad \forall u,v,z \in H.$$

Note that for $z = v$, partially relaxed strongly bimonotonicity reduces to relaxed strongly bimonotonicity.

3 Main Results

In this section, we suggest and analyze some iterative methods for solving trifunction bihemivariational inequality (2.1) using the auxiliary principle technique of Glowinski, Lions and Tremolieres [5] involving Bregman distance function as developed by Noor [11, 12, 13, 14, 15], Noor et al. [17, 18, 19, 20] and Zhu et al. [32].

For the readers convenience, we recall some basic properties of the Bregman convex functions [2]. For strongly convex functions f, we define the Bregman distance function as

$$B(v,u) = f(v) - f(u) - \langle f'(u), v-u\rangle \geq \alpha\|v-u\|^2, \quad \forall u,v \in K. \tag{3.1}$$

It is important to emphasize that various types of function f give different Bregman distance function. We give the following important examples of some practical important types of function f and their corresponding Bregman distance functions.

Examples

1. If $f(v) = \|v\|^2$, then $B(v,u) = \|v-u\|$, which is the squared Euclidean distance (SE).

2. If $f(v) = \sum_{i=1}^n a_i \log v_i$, which is known as Shannon entropy, then its corresponding Bregman distance is given as

$$B(v,u) = \sum_{i=1}^n \left(v_i \log(\frac{v_i}{u_i}) + u_i - v_i \right),$$

This distance is called Kullback-Leibler distance (KL) and has become a very important tool in several areas of applied mathematics such as machine learning.

3. If $f(v) = -\sum_{i=1}^{n} \log v_i$, which is called Burg entropy, then its corresponding Bregman distance is given as

$$B(v,u) = \sum_{i=1}^{n} \left(\log \frac{v_i}{u_i} + \frac{v_i}{u_i} - 1 \right).$$

This is called Itakura-Saito distance (IS), which is very important in the information theory, data analysis and machine learning.

Remark 3.1. It is a challenging problem to explore the applications of Bregman distance function for other types of nonconvex functions such as biconvex, k-convex functions, preinvex functions and harmonic functions.

For a given $u \in K_\beta$ satisfying (2.1), consider the auxiliary problem of finding $w \in K_\beta$ such that

$$\begin{aligned} \rho F(w, Tw, \beta(v-w)) &+ \langle E'(w) - E'(u), \beta(v-w)\rangle \\ &+ \rho \int_\Omega f^0(w; \beta(v-w)) d\Omega \geq 0, \quad \forall v \in K_\beta, \quad (3.2) \end{aligned}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential biconvex function $E(u)$ at $u \in K_\beta$.

We note that, if $w = u$, then clearly w is solution of the problem (2.1). This observation enables us to suggest and analyze the following iterative method for solving (2.1).

Algorithm 3.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho F(u_{n+1}, Tu_{n+1}, \beta(v-u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \\ &+\rho \int_\Omega f^0(u_{n+1}; \beta(v-u_{n+1})) d\Omega \geq 0, \quad \forall v \in K_\beta. \quad (3.3) \end{aligned}$$

Algorithm 3.1 is called the proximal method for solving problem (2.1). In passing, we remark that the proximal point method was suggested by Martinet [6] in the context of convex programming problems as regularization technique. For the recent developments and applications of the proximal point algorithms, see [11, 12, 13, 14, 15, 19, 32] and the references therein.

If $F(u, Tu, \beta(v-u)) = W(u, \beta(v-u))$, then Algorithm 3.1 collapses to the following method for solving the bifunction bihemivariational inequality (2.2).

Algorithm 3.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho W(u_{n+1}, \beta(v-u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), \beta(v-u_{n+1})\rangle$$
$$+\rho \int_\Omega f^0(u_{n+1}; \beta(v-u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta,$$

If $F(u, Tu, \beta(v-u)) = \langle Au, \beta(v-u)\rangle$, then Algorithm 3.1 reduces to:

Algorithm 3.3. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative schemes*

$$\langle \rho A u_{n+1} + E'(u_{n+1}) - E'(u_n), \beta(v-u_{n+1})\rangle$$
$$+\rho \int_\Omega f^0(u_{n+1}; \beta(v-u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta,$$

is called the proximal point method for solving bihemivariational inequalities (2.3) and appears to be a new one.

