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Coefficient Bounds for Al-Oboudi Type Bi-univalent Functions based on a Modified Sigmoid Activation Function and Horadam Polynomials

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Abstract

Using the Al-Oboudi type operator, we present and investigate two special families of bi-univalent functions in $\mathfrak{D}$, an open unit disc, based on $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, a modified sigmoid activation function (MSAF) and Horadam polynomials. We estimate the initial coefficients bounds for functions of the type $g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j$ in these families. Continuing the study on the initial cosfficients of these families, we obtain the functional of Fekete-Szegö for each of the two families. Furthermore, we present few interesting observations of the results investigated.

1 Preliminaries

Let the set of complex numbers be denoted by $\mathbb{C}$ and the set of normalized regular functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ that have the power series of the form

$$g(z) = z + d_2 z^2 + d_3 z^3 + ... = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1.1}$$

be indicated by $\mathcal{A}$ and the set of all functions of $\mathcal{A}$ that are univalent in $\mathfrak{D}$ is symbolized by $\mathcal{S}$. The famous Koebe theorem (see [12]) ensures that any function

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$g \in \mathcal{S}$ has an inverse g^{-1} satisfying $z = g^{-1}(g(z))$, $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$ and $r_0(g) \geq 1/4$, $z, \omega \in \mathfrak{D}$, where

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \dots. \quad (1.2)$$

A function g of $\mathcal{A}$ is said to be bi-univalent (or bi-schlicht) in $\mathfrak{D}$ if g and its inverse g^{-1} are both univalent (or schlicht) in $\mathfrak{D}$. The set of bi-univalent functions having the form (1.1) is indicated by $\sum$. Historically investigations of the family $\sum$ begun five decades ago by Lewin [23] and Brannan et al. [9]. After few years, Tan [40] found the initial coefficient bounds of bi-univalent functions. Later, Brannan and Taha [10] presented and investigated certain subsets of $\sum$ similar to convex and starlike functions of order σ $(0 \leq \sigma < 1)$ in $\mathfrak{D}$. Some interesting results concerning initial bounds for certain special sets of $\sum$ have been appreared in [11], [18] and [32].

Let the set of real numbers be $\mathbb{R} = (-\infty, \infty)$ and the set positive integers be $\mathbb{N} := \mathbb{N}_0 \backslash \{0\} = \{1, 2, 3, ...\}$.

Recently, Hörzum and Koçer [21] (see also [20]) examined the Horadam polynomials $H_j(x)$ (or $H_j(x, a, b; p, q)$). It is given by the recurrence relation

$$H_j(x) = pxH_{j-1}(x) + qH_{j-2}(x), \; H_1(x) = a, \; H_2(x) = bx, \quad (1.3)$$

where $j \in \mathbb{N}\backslash\{1, 2\}$, $x \in \mathbb{R}$, p, q, a and b are real constants. It is easy to see from (1.3) that $H_3(x) = pbx^2 + qa$. The generating function of the sequence $H_j(x)$, $j \in \mathbb{N}$, is as below (see [21]):

$$\mathcal{G}(x, z) := \sum_{j=1}^{\infty} H_j(x)z^{j-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}, \quad (1.4)$$

where $z \in \mathbb{C}$ is independent of the argument $x \in \mathbb{R}$, that is $\Re(z) \neq x$.

Few particular cases of $H_j(x, a, b; p, q)$ are:

i) $H_j(x, 1, 1; 1, 1) = F_j(x)$, *ii)* $H_j(x, 1, 2; 2, -1) = U_j(x)$,
iii) $H_j(x, 1, 1; 2, -1) = T_j(x)$, *iv)* $H_j(x, 2, 1; 1, 1) = L_j(x)$,
v) $H_j(x, 2, 2; 2, 1) = Q_j(x)$ and *vi)* $H_j(x, 1, 2; 2, 1) = P_j(x)$.

They are named as Fibonacci polynomials, second type Chebyshev polynomials, first type Chebyshev polynomials, Lucas polynomials, Pell-Lucas polynomials and Pell polynomials, respectively.

In the literature, the estimates on $|d_2|$, $|d_3|$ and the famous inequality of Fekete-Szegö were determined for bi-univalent functions related to certain polynomials like Fibonacci polynomials, (p, q)-Lucas polynomials, second kind Chebyshev polynomials and Horadam polynomials. We also note that the above polynomials and other special polynomials are potentially important in statistical, physical, mathematical and engineering sciences. Additional information about these polynomials can be found in [7], [8], [16], [17], [24] and [42]. More details about the famous Fekete-Szegö problem connected with Haradam polynomials are available with the works of [1], [2], [3], [26], [31], [38] and [41].

