

Subordination and Superordination Results for Normalized Analytic Functions Associated with Frasin Operator

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Abstract

In this paper, we derive some applications of first order differential subordination and superordination results involving Frasin operator for analytic functions in the open unit disk. Also by these results, we obtain sandwich results. Our results extend corresponding previously known results.

1. Introduction

Let $H = H(U)$ indicate the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, n]$ be the subclass of H consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C}, n \in \mathbb{N} = \{1, 2, \dots\}).$$

Also, let A be the subclass of H containing of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1.1)$$

Received: September 8, 2019; Accepted: November 13, 2019

2010 Mathematics Subject Classification: 30C45.

Keywords and phrases: analytic function, differential subordination, differential superordination, dominant, subordinant, Frasin operator.

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Let $f, g \in H$. The function f is said to be subordinate to g , or g is said to be superordinate to f , if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ ($z \in U$). It is well known that, if the function g is univalent in U , then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\xi, h \in H$ and $\psi(r, s, t; z) : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$. If ξ and $\psi(\xi(z), z\xi'(z), z^2\xi''(z); z)$ are univalent functions in U and if ξ satisfies the second-order differential superordination

$$h(z) \prec \psi(\xi(z), z\xi'(z), z^2\xi''(z); z), \quad (1.2)$$

then ξ is called a solution of the differential superordination (1.2). (If f is subordinate to g , then g is superordinate to f). An analytic function q is called a subordinant of (1.2), if $q \prec \xi$ for all ξ satisfying (1.2). A univalent subordinant \tilde{q} that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.2) is called the best subordinant.

For $m \in \mathbb{N}$, $\lambda, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leq \tau \leq 1$ and $f \in A$, the Frasin operator $D_{m,\tau}^\lambda : A \rightarrow A$ (see [4]) is defined by

$$D_{m,\tau}^\lambda f(z) = z + \sum_{k=2}^{\infty} \left(1 + (k-1) \sum_{j=1}^m \binom{m}{j} (-1)^{j+1} \tau^j \right)^\lambda a_k z^k. \quad (1.3)$$

It is readily verified from (1.3) that

$$C_j^m(\tau) z (D_{m,\tau}^\lambda f(z))'_z = D_{m,\tau}^{\lambda+1} f(z) - (1 - C_j^m(\tau)) D_{m,\tau}^\lambda f(z), \quad (1.4)$$

where $C_j^m(\tau) = \sum_{j=1}^m \binom{m}{j} (-1)^{j+1} \tau^j$.

Special cases of this operator include the generalized Sălăgean operator [1], the Sălăgean differential operator [10].

Very recently several authors, Shanmugam et al. [11], Goyal et al. [5], El-Ashwah et al. [3] and Wanas and Majeed [12] have obtained sandwich results for certain classes of analytic functions.

The main object of the present investigation is to find sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \left(\frac{D_{m,\tau}^\lambda f(z)}{z} \right)^\gamma \prec q_2(z)$$

and

$$q_1(z) \prec \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)} \right)^\gamma \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

In order to prove our results, we make use of the following known results.

Definition 1.1 [6]. Denote by Q the set of all functions f that are analytic and injective on $\bar{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1.1 [6]. Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that

(1) $Q(z)$ is starlike univalent in U ,

(2) $\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$ for $z \in U$.

If ξ is analytic in U , with $\xi(0) = q(0)$, $\xi(U) \subset D$ and

$$\theta(\xi(z)) + z\xi'(z)\phi(\xi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.5}$$

then $\xi \prec q$ and q is the best dominant of (1.5).

Lemma 1.2 [7]. Let q be a convex univalent function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} \setminus \{0\}$ with

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\left(\frac{\alpha}{\beta}\right)\right\}.$$

If ξ is analytic in U and

$$\alpha\xi(z) + \beta z\xi'(z) \prec \alpha q(z) + \beta zq'(z), \quad (1.6)$$

then $\xi \prec q$ and q is the best dominant of (1.6).

Lemma 1.3 [7]. Let q be convex univalent in U and let $\beta \in \mathbb{C}$. Further assume that $\operatorname{Re}(\beta) > 0$. If $\xi \in H[q(0), 1] \cap Q$ and $\xi(z) + \beta z\xi'(z)$ is univalent in U , then

$$q(z) + \beta zq'(z) \prec \xi(z) + \beta z\xi'(z), \quad (1.7)$$

which implies that $q \prec \xi$ and q is the best subordinator of (1.7).

