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The New Results in n-injective Modules and n-projective Modules

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Abstract

In this paper, we introduce and clarify a new presentation between the n-exact sequence and the n-injective module and n-projective module. Also, we obtain some new results about them.

1 Introduction

Category theory formalizes mathematical structures and their concepts in terms of a labeled directed graph called a category, whose nodes are called objects, and their edges called arrows (or morphisms). This category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. The language of category theory has been employed to formalize concepts of other high-level abstractions such as sets, rings, and groups. Several terms were utilized in category theory, including the âmorphismâ that is used differently from their usage in the rest of mathematics. In category theory, morphisms obey specific conditions of theory. Samuel Eilenberg and Saunders Mac Lane introduced the concepts of categories, functors, and natural transformations

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in 1942-45 in their study of algebraic topology, to understand the processes that preserve the mathematical structure. Category theory has practical applications in programming language theory, for example, the usage of monads in functional programming. It may also be used as an axiomatic foundation for mathematics, as an alternative to set theory and other proposed foundations. In mathematics, an abelian category is a category in which morphisms and objects can be added and in which kernels and cokernels exist and have desirable properties. The motivating prototype example of an abelian category is the category of abelian groups, Ab. The theory originated to unify several cohomology theories by Alexander Grothendieck and independently in the slightly earlier work of David Buchsbaum. Abelian categories are very stable categories. For example, they are regular and satisfy the snake lemma. The class of Abelian categories is closed under several categorical constructions, for instance, the category of chain complexes of an Abelian category, or the category of functors from a small category to an Abelian category are Abelian as well. These stability properties make them inevitable in homological algebra and beyond. This theory has significant applications in algebraic geometry, cohomology, and pure category theory. The Abelian categories are named after Niels Henrik Abel. An exact sequence is a concept in mathematics, especially in group theory, ring, and module theory, homological algebra, as well as in differential geometry. An exact sequence is a sequence, either finite or infinite, of objects and morphisms between them such that the image of one morphism equals the kernel of the next. Homological algebra is the branch of mathematics that studies homology in a general algebraic setting. It is a relatively young discipline, whose origins can be traced to investigations in combinatorial topology (a precursor to algebraic topology) and abstract algebra (theory of modules and syzygies) at the end of the 19th century, chiefly by Henri Poincaré and David Hilbert. The development of homological algebra has closely intertwined with the emergence of category theory. By and large, homological algebra is the study of homological functors and the intricate algebraic structures that they entail. One quite useful and ubiquitous concept in mathematics is that of chain complexes, which can be studied both through their homology and cohomology. Homological algebra affords the means to extract information contained in these complexes and present it in the form of homological invariants of rings, modules, topological spaces, and other âtangibleâ mathematical objects. A powerful tool for doing this is provided by spectral sequences. From its very origins, homological algebra has played an enormous role in algebraic topology. Its sphere of influence has gradually expanded and presently includes commutative algebra, algebraic geometry, algebraic number theory, representation theory, mathematical physics, operator algebras, complex analysis, and the theory of partial differential equations. K-theory is an independent discipline that draws upon methods of homological algebra, as does the noncommutative geometry of Alain Connes. This paper is organized as follows.

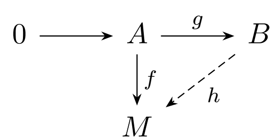
In this paper, we show to prove the important theorems of n-injective modules and n-projective modules. Finally, we recall the definition of n-projective module, and we give an open problem about some theorems of n-projective modules.

2 Preliminaries

All rings R in this paper are assumed to have an identity element 1 (or unit) (where $r1 = r = 1r$ for all $r \in R$). We do not insist that $1 \neq 0$; however, should $1 = 0$, then R is the zero ring having only one element.

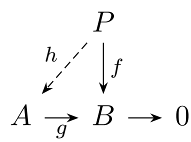
In this section, we recall some of the fundamental concepts and definitions, which are necessary for this paper. For details, we refer to [4,6,7,9,10,11].