The recent research trend is the study of bi-univalent functions linked with any one of the above mentioned polynomials using well-known operators, which can be seen in the research papers [4], [13], [25], [28], [34], [36], [37] and [39]. Generally interest was shown to estimate the initial Taylor-Maclaurin coefficients and the celebrated inequality of Fekete-Szegö for the special families of $\sum$ that are being introduced using known operators.

In this work, we present two special sets of $\sum$ using Al-Oboudi type operator which was precisely defined in the paper [19]. We determine the initial coefficient bounds and also obtain the relevant connection to the celebrated Fekete-Szegö functional for functions in the defined families.

Let $\mathcal{A}_\phi$ denote the set of regular functions of the form

$$g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j,$$

where $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, is a MSAF. Note that $\phi(0) = 1$ and hence $\mathcal{A}_1 := \mathcal{A}$ (see [14]).

Definition 1.1. For $g_\phi \in \mathcal{A}_\phi$, $k \in \mathbb{N}_0, \beta \geq 0$, an Al-Oboudi type operator D_β^k :

$\mathcal{A}_\phi \to \mathcal{A}_\phi$, is defined by

$$D_\beta^0 g_\phi(z) = g_\phi(z),\ D_\beta^1 g_\phi(z) = (1-\beta)g_\phi(z) + \beta z g'_\phi(z), ..., D_\beta^k g_\phi(z) = D_\beta(D_\beta^{k-1} g_\phi(z)),$$
$$z \in \mathfrak{D}.$$

Remark 1.1. If $g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j \in \mathcal{A}_\phi$, $z \in \mathfrak{D}$, then

$$D_\beta^k g_\phi(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\beta)^k \phi(s) d_j z^j,\ z \in \mathfrak{D}.$$

When $\phi(s) = 1$, we get the Al-Oboudi operator [5], which reduces to the Sălăgean operator [29], if $\beta = 1$.

For regular functions g and f in $\mathfrak{D}$, g is said to subordinate to f, if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0) = 0$, $|\psi(z)| < 1$ and $g(z) = f(\psi(z))$, $z \in \mathfrak{D}$. This subordination is indicated as $g \prec f$ or $g(z) \prec f(z)$. Specifically, when $f \in \mathcal{S}$ in $\mathfrak{D}$, then $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$.

Inspired by the articles [6], [33] and the trends on functions $\in \sum$, we present two special families of $\sum$ by using Al-Oboudi type operator, which is as in Definition 1.1 and Horadam polynomials $H_j(x)$ as in the relation (1.3) having the generating function (1.4).

Throughout this paper, $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2), p, q, a and b are as in (1.3) and $\mathcal{G}$ is as in (1.4), unless and otherwise mentioned.

Definition 1.2. A function g in $\sum$ having the power series (1.1) is said to be in the family $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$, $0 \le \gamma \le 1$, $\mu \ge 0$, $k \in \mathbb{N}_0$, $\beta \ge 0$ and $\phi(s)$ the MSAF, if

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2 (D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1-\gamma) z} \prec 1 - a + \mathcal{G}(x, z),\ z \in \mathfrak{D}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu \omega^2 (D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} \prec 1 - a + \mathcal{G}(x, \omega),\ \omega \in \mathfrak{D}.$$

We note that i) $\mu = 0$, ii) $\gamma = 0$ and iii) $\gamma = 1$ lead the family $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$ to the below mentioned subfamilies:

1. $SK_{\sum}(x, \gamma, k, \beta, \phi(s)) \equiv S\mathfrak{S}_{\sum}(x, \gamma, 0, k, \beta, \phi(s))$ is the set of functions $g \in \sum$ satisfying

$$\frac{z(D_\beta^k g_\phi(z))'}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} \prec 1-a+\mathcal{G}(x,z), \text{ and } \frac{\omega(D_\beta^k f_\phi(\omega))'}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} \prec 1-a+\mathcal{G}(x,\omega),$$

where $z, \omega \in \mathfrak{D}$.