If $f(x, u) = 0$, then Algorithm 3.1 collapses to:

Algorithm 3.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, Tu_{n+1}, \beta(v-u_{n+1})) + \langle E'(u_{n+1}) - E'(u_n), \beta(v-u_{n+1})\rangle \geq 0, \quad \forall v \in K_\beta.$$

In brief, for suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational-hemivariational inequalities and related problems.

We now study the convergence analysis of Algorithm 3.1, which is the main motivation of our next result.

Theorem 3.1. *Let $F(.,.,.)$ and the operator T be jointly pseudomonotone with respect to $\int_\Omega f^0(u;\beta(v-u))d\Omega$. Let E be differentiable strongly biconvex function with module $\mu > 0$. Then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to a solution $u \in K_\beta$ satisfying (2.1).*

Proof. Let $u \in K_\beta$ be a solution of (2.1). Then

$$F(u, Tu, \beta(v-u)) + \int_\Omega f^0(u;\beta(v-u))d\Omega \geq 0, \quad \forall v \in K_\beta,$$

implies that

$$-F(v, Tv, \beta(u-v)) - \int_\Omega f^0(x, u;\beta(v-u))d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{3.4}$$

since $F(.,.,.)$ is jointly pseudomonotone with respect to $\int_\Omega f^0(u;\beta(v-u))d\Omega$.

Taking $v = u$ in (3.3) and $v = u_{n+1}$ in (3.4), we have

$$\begin{aligned} \rho F(u_{n+1}, Tu_{n+1}, \beta(u-u_{n+1})) &+ \langle E'(u_{n+1}) - E'(u_n), \beta(u-u_{n+1})\rangle \\ &\geq -\rho \int_\Omega f^0(u_{n+1};\beta(u-u_{n+1}))d\Omega. \end{aligned} \tag{3.5}$$

and

$$-F(u_{n+1}, Tu_{n+1}, \beta(u-u_{n+1})) - \int_\Omega f^0(u;\beta(u_{n+1}-u))d\Omega \geq 0. \tag{3.6}$$

We now consider the function Bregman distance function

$$\begin{aligned} B(u,w) &= E(u) - E(w) - \langle E'(w), \beta(u-w)\rangle \\ &\geq \mu\|\beta(u-w)\|^2, \quad \text{(using strongly biconvexity of } E). \end{aligned} \tag{3.7}$$

where $\mu > 0$ is a constant.

Now combining (3.7) and (3.4), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_{n+1}), \beta(u_{n+1} - u_n)\rangle \\ &\quad + \langle E'(u_{n+1}) - E'(u_n), \beta(u - u_{n+1})\rangle \\ &\geq \mu\|\beta(u_{n+1} - u_n)\|^2 + \langle E'(u_{n+1}) - E'(u_n), \beta(u - u_{n+1})\rangle \\ &\geq \mu\|\beta(u_{n+1} - u_n)\|^2 - \rho F(u_{n+1}, Tu_{n+1}, \beta(u - u_{n+1})) \\ &\quad - \rho\int_\Omega f^0(u_n; \beta(u - u_{n+1}))d\Omega \\ &\geq \mu\|\beta(u_{n+1} - u_n)\|^2, \end{aligned}$$

where we have used (3.6).

If $u_{n+1} = u_n$, then clearly u_n is a solution of the trifunction bihemivariational inequality (2.1). Otherwise, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|\beta(u_{n+1} - u_n)\| = 0 \Rightarrow \lim_{n\to\infty} u_{n+1} = u.$$

Now using the technique of Zhu and Marcotte [20], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the trifunction bihemivariational inequality (2.1). □

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving (2.1) using the auxiliary principle technique.