Lemma 1.4 [2]. Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

$$(1) \operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0 \text{ for } z \in U,$$

$$(2) Q(z) = zq'(z)\phi(q(z)) \text{ is starlike univalent in } U.$$

If $\xi \in H[q(0), 1] \cap Q$, with $\xi(U) \subset D$, $\theta(\xi(z)) + z\xi'(z)\phi(\xi(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(\xi(z)) + z\xi'(z)\phi(\xi(z)), \quad (1.8)$$

then $q \prec \xi$ and q is the best subordinator of (1.8).

2. Main Results

Theorem 2.1. Let q be convex univalent in U with $q(0) = 1$, $\sigma \in \mathbb{C}/\{0\}$, $\gamma > 0$ and suppose that q satisfies

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\left(\frac{\gamma}{\sigma}\right)\right\}. \quad (2.1)$$

If $f \in A$ satisfies the subordination

$$\left(1 - \frac{\sigma}{C_j^m(\tau)}\right) \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma + \frac{\sigma}{C_j^m(\tau)} \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)}\right) \prec q(z) + \frac{\sigma}{\gamma} zq'(z), \quad (2.2)$$

then

$$\left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma \prec q(z) \quad (2.3)$$

and q is the best dominant of (2.2).

Proof. Define the function ξ by

$$\xi(z) = \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma, \quad (z \in U). \quad (2.4)$$

Differentiating (2.4) logarithmically with respect to z , we get

$$\frac{z\xi'(z)}{\xi(z)} = \gamma \left(\frac{z(D_{m,\tau}^\lambda f(z))'}{D_{m,\tau}^\lambda f(z)} - 1\right).$$

Now, in view of (1.4), we obtain the following subordination

$$\frac{z\xi'(z)}{\xi(z)} = \frac{\gamma}{C_j^m(\tau)} \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)} - 1\right).$$

Therefore,

$$\frac{z\xi'(z)}{\gamma} = \frac{1}{C_j^m(\tau)} \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)} - 1\right).$$

The subordination (2.2) from the hypothesis becomes

$$\xi(z) + \frac{\sigma}{\gamma} z\xi'(z) \prec q(z) + \frac{\sigma}{\gamma} zq'(z).$$

Hence, an application of Lemma 1.2 with $\alpha = 1$ and $\beta = \frac{\sigma}{\gamma}$, we obtain (2.3).

Theorem 2.2. Let $\eta_i \in \mathbb{C}$ ($i = 1, 2, 3, 4$), $\gamma > 0$, $\delta \in \mathbb{C}/\{0\}$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$ ($z \in U$) and assume that q satisfies

$$\operatorname{Re} \left\{ 1 + \frac{\eta_2}{\delta} q(z) + \frac{2\eta_3}{\delta} q^2(z) + \frac{3\eta_4}{\delta} q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0. \tag{2.5}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U . If $f \in A$ satisfies

$$\begin{aligned} &\Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z) \\ &< \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + \delta \frac{zq'(z)}{q(z)}, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} &\Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z) \\ &= \eta_1 + \eta_2 \left(\frac{D_{m, \tau}^{\lambda+1} f(z)}{D_{m, \tau}^{\lambda} f(z)} \right)^{\gamma} + \eta_3 \left(\frac{D_{m, \tau}^{\lambda+1} f(z)}{D_{m, \tau}^{\lambda} f(z)} \right)^{2\gamma} + \eta_4 \left(\frac{D_{m, \tau}^{\lambda+1} f(z)}{D_{m, \tau}^{\lambda} f(z)} \right)^{3\gamma} \\ &\quad + \frac{\gamma\delta}{C_j^m(\tau)} \left(\frac{D_{m, \tau}^{\lambda+2} f(z)}{D_{m, \tau}^{\lambda+1} f(z)} - \frac{D_{m, \tau}^{\lambda+1} f(z)}{D_{m, \tau}^{\lambda} f(z)} \right), \end{aligned} \tag{2.7}$$

then

$$\left(\frac{D_{m, \tau}^{\lambda+1} f(z)}{D_{m, \tau}^{\lambda} f(z)} \right)^{\gamma} < q(z)$$

and q is the best dominant of (2.6).