2. $SL_{\sum}(x, \mu, k, \beta, \phi(s)) \equiv S\mathfrak{S}_{\sum}(x, 0, \mu, k, \beta, \phi(s))$ is the family of functions $g \in \sum$ satisfying

$$(D_\beta^k g_\phi(z))' + \mu z(D_\beta^k g_\phi(z))'' \prec 1 - a + \mathcal{G}(x,z)$$

and

$$(D_\beta^k f_\phi(\omega))' + \mu\omega(D_\beta^k f_\phi(\omega))'' \prec 1 - a + \mathcal{G}(x,\omega),$$

where $z, \omega \in \mathfrak{D}$.

3. $SM_{\sum}(x, \mu, k, \beta, \phi(s)) \equiv S\mathfrak{S}_{\sum}(x, 1, \mu, k, \beta, \phi(s))$ is the family of functions $g \in \sum$ satisfying

$$\left(\frac{z(D_\beta^k g_\phi(z))'}{D_\beta^k g_\phi(z)}\right) + \mu\left(\frac{z(D_\beta^k g_\phi(z))''}{D_\beta^k g_\phi(z)}\right) \prec 1 - a + \mathcal{G}(x,z)$$

and

$$\left(\frac{\omega(D_\beta^k f_\phi(\omega))'}{D_\beta^k f_\phi(\omega)}\right) + \mu\left(\frac{\omega(D_\beta^k f_\phi(\omega))''}{D_\beta^k f_\phi(\omega)}\right) \prec 1 - a + \mathcal{G}(x,\omega),$$

where $z, \omega \in \mathfrak{D}$.

Letting $k = 0$ and $\phi(s) = 1$ in the Definition 1.2, we obtain the family $SN_{\sum}(x, \gamma, \mu) \equiv S\mathfrak{S}_{\sum}(x, \gamma, \mu, 0, \beta, 1)$ of functions $g \in \sum$ satisfying

$$\frac{zg'(z) + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} \prec 1 - a + \mathcal{G}(x,z) \quad \text{and} \quad \frac{\omega f'(\omega) + \mu\omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} \prec 1 - a + \mathcal{G}(x,\omega),$$

where $z, \omega \in \mathfrak{D}$, $f(\omega) = g^{-1}(\omega)$ is as given by (1.2), a is as in (1.3) and $\mathcal{G}$ is as in (1.4).

Definition 1.3. A function $g \in \sum$ having the power series (1.1) is said to be in the family $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s))$, $0 \le \gamma \le 1$, $\tau \ge 1$, $k \in \mathbb{N}_0$, $\beta \ge 0$ and $\phi(s)$ the MSAF, if

$$\frac{z[(D_\beta^k g_\phi(z))']^\tau}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} \prec 1 - a + \mathcal{G}(x, z),\ z \in \mathfrak{D}$$

and

$$\frac{\omega[(D_\beta^k f_\phi(\omega))']^\tau}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} \prec 1 - a + \mathcal{G}(x, \omega),\ \omega \in \mathfrak{D}.$$

Note that the certain choices of γ lead the family $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s))$ to the following two subclasses:

1. $SP_{\sum}(x, \tau, k, \beta, \phi(s)) \equiv S\mathfrak{B}_{\sum}(x, 0, \tau, k, \beta, \phi(s))$ is the set of functions $g \in \sum$ satisfying

$$[(D_\beta^k g_\phi(z))']^\tau \prec 1-a+\mathcal{G}(x, z),\ z \in \mathfrak{D} \quad \text{and} \quad [(D_\beta^k f_\phi(\omega))']^\tau \prec 1-a+\mathcal{G}(x, \omega),\ \omega \in \mathfrak{D},$$

2. $S\mathfrak{N}_{\sum}(x, \tau, k, \beta, \phi(s)) \equiv S\mathfrak{B}_{\sum}(x, 1, \tau, k, \beta, \phi(s))$ is the class of functions $g \in \sum$ satisfying

$$\frac{z[(D_\beta^k g_\phi(z))']^\tau}{D_\beta^k g_\phi(z)} \prec 1-a+\mathcal{G}(x, z),\ z \in \mathfrak{D} \text{ and } \frac{\omega[(D_\beta^k f_\phi(\omega))']^\tau}{D_\beta^k f_\phi(\omega)} \prec 1-a+\mathcal{G}(x, \omega),\ \omega \in \mathfrak{D},$$

$S\mathfrak{N}_{\sum}(x, \tau, k, \beta, \phi(s))$ is the family of Al-Oboudi type τ-bi-pseudo-starlike functions associated with Horadam polynomials involving the MSAF.