Definition 2.1. An R-module M is injective provided that for every R-monomorphism $g : A \longrightarrow B$ between R-modules, any R-homomorphism $f : A \longrightarrow M$ can be extended to an R-homomorphism $h : B \longrightarrow M$ such that $hg = f$; i.e., the following diagram commutes



Definition 2.2. An R-module P is projective provided that for every R-epimorphism $g : A \longrightarrow B$ between R-modules and R-homomorphism f :

$P \longrightarrow B$, there exists an R-homomorphism $f : P \longrightarrow B$, there exists an R-homomorphism $h : P \longrightarrow A$ such that $gh = f$; i.e., the following diagram commutes



Definition 2.3. A left R-module F is a free left R-module if F is isomorphic to a direct sum of copies of R; that is, there is a (possibly infinite) index set B with $F = \oplus_{b \in B} R_b$, where $R_b = \langle b \rangle \cong R$ for all $b \in B$. We call B a basis of F.

Definition 2.4. Let M be an R-module. An element $m \in M$ is divisible provided that for any $r \in R$ that is not a righ zero-divisor, there exists an $x \in M$ such that $m = rx$. We also say that M is a divisible module module provided that every element of M is divisible. Note that a divisible group is a divisible $\mathbb{Z}$-module.

Definition 2.5. Let $\mathcal{C}$ be an additive category and $f : A \longrightarrow B$ a morphism in $\mathcal{C}$. A weak cokernel of f is a morphism $g : B \longrightarrow C$ such that for all $C' \in \mathcal{C}$ the sequence of abelian groups

$$\mathcal{C}(C, C') \xrightarrow{g^*} \mathcal{C}(B, C') \xrightarrow{f^*} \mathcal{C}(A, C')$$

Definition 2.6. A category $\mathcal{C}$ is abelian if

1. $\mathcal{C}$ has a zero object.

2. For every pair of objects there is a product and a sum.

3. $\mathcal{C}$ Every map has a kernel and cokernel.

4. $\mathcal{C}$ Every monomorphism is a kernel of a map.

5. $\mathcal{C}$ Every epimorphism is a cokernel of a map.

Definition 2.7. A category $\mathcal{C}$ is additive if

1. $Hom(A, B)$ is an (additive) abelian group for every $A, B \in obj(\mathcal{C})$

2. the distributive laws hold: given morphisms

$$X \xrightarrow{a} A \xrightarrow{f} B \xrightarrow{b} Y$$

and

$$X \xrightarrow{a} A \xrightarrow{g} B \xrightarrow{b} Y$$

where X and $Y \in obj(\mathcal{C})$, then

$$b(f+g) = bf + bg$$

and $a \leftrightarrow [under]overb$

$$(f+g)a = fa + ga$$

3. $\mathcal{C}$ has a zero object.

4. $\mathcal{C}$ has finite product and finite coproduct.

Definition 2.8. An abelian group D is said to be divisible if given any $y \in D$ and $0 \neq n \in \mathbb{Z}$, there exists $x \in D$ such that $nx = y$.

Example 2.9.

1. *Note that Q is a divisible Z-module since for every $q \in Q$, where $q = \frac{a}{b}$ for integers $a, b \in Z$ with $b \neq 0$, and for every $0 \neq z \in Z$, there exists $x \in Q$ such that $x = \frac{a}{zb}$ so that $q = zx$.*

2. *Note that Z is not a divisible Z-module since there is no $x \in Z$ with $3 = 2x$.*

Definition 2.10. Let $M_{i i \in \mathbb{Z}}$ be a family of R-modules, and let $f_{i i \in \mathbb{Z}}$ be a family of R-homomorphisms such that $M_{i-1} \xrightarrow{f_i} Mi$ for every $i \in \mathbb{Z}$. Then the sequence

$$... \xrightarrow{f_{-1}} M_{-1} \xrightarrow{f_0} M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} ... \tag{2.1}$$

is said to be exact provided that $Im(fi-1) = Ker(f_i)$ for every $i \in \mathbb{Z}$. Note that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.2}$$

is exact if and only if f is an R-monomorphism, g is an R-epimorphism, and

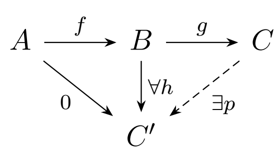
$$Im(f) = Ker(g)$$

This type of sequence is called short exact.