Proof. Define the function ξ by

$$\xi(z) = \left(\frac{D_{m, \tau}^{\lambda+1} f(z)}{D_{m, \tau}^{\lambda} f(z)} \right)^{\gamma}, \quad (z \in U). \tag{2.8}$$

By a straightforward computation and using (1.4), we have

$$\begin{aligned} & \eta_1 + \eta_2 \xi(z) + \eta_3 \xi^2(z) + \eta_4 \xi^3(z) + \delta \frac{z\xi'(z)}{\xi(z)} \\ & = \Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z), \end{aligned} \quad (2.9)$$

where $\Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z)$ is given by (2.7).

From (2.6) and (2.9), we obtain

$$\begin{aligned} & \eta_1 + \eta_2 \xi(z) + \eta_3 \xi^2(z) + \eta_4 \xi^3(z) + \delta \frac{z\xi'(z)}{\xi(z)} \\ & < \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + \delta \frac{zq'(z)}{q(z)}. \end{aligned}$$

By setting

$$\theta(w) = \eta_1 + \eta_2 w + \eta_3 w^2 + \eta_4 w^3 \text{ and } \phi(w) = \frac{\delta}{w}, \quad w \neq 0,$$

we see that $\theta(w)$ is analytic in \mathbb{C} , $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \delta \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + \delta \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U ,

$$\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ 1 + \frac{\eta_2}{\delta} q(z) + \frac{2\eta_3}{\delta} q^2(z) + \frac{3\eta_4}{\delta} q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0.$$

Thus, by Lemma 1.1, we get $\xi(z) < q(z)$. By using (2.8), we obtain the desired result.

Theorem 2.3. *Let q be convex univalent in U with $q(0) = 1$, $\gamma > 0$ and $\operatorname{Re}\{\sigma\} > 0$. Let $f \in A$ satisfy*

$$\left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma \in H[q(0), 1] \cap \mathcal{Q}$$

and

$$\left(1 - \frac{\sigma}{C_j^m(\tau)}\right) \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma + \frac{\sigma}{C_j^m(\tau)} \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)}\right)$$

be univalent in U . If

$$q(z) + \frac{\sigma}{\gamma} zq'(z) \prec \left(1 - \frac{\sigma}{C_j^m(\tau)}\right) \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma + \frac{\sigma}{C_j^m(\tau)} \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)}\right), \tag{2.10}$$

then

$$q(z) \prec \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma \tag{2.11}$$

and q is the best subordinant of (2.10).

Proof. Let the function ξ be defined by (2.4). After some computations and using (1.4), it is evident that

$$\begin{aligned} &\left(1 - \frac{\sigma}{C_j^m(\tau)}\right) \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma + \frac{\sigma}{C_j^m(\tau)} \left(\frac{D_{m,\tau}^\lambda f(z)}{z}\right)^\gamma \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)}\right) \\ &= \xi(z) + \frac{\sigma}{\gamma} z\xi'(z). \end{aligned} \tag{2.12}$$

From (2.10) and (2.12), we get

$$q(z) + \frac{\sigma}{\gamma} zq'(z) \prec \xi(z) + \frac{\sigma}{\gamma} z\xi'(z).$$

Hence, an application of Lemma 1.3 with $\alpha = 1$ and $\beta = \frac{\sigma}{\gamma}$, we obtain (2.11).

Theorem 2.4. Let $\eta_i \in \mathbb{C}$ ($i = 1, 2, 3, 4$), $\gamma > 0$, $\delta \in \mathbb{C}/\{0\}$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$ ($z \in U$) and assume that q satisfies

$$\operatorname{Re}\left\{\frac{\eta_2}{\delta} q(z) + \frac{2\eta_3}{\delta} q^2(z) + \frac{3\eta_4}{\delta} q^3(z)\right\} > 0. \tag{2.13}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U . Let $f \in A$ satisfy

$$\left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)}\right)^\gamma \in H[q(0), 1] \cap \mathcal{Q}$$

and $\Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z)$ is univalent in U , where $\Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z)$ is given by (2.7). If

$$\begin{aligned} & \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + \delta \frac{zq'(z)}{q(z)} \\ & \prec \Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z), \end{aligned} \tag{2.14}$$

then

$$q(z) \prec \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)}\right)^\gamma$$

and q is the best subordinator of (2.14).

Proof. Let the function ξ be defined by (2.8). By a straightforward computation and using (1.4), we find that

$$\begin{aligned} & \Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z) \\ & = \eta_1 + \eta_2 \xi(z) + \eta_3 \xi^2(z) + \eta_4 \xi^3(z) + \delta \frac{z\xi'(z)}{\xi(z)}, \end{aligned} \tag{2.15}$$

where $\Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z)$ is given by (2.7).