For a given $u \in K_\beta$ satisfying (2.1), find $w \in K_\beta$ such that

$$\begin{aligned} \rho F(u, Tu, \beta(v - w)) &+ \langle E'(w) - E'(u), \beta(v - w)\rangle \\ &+ \rho\int_\Omega f^0(u; \beta(v - w))d\Omega, \quad \forall v \in K_\beta, \end{aligned} \tag{3.8}$$

where $E'(u)$ is the differential of a strongly biconvex function $E(u)$ at $u \in K_\beta$.

Note that problems (3.2) and (3.8) are quite different problems.It is clear that for $w = u$, w is a solution of (2.1). This fact allows us to suggest and analyze another iterative method for solving trifunction bihemivariational inequality (2.1).

Algorithm 3.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho F(w_n, Tw_n, \beta(v-u_{n+1})) &+ \langle E'(u_{n+1}) - E'(w_n), \beta(v-u_{n+1})\rangle \\ &\geq -\rho \int_\Omega (w_n; \beta(v-u_{n+1}))d\Omega, \quad \forall v \in K_\beta, \quad (3.9) \\ \mu F(u_n, Tu_n, \beta(v-w_n)) &+ \langle E'(w_n) - E'(u_n), \beta(v-w_n)\rangle \\ &\geq -\mu \int_\Omega (u_n; \beta(v-w_n))d\Omega, \quad \forall v \in K_\beta. \quad (3.10) \end{aligned}$$

Note that for $F(u, Tu, \beta(v-u)) = W(u, \beta(v-u))$, Algorithm 3.5 reduces to:

Algorithm 3.6. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho W(w_n, \beta(v-u_{n+1})) &+ \langle E'(u_{n+1}) - E'(w_n), \beta(v-u_{n+1})\rangle \\ &\geq -\rho \int_\Omega (w_n; \beta(v-u_{n+1}))d\Omega, \quad \forall v \in K_\beta, \\ \mu W(u_n, \beta(v-w_n)) &+ \langle E'(w_n) - E'(u_n), \beta(v-w_n)\rangle \\ &\geq -\mu \int_\Omega (u_n; \beta(v-w_n))d\Omega, \quad \forall v \in K_\beta, \end{aligned}$$

which is called the predictor-corrector method for solving the bifunction bihemivariational inequality (2.3).

For $F(u, Tu, \beta(v-u)) = \langle Au, \beta(v-u)\rangle$ Algorithm 3.5 collapses to the method for solving the bivariational inequalities (2.2).

Algorithm 3.7. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Aw_n + E'(u_{n+1}) - E'(w_n), \beta(v-u_{n+1})\rangle \geq -\rho \int_\Omega (w_n; \beta(v-u_{n+1}))d\Omega, \forall v \in K_\beta,$$

$$\langle \mu Au_n + E'(w_n) - E'(u_n), \beta(v-w_n)\rangle \geq -\mu \int_\Omega (u_n; \beta(v-w_n))d\Omega, \quad \forall v \in K_\beta,$$

which is called the predictor-corrector method for solving the bihemivariational inequalities (2.2).

If $f(.;.) = 0$, then Algorithm 3.5 reduces to the following iterative method for solving trifunction bivaraiational inequalities (2.5).

Algorithm 3.8. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(w_n, Tw_n, \beta(v - u_{n+1})) + \langle E'(u_{n+1}) - E'(w_n), \beta(v - u_{n+1})\rangle \geq 0, \quad \forall v \in K_\beta,$$
$$\mu F(u_n, Tu_n, \beta(v - w_n)) + \langle E'(w_n) - E'(u_n), \beta(v - w_n)\rangle \geq 0, \quad \forall v \in K_\beta.$$

Similarly for suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving hemivariational and variational inequalities.

We now consider the convergence analysis of Algorithm 3.5 using essentially the technique of Theorem 3.1. For the sake of completeness and to convey an idea of the technique, we sketch the main points.