Definition 2.11. Let $\mathcal{C}$ be an additive category and $f : A \longrightarrow B$ a morphism in $\mathcal{C}$. A weak cokernel of f is a morphism $g : B \longrightarrow C$ such that for all $C' \in \mathcal{C}$ the sequence of abelian groups

$$\mathcal{C}(C, C') \xrightarrow{\hat{g}} \mathcal{C}(B, C') \xrightarrow{\hat{f}} \mathcal{C}(A, C')$$

is exact. Equivalently, g is a weak cokernel of f if $fg = 0$ and for each morphism $h : B \longrightarrow C'$ such that $fh = 0$ there exists a (not necessarily unique) morphism $p : C \longrightarrow C'$ such that $h = gp$. These properties are subsumed in the following commutative diagram:



Clearly, a weak cokernel g of f is a cokernel of f if and only if g is an epimorphism. The concept of weak kernel is defined dually.

Definition 2.12. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is called $\mathcal{X}$-monic if

$$\mathcal{C}(B, X) \xrightarrow{\mathcal{C}(f,X)} \mathcal{C}(A, X) \longrightarrow 0$$

is exact for any object $X \in \mathcal{X}$. A morphism $f : A \longrightarrow X$ in $\mathcal{C}$ is called a left $\mathcal{X}$-approximation of A if f is $\mathcal{X}$-monic and $X \in \mathcal{X}$. The subcategory $\mathcal{X}$ is said to be a covariantly finite subcategory of $\mathcal{C}$ if any object A of $\mathcal{C}$ has a left $\mathcal{X}$-approximation. We can defined $\mathcal{X}$-epic morphism, right $\mathcal{X}$- approximation and contravariantly finite subcategory dually. The subcategory $\mathcal{X}$ is called functorially finite if it is both contravariantly finite and covariantly finite.

Definition 2.13. Let $\mathcal{C}$ be an additive category and $d^0 : X^0 \longrightarrow X^1$ a morphism in $\mathcal{C}$. An n-coker of d^0 is a sequence

$$(d^1, ..., d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^n} X^{n+1}$$

such that, for all $Y \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{C}(X^{n+1}, Y) \xrightarrow{\hat{d^n}} \mathcal{C}(X^n, Y) \xrightarrow{\hat{d^{n-1}}} ... \xrightarrow{\hat{d^1}} \mathcal{C}(X^1, Y) \xrightarrow{\hat{d^0}} \mathcal{C}(X^0, Y)$$

is exact. Equivalently, the sequence $(d^1, ..., d^n)$ is an n-coker of d^0 if, for all $1 \leq k \leq n-1$ the morphism d^k is a weak cokernel of d^{k-1}, and d^n is moreover a cokernel of d^{n-1}. In this case, we say the sequence

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^n} X^{n+1}$$

is right n-exact.

Remark 2.14. When we say n-cokernel we always means that n is a positive integer. We note that the notion of 1-cokernel is the same as cokernel. we can define n- kernel and left n-exact sequence dually.

Definition 2.15. Let $\mathcal{C}$ be an additive category. An n-exact sequence in $\mathcal{C}$ is a complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \tag{2.3}$$

in $Ch^n(\mathcal{C})$ such that $(d^0, ..., d^{n-1})$ is an n-ker of d^n, and $(d^1, ..., d^n)$ is an n-coker of d^0. The sequence (3.1) is called n-exact if it is both right n-exact and left n-exact.

Theorem 2.16. *Let A, B, $\{B_i | i \in I\}$, $\{A_j | j \in J, Jisfinite\}$ be modules over a ring R . Then there is isomorphisms of abelian groups:*

1. $Hom_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} Hom_R(A, B_i)$.
2. $Hom_R(\oplus_{j\in J} A_j, B) \cong \oplus_{j\in J} Hom_R(A_j, B)$.