From (2.14) and (2.15), we obtain

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + \delta \frac{zq'(z)}{q(z)}$$

$$\prec \eta_1 + \eta_2 \xi(z) + \eta_3 \xi^2(z) + \eta_4 \xi^3(z) + \delta \frac{z \xi'(z)}{\xi(z)}.$$

By setting $\theta(w) = \eta_1 + \eta_2 w + \eta_3 w^2 + \eta_4 w^3$ and $\phi(w) = \frac{\delta}{w}$, $w \neq 0$, we see that $\theta(w)$ is analytic in \mathbb{C} , $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \delta \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U ,

$$\operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} = \operatorname{Re} \left\{ \frac{\eta_2}{\delta} q(z) + \frac{2\eta_3}{\delta} q^2(z) + \frac{3\eta_4}{\delta} q^3(z) \right\} > 0.$$

Thus, by Lemma 1.4, we get $q(z) \prec \xi(z)$. By using (2.8), we obtain the desired result.

Concluding the results of differential subordination and superordination, we state the following “sandwich results”.

Theorem 2.5. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_2 satisfies (2.1), $\gamma > 0$ and $\operatorname{Re}\{\sigma\} > 0$. Let $f \in A$ satisfy*

$$\left(\frac{D_{m,\tau}^\lambda f(z)}{z} \right)^\gamma \in H[1, 1] \cap Q$$

and

$$\left(1 - \frac{\sigma}{C_j^m(\tau)} \right) \left(\frac{D_{m,\tau}^\lambda f(z)}{z} \right)^\gamma + \frac{\sigma}{C_j^m(\tau)} \left(\frac{D_{m,\tau}^\lambda f(z)}{z} \right)^\gamma \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)} \right)$$

be univalent in U . If

$$q_1(z) + \frac{\sigma}{\gamma} zq_1'(z) \prec \left(1 - \frac{\sigma}{C_j^m(\tau)} \right) \left(\frac{D_{m,\tau}^\lambda f(z)}{z} \right)^\gamma + \frac{\sigma}{C_j^m(\tau)} \left(\frac{D_{m,\tau}^\lambda f(z)}{z} \right)^\gamma \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^\lambda f(z)} \right)$$

$$\prec q_2(z) + \frac{\sigma}{\gamma} zq_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D_{m,\tau}^{\lambda} f(z)}{z} \right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinator and the best dominant.

Theorem 2.6. Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies (2.13) and q_2 satisfies (2.5). Let $f \in A$ satisfies

$$\left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^{\lambda} f(z)} \right)^{\gamma} \in H[1, 1] \cap \mathcal{Q}$$

and $\Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z)$ is univalent in U , where $\Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z)$ is given by (2.7). If

$$\begin{aligned} & \eta_1 + \eta_2 q_1(z) + \eta_3 q_1^2(z) + \eta_4 q_1^3(z) + \delta \frac{zq_1'(z)}{q_1(z)} \\ & \prec \Omega(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, \lambda, m, \tau; z) \\ & \prec \eta_1 + \eta_2 q_2(z) + \eta_3 q_2^2(z) + \eta_4 q_2^3(z) + \delta \frac{zq_2'(z)}{q_2(z)}, \end{aligned}$$

then

$$q_1(z) \prec \left(\frac{D_{m,\tau}^{\lambda+1} f(z)}{D_{m,\tau}^{\lambda} f(z)} \right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinator and the best dominant.

Remark 2.1. By selecting the particular values of m , τ and λ , we can derive a number of known results. Some of them are given below.

(1) Taking $m = 1$ in Theorems 2.1, 2.3 and 2.5, we have the results obtained by Răducanu and Nechita [9, Theorem 3.1, Theorem 3.6, Theorem 3.9].

(2) Putting $m = \tau = 1$ in Theorems 2.1, 2.3 and 2.5, we get the results obtained by Răducanu and Nechita [9, Corollary 3.3, Corollary 3.8, Corollary 3.11].

(3) Setting $m = \tau = 1$ and $\lambda = 1$ in Theorem 2.1, we obtain the results obtained by Murugusundaramoorthy and Magesh [8, Corollary 3.3].

(4) Taking $m = \tau = 1$ and $\lambda = 1$ in Theorems 2.3 and 2.5, we have the results obtained by Răducanu and Nechita [9, Corollary 3.7, Corollary 3.10].

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