Theorem 3.2. *Let $F(.,.,.)$ and the operator T be partially relaxed strongly jointly bimonotone with a constant $\gamma > 0$ and let $\int_\Omega f^0(u; v - u)d\Omega$ be partially relaxed strongly bimonotone with a constant $\alpha > 0$. If E is strongly biconvex function with modulus $\beta > 0$ and $0 < \rho < \beta/(\alpha + \gamma)$, $0 < \mu < \beta/(\alpha + \gamma)$, then the approximate solution u_{n+1} obtained from Algorithm 3.5 converges to a solution $u \in K_\beta$ of (2.1).*

Proof. Let $u \in K_\beta$ be solution of (2.1). Then

$$\rho\{F(u, Tu, \beta(v - u)) + \int_\Omega f^0(u; \beta(v - u))d\Omega\} \geq 0, \quad \forall v \in K_\beta \quad (3.11)$$
$$\mu\{F(u, Tu, \beta(v - u)) + \int_\Omega f^0(u; \beta(v - u))d\Omega\} \geq 0, \quad \forall v \in K_\beta, \quad (3.12)$$

where $\rho > 0$ and $\mu > 0$ are constants.

Setting $v = u_{n+1}$ in (3.11) and $v = u$ in (3.9), we have

$$\rho\{F(u, Tu, \beta(u_{n+1} - u)) + \int_\Omega f^0(x, u; \beta(u_{n+1} - u))d\Omega\} \geq 0. \quad (3.13)$$

and

$$\begin{aligned}\rho F(w_n, Tw_n, \beta(u-u_{n+1})) &+ \langle E'(u_{n+1}) - E'(w_n), \beta(u-u_{n+1})\rangle \\ &\geq -\rho \int_\Omega f^0(x, w_n; \beta(u-u_{n+1}))d\Omega. \quad (3.14)\end{aligned}$$

As in Theorem 3.1 and from (3.13) and (3.13), we have

$$\begin{aligned} & B(u, w_n) - B(u, u_{n+1}) \\ = \ & E(u_{n+1}) - E(w_n) - \langle E'(u_{n+1}), \beta(u_{n+1} - w_n)\rangle \\ & + \langle E'(u_{n+1}) - E'(w_n), \beta(u - u_{n+1})\rangle \\ \geq \ & \mu\|\beta(u_{n+1} - w_n)\|^2 + \langle E'(u_{n+1}) - E'(w_n), \beta(u - u_{n+1})\rangle \\ \geq \ & \mu\|\beta(u_{n+1} - w_n)\|^2 - \rho F(w_n, Tw_n, \beta(u - u_{n+1})) \\ & -\rho \int_\Omega f^0(w_n; \beta(u - u_{n+1}))d\Omega \\ \geq \ & \mu\|\beta(u_{n+1} - w_n)\|^2 \\ & -\rho\{F(w_n, Tw_n, \beta(u - u_{n+1})) + F(u, Tu, \beta(u_{n+1} - u))\} \\ & -\rho\{\int_\Omega f^0(u; \beta(u_{n+1} - u))d\Omega + \int_\Omega f^0(w_n; \beta(u - u_{n+1}))d\Omega\} \\ \geq \ & \mu\|\beta(u_{n+1} - w_n)\|^2 - \rho(\alpha + \gamma)\|\beta(u_{n+1} - w_n)\|^2 \\ = \ & \{\mu - \rho(\alpha + \gamma)\}\|\beta(u_{n+1} - w_n)\|^2, \end{aligned}$$

where we have used the fact that $F(.,.,.)$ and $\int_\Omega f^0(.;.)d\Omega$ are partially relaxed strongly bimonotone with constants $\alpha > 0$ and $\gamma > 0$ respectively.