Theorem 2.17. *Let A, B, $\{B_i | i \in I\}$ be modules over a ring R. Then if I is finite there is isomorphisms of abelian groups: $Hom_R(A, \oplus_{i\in I} B_i) \cong \oplus_{i\in I} Hom_R(A, B_i)$.*

Proposition 2.18. *A direct product of R-modules $\prod_{i\in\mathbb{I}} J_i$ is injective if only if J_i is injective for every $i \in \mathbb{I}$.*

Corollary 2.19. *Let R be an integral domain and let K the field of fractions of R. Then K is an injective R-module.*

Corollary 2.20. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules. If Λ is finite and M_λ is injective for every $\lambda \in \Lambda$, then $\oplus_{\lambda\in\Lambda} M_\lambda$ is also injective.*

Theorem 2.21. *Let M be an R-module. Then M is injective if and only if for every short exact sequence $0 \longrightarrow A \xrightarrow{\theta} B \xrightarrow{\psi} C \longrightarrow 0$ of R-modules,*

$$0 \longrightarrow Hom_R(C, M) \xrightarrow{\Psi} Hom_R(B, M) \xrightarrow{\Theta} Hom_R(A, M) \longrightarrow 0$$

is also a short exact sequence, where $\Psi(f) = f\psi$ and $\Theta(f) = f\theta$.

Proposition 2.22. *Let R be a ring. A direct sum of R-modules $\sum_{i\in\mathbb{I}} P_i$ is projective if only if each P_i is projective.*

Proposition 2.23. *Every free left R-module is projective.*

Theorem 2.24. *A left R-module P is projective if and only if P is a direct summand of a free left R-module.*

Corollary 2.25.

1. *Every direct summand of a projective module is itself projective.*

2. *Every direct sum of projective modules is projective.*

Lemma 2.26. *Let R be a ring with identity. A unitary R-module J is injective if and only if for every left ideal L of R, any R-module homomorphism $L \longrightarrow J$ may be extended to an R-module homomorphism $R \longrightarrow J$:*

Example 2.27.

1. *Q is an injective Z-module by Lemma (2.26) since for every Z-homomorphism $f : nZ \longrightarrow Q$, where nZ is an ideal of Z for $0 \neq n \in Z$, there exists a Z-homomorphism $g : Z \longrightarrow Q$ defined by $g(z) = \frac{zf(n)}{n}$, so $g(nz) = \frac{(nz)f(n)}{n} = zf(n) = f(nz)$ for every $nz \in Z$.*

2. *Note that Z is not an injective Z-module since using the Z-homomorphism $f : 2Z \longrightarrow Z$ given by $f(2z) = z$, there is no Z-homomorphism $g : Z \longrightarrow Z$ such that $g(2z) = f(2z)$ for every $2z \in 2Z$. Otherwise, $1 = f(2) = g(2) = 2g(1)$, implying that $g(1) = \frac{1}{2}$. However, since $g(1) \in Z$, this is impossible.*

3 n-injective Module

Definition 3.1. Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. An R-module M is n-injective if the sequence of R-module in $\mathcal{C}$ is left n-exact

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

if there is $M \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, M) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, M) \xrightarrow{\hat{d^{n-1}}}$$

$$... \xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, M) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, M)$$

is right n-exact.

Proposition 3.2. *Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. A direct product of R-modules $\prod_{i\in\mathbb{I}} J_i$ is n-injective if only if J_i is n-injective for every $i \in \mathbb{I}$.*

Proof. Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n$, and d^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. The sequence of R-module in $\mathcal{C}$

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

is left n-exact.