On taking $k = 0$ and $\phi(s) = 1$ in Definition 1.3, we get the family $SQ_{\sum}(x, \gamma, \tau) \equiv S\mathfrak{B}_{\sum}(x, \gamma, \tau, 0, \beta, 1)$ of functions $g(z)$ in $\sum$ satisfying

$$\frac{z(g'(z))^\tau}{\gamma g(z) + (1-\gamma)z} \prec 1 - a + \mathcal{G}(x, z) \quad \text{and} \quad \frac{\omega(f'(\omega))^\tau}{\gamma f(\omega) + (1-\gamma)\omega} \prec 1 - a + \mathcal{G}(x, \omega),$$

where $z, \omega \in \mathfrak{D}$, $f(\omega) = g^{-1}(\omega)$ is as given by (1.2), a is as in (1.3) and $\mathcal{G}$ is as in (1.4).

Remark 1.2. We note that *i)* $S\mathfrak{B}_{\sum}(x, \gamma, 1, k, \beta, \phi(s)) \equiv SK_{\sum}(x, \gamma, k, \beta, \phi(s))$, ii) $S\mathfrak{N}_{\sum}(x, 1, k, \beta, \phi(s)) \equiv SK_{\sum}(x, 1, k, \beta, \phi(s)) \equiv SM_{\sum}(x, 0, k, \beta, \phi(s))$ and iii) $SP_{\sum}(x, 1, k, \beta, \phi(s)) \equiv SK_{\sum}(x, 0, k, \beta, \phi(s)) \equiv SL_{\sum}(x, 0, k, \beta, \phi(s))$.

Remark 1.3. i) For $\mu = \gamma = 0$, the class $SN_\Sigma(x, 0, 0) \equiv \mathcal{H}_\Sigma(x)$ was studied by Alamoush [2] and ii) For $\mu = 0$ and $\gamma = 1$, the family $SN_\Sigma(x, 1, 0) \equiv S^*_\Sigma(x)$ was investigated by Srivastava et al. [32].

Remark 1.4. i) For $\gamma = 0$, the family $SQ_\Sigma(x, 0, \tau) \equiv S^*_\Sigma(x, \tau)$ was investigated by Abirami et al. [1] and ii) For $\beta = 1$, the family $S\mathfrak{N}_{\sum}(x, \tau, k, 1, \phi(s)) \equiv M_\Sigma(x, \tau, k, \phi(s))$ was considered in [25].

Remark 1.5. In a special situation, if we choose $a = 1, b = p = 2, q = -1$ and $x \to t$, the generating function (1.4) reduces to the second type Chebyshev polynomials $U_j(t)$, which is explicitly given by

$$U_j(t) = (j+1)\ {}_2F_1\left(-j, j+2; \frac{3}{2}; \frac{1-t}{2}\right) = \frac{sin(j+1)\psi}{sin\psi},\ (t = sin\psi)$$

in terms of the Gauss hypergeometric function ${}_2F_1$. In this particular situation, the bi-univalent function families $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$ and $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s))$ would become the families $S\mathfrak{S}_{\sum}(t, \gamma, \mu, k, \beta, \phi(s))$ and $S\mathfrak{B}_{\sum}(t, \gamma, \tau, k, \beta, \phi(s))$, respectively. The families $S\mathfrak{B}_{\sum}(t, 1, \tau, 0, \beta, \phi(s)) \equiv AO_{\sum}(t, \tau, \phi(s))$ and $S\mathfrak{B}_{\sum}(t, 1, \tau, 0, \beta, 1) \equiv AY_{\sum}(t, \tau)$ were studied earlier in [8] and [7], respectively.

In Section 2, we derive the estimates for $|d_2|$, $|d_3|$ and the inequality of Fekete-Szegö [15] for functions of the form (1.1) $\in S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$ and we also present some observations of our result. In Section 3, we derive the estimates for $|d_2|$, $|d_3|$ and the Fekete-Szegö inequality for functions of the form (1.1) $\in S\mathfrak{B}_{\sum}(x, \gamma, \tau, k, \beta, \phi(s))$. Few interesting consequences of the result are also considered.

2 Estimates for Function Family $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$

In the following theorem, we determine the initial coefficients bounds and the inequality of Szegö for functions in $S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$.