In a similar way, we can obtain

$$B(u, u_n) - B(u, w_n) \geq \{\beta - \mu(\alpha + \gamma)\}\|\beta(w_n - u_n)\|^2.$$

If $u_{n+1} = w_n = u_n$, then clearly u_n is a solution of the trifunction hemivariational inequality (2.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha+\gamma}$ and $0 < \mu < \frac{\beta}{\alpha+\gamma}$, it follows that the sequences $B(u, w_n) - B(u, u_{n+1})$ and $B(u, u_n) - B(u, w_n)$ are nonnegative and we must have

$$\lim_{n\to\infty} \|\beta(u_{n+1} - w_n)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\beta(w_n - u_n)\| = 0.$$

Thus

$$\lim_{n\to\infty}\|\beta(u_{n+1}-u_n)\| = \lim_{n\to\infty}\|\beta(u_{n+1}-w_n)\| + \lim_{n\to\infty}\|\beta(w_n-u_n)\| = 0$$

Now using the technique of Zhu and Marcotte [20], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the trifunction bihemivariational inequality (2.1). $\square$

We now suggest and analyze some new iterative methods for solving the trifunction bihemivariational inequality (2.1) using the auxiliary principle technique of Glowinski, Lions and Tremolieres [10] without the Bregman distance function as developed by Noor [16-24].

For a given $u \in K_\beta$ satisfying (2.1), find $w \in K_\beta$ such that

$$\begin{aligned} \rho F(u, Tu, \beta(v-w)) &+ \langle w-u, v-w\rangle \\ &+ \rho\int_\Omega f^0(u;\beta(v-w))d\Omega \geq 0, \quad \forall v \in K_\beta, \quad (3.15) \end{aligned}$$

where $\rho > 0$ is a constant. Problem (3.15) is known as the auxiliary trifunction bihemivariational inequality. We note that if $w = u$, then clearly w is a solution of the (2.1). This observation enables us to suggest and analyze the following iterative method for solving (2.1).

Algorithm 3.9. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho F(w_n, Tw_n, \beta(v-w_n)) &+ \langle u_{n+1}-w_n, v-u_{n+1}\rangle \\ &+ \rho\int_\Omega f^0(u;\beta(v-u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta \quad (3.16) \\ \eta F(u_n, Tu_n, v-u_n) &+ \langle w_n-u_n, v-w_n\rangle \\ &+ \eta\int_\Omega f^0(u;\beta(v-w_n))d\Omega \geq 0, \quad \forall v \in K_\beta, \quad (3.17) \end{aligned}$$

where $\rho > 0$ and $\eta > 0$ are constants. Algorithm 3.9 is called the predictor-corrector method for solving the trifunction bihemivariational inequality (2.1).

We now study the convergence analysis of Algorithm 3.9.

Theorem 3.3. *Let $\bar{u} \in K_\beta$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.9 . If $F(.,.)$ is partially relaxed strongly monotone with a constant $\alpha > 0$ and the operator $\int_\Omega f^0(u;-,)d\Omega$ is partially relaxed strongly monotone with a constant $\gamma > 0$, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - 2\rho(\alpha + \gamma))\|u_{n+1} - w_n\|^2 \tag{3.18}$$

$$\|w_n - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\beta(\alpha + \gamma))\|w_n - u_n\|^2. \tag{3.19}$$

Proof. Let $\bar{u} \in K_\beta$ be a solution of (2.1). Then

$$\rho F(\bar{u}, T\bar{u}, \beta(v - \bar{u}) \quad + \quad \rho \int_\Omega f^0(\bar{u}; \beta(v - \bar{u}))d\Omega \geq 0, \quad \forall v \in K_\beta \tag{3.20}$$

$$\eta F(\bar{u}, T\bar{u}, \beta(v - \bar{u}) \quad + \quad \eta \int_\Omega f^0(\bar{u}; \beta(v - \bar{u}))d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{3.21}$$

where $\rho > 0$ and $\eta > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.20) and $v = \bar{u}$ in (3.16), we have

$$\rho F(\bar{u}, T\bar{u}, u_{n+1} - \bar{u}) + \rho \int_\Omega f^0(u; \beta(u_{n+1} - u))d\Omega \geq 0 \tag{3.22}$$

$$\begin{aligned} &\rho F(w_n, Tw_n, \bar{u} - w_n) + \langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle \\ &+\rho \int_\Omega f^0(u; \beta(\bar{u} - u_{n+1}))d\Omega \geq 0. \end{aligned} \tag{3.23}$$