Suppose that $\prod_{i\in\mathbb{I}} J_i$ is n-injective. To show that, J_i is n-injective for each $i \in \mathbb{I}$. Now if there is $\prod_{i\in\mathbb{I}} J_i$ the induced sequence of abelian groups this sequence is

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^{n-1}}} Hom_{\mathcal{C}}(X^{n-1}, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^{n-2}}}$$

$$... \xrightarrow{\hat{d^2}} Hom_{\mathcal{C}}(X^2, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, \prod_{i\in\mathbb{I}} J_i)$$

is right n-exact. By Theorem 2.16, (1),

$$Hom_{\mathcal{C}}(X^i, \prod_{i\in\mathbb{I}} J_i) \cong \prod_{i\in I} Hom_R(X^i, J_i)$$

for each $i \in \mathbb{I}$. Then this sequence

$$0 \longrightarrow \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^{n+1}, J_i) \xrightarrow{\hat{d^n}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^n, J_i) \xrightarrow{\hat{d^{n-1}}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^{n-1}, J_i) \xrightarrow{\hat{d^{n-2}}}$$

$$... \xrightarrow{\hat{d^1}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^1, J_i) \xrightarrow{\hat{d^2}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^2, J_i) \xrightarrow{\hat{d^0}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^0, J_i)$$

is right n-exact. Then J_i is n-injective for each $i \in \mathbb{I}$.

Conversely, suppose that J_i is n-injective. To show that, $\prod_{i\in\mathbb{I}} J_i$ is n-injective for each $i \in \mathbb{I}$. Now if there is J_i the induced sequence of abelian groups this sequence is

$$0 \longrightarrow \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^{n+1}, J_i) \xrightarrow{\hat{d^n}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^n, J_i) \xrightarrow{\hat{d^{n-1}}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^{n-1}, J_i) \xrightarrow{\hat{d^{n-2}}}$$

$$... \xrightarrow{\hat{d^1}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^1, J_i) \xrightarrow{\hat{d^2}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^2, J_i) \xrightarrow{\hat{d^0}} \prod_{i\in\mathbb{I}} Hom_{\mathcal{C}}(X^0, J_i)$$

is right n-exact. By Theorem 2.16, (1). Then this sequence

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^{n-1}}} Hom_{\mathcal{C}}(X^{n-1}, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^{n-2}}}$$

$$... \xrightarrow{\hat{d^2}} Hom_{\mathcal{C}}(X^2, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, \prod_{i\in\mathbb{I}} J_i) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, \prod_{i\in\mathbb{I}} J_i)$$

is right n-exact. Then $\prod_{i\in\mathbb{I}} J_i$ is also n-injective. □

Corollary 3.3. *Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \le i \le n$, and d^i for all $0 \le i \le n-1$ is a morphism in $\mathcal{C}$. Let R be an integral domain and let K the field of fractions of R. Then K is an n-injective R-module.*

Proof. By Corollary 2.19, k is injective R-module. Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \le i \le n$, and d^i for all $0 \le i \le n-1$ is a morphism in $\mathcal{C}$. The sequence of R-module in $\mathcal{C}$

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \dots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

is left n-exact. By Theorem 2.21

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, K) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, M) \xrightarrow{\hat{d^{n-1}}}$$

$$\dots \xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, K) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, K)$$

is right n-exact. Then K is n-injective module.

□

Corollary 3.4. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules. If Λ is finite and M_λ is n-injective for every $\lambda \in \Lambda$, then $\oplus_{\lambda\in\Lambda} M_\lambda$ is also n-injective.*

Proof. Let $\mathcal{C}$ be an category of R-modules, $X^i \in obj(\mathcal{C})$ for all $0 \le i \le n$, and d^i for all $0 \le i \le n-1$ is a morphism in $\mathcal{C}$. The sequence of R-module in $\mathcal{C}$

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \dots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

is left n-exact.

Suppose that M_λ is n-injective. To show that, $\oplus_{\lambda\in\Lambda} M_\lambda$ is n-injective for each $\lambda \in \Lambda$. Now if there is M_λ the induced sequence of abelian groups this sequence is

$$0 \longrightarrow \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^{n+1}, M_\lambda) \xrightarrow{\hat{d^n}} \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^n, M_\lambda) \xrightarrow{\hat{d^{n-1}}}$$

$$\oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^{n-1}, M_\lambda) \xrightarrow{\hat{d^{n-2}}} \dots \xrightarrow{\hat{d^2}} \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^2, M_\lambda)$$

$$\xrightarrow{\hat{d^1}} \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^1, M_\lambda) \xrightarrow{\hat{d^0}} \oplus_{\lambda\in\Lambda} Hom_{\mathcal{C}}(X^0, M_\lambda)$$

is right n-exact. If Λ is finite by Theorem 2.17,

$$Hom_{\mathcal{C}}(X^i, \oplus_{\lambda\in\Lambda} M_\lambda) \cong \oplus_{\lambda\in\Lambda} Hom_R(X^i, M_\lambda)$$

for every $\lambda \in \Lambda$, Then this sequence

$$0 \longrightarrow Hom_{\mathcal{C}}(X^{n+1}, \oplus_{\lambda\in\Lambda} M_\lambda) \xrightarrow{\hat{d^n}} Hom_{\mathcal{C}}(X^n, \oplus_{\lambda\in\Lambda} M_\lambda)$$

$$\xrightarrow{\hat{d^{n-1}}} Hom_{\mathcal{C}}(X^{n-1}, \oplus_{\lambda\in\Lambda} M_\lambda) \xrightarrow{\hat{d^{n-2}}} ... \xrightarrow{\hat{d^2}} Hom_{\mathcal{C}}(X^2, \oplus_{\lambda\in\Lambda} M_\lambda)$$

$$\xrightarrow{\hat{d^1}} Hom_{\mathcal{C}}(X^1, \oplus_{\lambda\in\Lambda} M_\lambda) \xrightarrow{\hat{d^0}} Hom_{\mathcal{C}}(X^0, \oplus_{\lambda\in\Lambda} M_\lambda)$$

is right n-exact. Then $\oplus_{\lambda\in\Lambda} M_\lambda$ is also n-injective. □

Proposition 3.5. *Every R-module injective is not n-injective.*

Proof. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules. If $\oplus_{\lambda\in\Lambda} M_\lambda$ is injective, then M_λ is injective for every $\lambda \in \Lambda$ but $\oplus_{\lambda\in\Lambda} M_\lambda$ is not n-injective for every $\lambda \in \Lambda$ and then, M_λ is n-injective for every $\lambda \in \Lambda$. □

Definition 3.6. Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a morphism in $\mathcal{C}$. An R-module P is n-projective if the sequence of R-module in $\mathcal{C}$ is rightt n-exact

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}$$

if there is $P \in \mathcal{C}$ the induced sequence of abelian groups

$$Hom_{\mathcal{C}}(P, Y^0) \xrightarrow{\hat{f^0}} Hom_{\mathcal{C}}(P, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(P, Y^2) \xrightarrow{\hat{f^2}}$$

$$... \xrightarrow{\hat{f^{n-2}}} Hom_{\mathcal{C}}(P, Y^{n-1}) \xrightarrow{\hat{f^{n-1}}} Hom_{\mathcal{C}}(P, Y^n) \xrightarrow{\hat{f^n}} Hom_{\mathcal{C}}(P, Y^{n+1}) \longrightarrow 0$$

is left n-exact.

Proposition 3.7. *Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n$ is a morphism in $\mathcal{C}$. A direct sum of R-modules $\oplus_{i\in\mathbb{I}} P_i$ is n-projective if only if P_i is n-projective for every $i \in \mathbb{I}$ and $\mathbb{I}$ is finite.*

Proof. Let $\mathcal{C}$ be an category of R-modules, $Y^i \in obj(\mathcal{C})$ for all $0 \leq i \leq n+1$, and f^i for all $0 \leq i \leq n-1$ is a morphism in $\mathcal{C}$. The sequence of R-module in $\mathcal{C}$

$$Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} Y^2 \xrightarrow{f^2} ... \xrightarrow{f^{n-1}} Y^n \xrightarrow{f^n} Y^{n+1}$$

is right n-exact.