Adding (3.22) and (3.23), we have

$$\begin{aligned} & \langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle \\ \geq \quad & -\rho\{F(w_n, Tw_n, \beta(\bar{u} - w_n)) + F(\bar{u}, T\bar{u}, \beta(u_{n+1} - \bar{u}))\} \\ & -\rho\{\int_\Omega f^0(u; \beta(u_{n+1} - \bar{u}))d\Omega + \int_\Omega f^0(u; \beta(\bar{u} - u_{n+1}))d\Omega\} \\ \geq \quad & -(\alpha + \gamma)\rho\|u_{n+1} - w_n\|^2, \end{aligned} \tag{3.24}$$

where we have used the fact that $F(.,.,.)$ is relaxed strongly monotone with constants $\alpha > 0$.

Recall the following result,

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \quad \forall a, b \in H, \tag{3.25}$$

Setting $u = \bar{u} - u_{n+1}$ and $v = u_{n+1} - w_n$ in (3.25), (3.24) can be written as

$$\|u_{n+1} - \bar{u}\|^2 \leq \|\bar{u} - w_n\|^2 - (1 - 2(\alpha + \gamma)\rho)\|u_{n+1} - w_n\|^2,$$

the required (3.18).

Taking $v = \bar{u}$ in (3.21) and $v = w_n$ in (3.17), we obtain

$$\eta F(\bar{u}, T\bar{u}, \beta(w_n - \bar{u})) + \eta \int_\Omega f^0(u; \beta(w_n - \bar{u}))d\Omega \geq 0 \tag{3.26}$$

$$\begin{aligned} &\eta F(u_n, Tu_n, \beta(\bar{u} - u_n)) + \langle w_n - u_n, \bar{u} - w_n\rangle \\ &\quad + \eta \int_\Omega f^0(u_n; \beta(\bar{u} - w_n))d\Omega \geq 0. \end{aligned} \tag{3.27}$$

Adding (3.26), (3.27) and rearranging the terms, we have

$$\langle w_n - u_n, \bar{u} - w_n\rangle \geq -\beta(\alpha + \gamma)\|u_n - w_n\|^2, \tag{3.28}$$

since $F(.,.,.)$ and $\int_\Omega f^0(u; -))d\Omega$ are partially relaxed strongly monotone with constants $\alpha > 0$ and $\gamma > 0$ respectively.

Now taking $v = w_n - u_n$ and $u = \bar{u} - w_n$ in (3.25), (3.28) can be written as

$$\|w_n - \bar{u}\|^2 \leq \|\bar{u} - u_n\|^2 - (1 - 2(\alpha + \gamma)\beta)\|w_n - u_n\|^2,$$

the required (3.19). □

Theorem 3.4. *Let H be a finite dimensional space and let $0 < \rho < 1/2(\alpha + \gamma)$, $0 < \beta < 1/2(\alpha + \gamma)$. If $\bar{u} \in K_\beta$ is a solution of (1) and u_{n+1} is an approximate solution obtained from Algorithm 3.10, then*

$$\lim_{n \longrightarrow \infty}(u_n) = \bar{u}.$$

Proof. Let $\bar{u} \in K_\beta$ be a solution of (2.1). Since $0 < \rho < 1/2(\alpha + \gamma)$ and $0 < \beta < 1/2(\alpha + \gamma)$, it follows from (3.18)and (3.19) that the sequences $\{\|w_n - \bar{u}\|\}$ and $\{\|\bar{u} - u_n\|\}$ are nonincreasing and consequently $\{u_n\}$ and $\{w_n\}$ are bounded.

Furthermore, we have

$$\sum_{n=0}^{\infty}(1-2(\alpha+\gamma)\rho)\|u_{n+1}-w_n\|^2 \leq \|w_0-\bar{u}\|^2$$
$$\sum_{n=0}^{\infty}(1-2(\alpha+\gamma)\beta)\|w_n-u_n\|^2 \leq \|u_0-\bar{u}|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-w_n\|=0 \quad \text{and} \quad \lim_{n\longrightarrow\infty}\|w_n-u_n\|=0.$$

Thus

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\| \leq \lim_{n\longrightarrow\infty}\|u_{n+1}-w_n\|+\lim_{n\longrightarrow\infty}\|w_n-u_n\|=0. \tag{3.29}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u}\in H$. Replacing w_n by u_{n_j} in (3.15), (3.16) and taking the limit $n_j \longrightarrow \infty$ and using (3.29), we have

$$F(\hat{u},T\hat{u},v-\hat{u})+\int_{\Omega} f^0(\hat{u};\beta(v-\hat{u}))d\Omega \geq 0, \quad \forall v\in K,$$

which implies that $\hat{u}$ solves the trifunction bihemivariational inequality (2.1) and

$$\|u_{n+1}-\hat{u}\|^2 \leq \|u_n-\hat{u}\|^2.$$

Thus, it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\longrightarrow\infty}(u_n)=\hat{u},$$

the required result. □

In recent years, inertial proximal methods [1] have been suggested and analyzed for maximal monotone operators associated with the discretizations of the differential equations in times, whereas Noor [12] has used the auxiliary principle technique to suggest an inertial method for variational inequalities, the converges of which requires only pseudomonotonicity, which is a weaker condition than monotonicity. This clearly improves the convergence criteria of the inertial proximal method. We again use the auxiliary principle to suggest and analyze an inertial proximal method for solving the trifunction bihemivariational inequality(2.1).

For a given $u \in K_\beta$ satisfying (2.1), consider the problem of finding $w \in K_\beta$ such that

$$\begin{aligned}\rho F(w, Tw, \beta(v-w)) &+ \langle w - u - \alpha(u-u), v - w\rangle \\ &+\rho \int_\Omega f^0(u; \beta(v-w))d\Omega \geq 0, \quad \forall v \in K_\beta,\end{aligned} \tag{3.30}$$

where $\rho > 0$ and $\alpha > 0$ are constants.

It is clear that, if $w = u$, then u is a solution of (2.1). This fact allows us to suggest and analyze an iterative method for solving the trifunction bihemivariational inequality (2.1).

Algorithm 3.10. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho F(u_{n+1}, Tu_{n+1}, \beta(v-u_{n+1})) \\ &+\langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), v - u_{n+1}\rangle \\ &+\rho \int_\Omega f^0(u; \beta(v-u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta.\end{aligned} \tag{3.31}$$

For $\alpha_n = 0$, Algorithm 3.11 reduces to :

Algorithm 3.11. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\rho F(u_{n+1}, Tu_{n+1}, v - u_{n+1}) &+ \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &+\rho \int_\Omega f^0(u; \beta(v-u_{n+1}))d\Omega \geq 0, \quad \forall v \in K_\beta,\end{aligned}$$

which is known as the proximal method for solving trifunction bihemivariational inequality (2.1).

In a similar way for $F(u, Tu, v-u) = \langle Au, v-u\rangle$, one can obtain a number of new and known proximal methods from Algorithm 3.11 for solving bihemivariational inequalities (2.2) and its special cases. This shows that the new methods suggested in this paper are unifying one and more general than the previous ones.

For the convergence analysis of Algorithm 3.11, we need the following result.

Theorem 3.5. *Let $\bar{u} \in K_\beta$ be a solution of (2.1) and let u_{n+1} be the approximate solution obtained from Algorithm 3.10. If the trifunction $F(.,.,.)$ is pseudomonotone with respect to $\int_\Omega f^0(.;-))d\Omega$ and the operator $\int_\Omega f^0(.;.))d\Omega$ is monotone, then*

$$\begin{aligned}\|u_{n+1}-\bar{u}\|^2 \quad \leq \quad & \|u_n-\bar{u}\|^2 - \|u_{n+1}-u_n-\alpha_n(u_n-u_{n-1})\|^2 \\ & +\alpha_n\{\|u_n-\bar{u}\|^2 - \|u_{n-1}-\bar{u}\|^2 + 2\|u_n-u_{n-1}\|^2\}. \quad (3.32)\end{aligned}$$

Proof. Let $\bar{u} \in K_\beta$ be a solution of (2.1). Then

$$-F(v, Tv, \beta(\bar{u}-v)) + \int_\Omega f^0(\bar{u}; \beta(v-\bar{u})d\Omega \geq 0, \quad \forall v \in K_\beta, \tag{3.33}$$

since $F(.,.,.)$ is pseudomonotone with respect to $\int_\Omega f^0(.;.))d\Omega$.