Suppose that $\oplus_{i\in\mathbb{I}}P_i$ is n-projective. To show that, P_i is n-projective for each $i\in\mathbb{I}$. Now if there is $\oplus_{i\in\mathbb{I}}P_i$ the induced sequence of abelian groups this sequence is

$$Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^0) \xrightarrow{\hat{f^0}} Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^2) \xrightarrow{\hat{f^2}}$$
$$... \xrightarrow{\hat{f^{n-1}}} Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^n) \xrightarrow{\hat{f^n}} Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^{n+1}) \longrightarrow 0$$

is left n-exact. If $\mathbb{I}$ is finite by Theorem 2.16, (2)

$$Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^j) \cong \oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^j)$$

for every $i\in\mathbb{I}$, Then this sequence

$$\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^0) \xrightarrow{\hat{f^0}} \oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(P_i, Y^2) \xrightarrow{\hat{f^2}}$$
$$... \xrightarrow{\hat{f^{n-1}}} \oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^n) \xrightarrow{\hat{f^n}} \oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^{n+1}) \longrightarrow 0$$

is left n-exact. Then P_i is n-projective for each $i\in\mathbb{I}$.

Conversely, suppose that P_i is n-projective. To show that, $\oplus_{i\in\mathbb{I}}P_i$ is n-projective for every $i\in\mathbb{I}$ and $\mathbb{I}$ is finite. Now if there is P_i the induced sequence of abelian groups this sequence is

$$\oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^0) \xrightarrow{\hat{f^0}} \oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(P_i, Y^2) \xrightarrow{\hat{f^2}}$$
$$... \xrightarrow{\hat{f^{n-1}}} \oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^n) \xrightarrow{\hat{f^n}} \oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^{n+1}) \longrightarrow 0$$

is left n-exact. If $\mathbb{I}$ is finite by Theorem 2.16, (2)

$$Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^j) \cong \oplus_{i\in\mathbb{I}}Hom_{\mathcal{C}}(P_i, Y^j)$$

for every $i\in\mathbb{I}$. Then this sequence

$$Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^0) \xrightarrow{\hat{f^0}} Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^1) \xrightarrow{\hat{f^1}} Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^2) \xrightarrow{\hat{f^2}}$$
$$... \xrightarrow{\hat{f^{n-1}}} Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^n) \xrightarrow{\hat{f^n}} Hom_{\mathcal{C}}(\oplus_{i\in\mathbb{I}}P_i, Y^{n+1}) \longrightarrow 0$$

is left n-exact. Then $\oplus_{i\in\mathbb{I}}P_i$ is also n-projective. □

4 One Open Problem

Using the following definitions, can we prove the following theorems about n-projective module and free R-module.

Proposition 4.1. *Every free left R-module is n-projective.*

Proposition 4.2.

1. *Every finite direct summand of a n-projective module is itself n-projective.*
2. *Every finite direct sum of n-projective modules is n-projective.*

Definition 4.3. Let n be a positive integer. An n-abelian category is an additive category $\mathcal{C}$ which satisfies the following axioms;

($A0$) The category $\mathcal{C}$ is idempotent complete.

($A1$) Every morphism in $\mathcal{C}$ has n-ker and n-coker.

($A2$) for every monomorphism $f^0 : X^0 \longrightarrow X^1$ in $\mathcal{C}$ and, for every n-coker $(f^0, f^1, ..., f^{n-1})$ of f^0, the following sequence n-exact:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} ... \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

($A2^{op}$) for every epimorphism $g^n : X^n \longrightarrow X^{n+1}$ in $\mathcal{C}$ and, for every n-ker $(g^0, g^1, ..., g^{n-1})$ of g^n, the following sequence n-exact:

$$X^0 \xrightarrow{g^0} X^1 \xrightarrow{g^1} ... \xrightarrow{g^{n-1}} X^n \xrightarrow{g^n} X^{n+1}.$$

Now one can investigate a divisible modules in n-additive abelian category. Next one can obtain all of the result of them as we obtained in this paper, and it is an open problem.

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