Taking $v = u_{n+1}$ in (3.33), we have

$$F(u_{n+1}, Tu_{n+1}, \beta(\bar{u}-u_{n+1})) + \int_\Omega f^0(\bar{u}; \beta(\bar{u}-u_{n+1}))d\Omega \geq 0. \tag{3.34}$$

Now taking $v = \bar{u}$ in (3.31), we obtain

$$\begin{aligned}\rho F(u_{n+1}, Tu_{n+1}, \bar{u}-u_{n+1}) \quad + \quad & \langle u_{n+1}-u_n-\alpha_n(u_n-u_{n-1}), \bar{u}-u_{n+1}\rangle \\ + \quad & \rho\int_\Omega f^0(u_{n+1}; \beta(\bar{u}-u_{n+1}))d\Omega \geq 0. \quad (3.35)\end{aligned}$$

From (23), (24) and using the monotonicity of $\int_\Omega f^0(.;.))d\Omega$ we have

$$
\begin{aligned}
& \langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), \bar{u} - u_{n+1}\rangle \\
\geq\ & -\rho F(u_{n+1}, Tu_{n+1}, \bar{u} - u_{n+1}) - \rho J^0(u_{n+1}; \hat{u} - u_{n+1}) \\
\geq\ & -\rho \int_\Omega f^0(u_{n+1}; \beta(\hat{u} - u_{n+1}))d\Omega \\
& + \int_\Omega f^0(\hat{u}; \beta(u_{n+1} - \hat{u}))d\Omega\} \geq 0,
\end{aligned} \tag{3.36}
$$

which implies that

$$
\langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq \alpha_n \langle u_n - u_{n-1}, \bar{u} - u_n + u_n - u_{n+1}\rangle. \tag{3.37}
$$

Using (3.25) and rearranging the terms in (3.37), one can easily obtain (3.32), the required result. □

Theorem 3.6. *Let H be a finite dimensional space. Let u_{n+1} be the approximate solution obtained from Algorithm 3.9 and $\bar{u} \in K_\beta$ be a solution of (2.1). If there exists $\alpha \in (0,1)$ such that $0 \leq \alpha_n \leq \alpha, \quad \forall n \in N$ and $\quad \sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\|^2 \leq \infty$, then $\lim_{n\longrightarrow\infty} u_n = \bar{u}$.*

Proof. Let $\hat{u} \in K_\beta$ be a solution of (2.1). First we consider the case $\alpha_n = 0$. Using the technique of Theorem 3.3, we can prove that $\lim_{n\longrightarrow\infty} u_n = \hat{u}$.

Now we consider the case $\alpha_n > 0$. From (3.32), we have

$$
\begin{aligned}
& \sum_{n+1}^{\infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \\
\leq\ & \|u_0 - \bar{u}\|^2 + \sum_{n=1}^{\infty} \{\alpha\|u_n - \bar{u}\|^2 + 2\|u_n - u_{n-1}\|^2\} \leq \infty,
\end{aligned}
$$

which implies that

$$
\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 = 0.
$$

Repeating the arguments as in Theorem 3.3, one can easily show that $\lim_{n\to\infty} u_n = \hat{u}$, the required result. □

Conclusion

In this paper, we have introduced and studied the trifunction bihemivariational inequalities. Several special cases are discussed as applications of the trifunction bihemivariational inequalities. The auxiliary principle technique is used to suggest several implicit and explicit iterative methods for solving the trifunction bihemivariational inequalities, Convergence criteria of the proposed methods is discussed under suitable mild conditions. Results obtained in this paper continue to hold for the special cases. Comparison of the proposed methods with other methods need further efforts. The ideas and techniques of this paper stimulate further research in these dynamic fields

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