Theorem 2.1. *Let $0 \leq \gamma \leq 1, \mu \geq 0, k \in \mathbb{N}_0, \beta \geq 0$ and $\phi(s)$ be the MSAF. If the function $g \in S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(bx)^2-(2(\mu+1)-\gamma)^2(pbx^2+qa)|}}, \tag{2.1}$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{(bx)^2}{(2(\mu+1)-\gamma)^2}+\frac{|bx|}{(3(2\mu+1)-\gamma)}\right] \tag{2.2}$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|bx|}{(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J \\ \frac{|bx|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(bx)^2-(2(\mu+1)-\gamma)^2(pbx^2+qa)|} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J, \end{cases} \tag{2.3}$$

where

$$J = \frac{1}{(3(2\mu+1)-\gamma)}\left|\gamma^2-(2\mu+3)\gamma+3(2\mu+1)-(2(\mu+1)-\gamma)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|. \tag{2.4}$$

Proof. Let $g \in S\mathfrak{S}_{\sum}(x, \gamma, \mu, k, \beta, \phi(s))$. Then, for two regular functions $\mathfrak{M}$, $\mathfrak{N}$ with $\mathfrak{M}(0) = 0, |\mathfrak{M}(z)| < 1, \mathfrak{N}(0) = 0$ and $|\mathfrak{N}(\omega)| < 1$, $z, \omega \in \mathfrak{D}$ and on account of Definition 1.2, we can write

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2(D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} = 1 - a + \mathcal{G}(x, \mathfrak{M}(z))$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu\omega^2(D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} = 1 - a + \mathcal{G}(x, \mathfrak{N}(\omega)).$$

Or, equivalently

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2(D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} = 1 - a + H_1(x) + H_2(x)\mathfrak{m}(z) + H_3(x)(\mathfrak{m}(z))^2 + ... \tag{2.5}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu\omega^2(D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} = 1-a+H_1(x)+H_2(x)\mathfrak{n}(\omega)+H_3(x)(\mathfrak{n}(\omega))^2+\dots. \tag{2.6}$$

From (2.5) and (2.6), in view of (1.3), we find

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2(D_\beta^k g_\phi(z))''}{\gamma D_\beta^k g_\phi(z) + (1-\gamma)z} = 1+H_2(x)\mathfrak{m}_1 z+[H_2(x)\mathfrak{m}_2+H_3(x)\mathfrak{m}_1^2]z^2+\dots \tag{2.7}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu\omega^2(D_\beta^k f_\phi(\omega))''}{\gamma D_\beta^k f_\phi(\omega) + (1-\gamma)\omega} = 1+H_2(x)\mathfrak{n}_1\omega+[H_2(x)\mathfrak{n}_2+H_3(x)\mathfrak{n}_1^2]\omega^2+\dots. \tag{2.8}$$

It is well known that if $|\mathfrak{M}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + \dots| < 1$, $z \in \mathfrak{D}$ and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + \dots| < 1$, $\omega \in \mathfrak{D}$, then

$$|\mathfrak{m}_i| \le 1 \,\text{and}\, |\mathfrak{n}_i| \le 1 \,(i \in \mathbb{N}). \tag{2.9}$$

We easily get the following by equating the corresponding coefficients in (2.7) and (2.8):

$$(1+\beta)^k\phi(s)\,(2(\mu+1)-\gamma)d_2 = H_2(x)\mathfrak{m}_1 \tag{2.10}$$

$$(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)d_3-(1+\beta)^{2k}\phi^2(s)(2(\mu+1)-\gamma)\gamma\, d_2^2 = H_2(x)\mathfrak{m}_2+H_3(x)\mathfrak{m}_1^2 \tag{2.11}$$

$$-(1+\beta)^k\phi(s)\,(2(\mu+1)-\gamma)\,d_2 = H_2(x)\mathfrak{n}_1 \tag{2.12}$$

$$-(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)d_3+(1+\beta)^{2k}\phi^2(s)(\gamma^2-2(\mu+2)\gamma+6(2\mu+1))d_2^2 = H_2(x)\mathfrak{n}_2+H_3(x)\mathfrak{n}_1^2. \tag{2.13}$$

From (2.10) and (2.12), we easily obtain

$$\mathfrak{m}_1 = -\mathfrak{n}_1 \tag{2.14}$$

and also

$$2(1+\beta)^{2k}\phi^2(s)(2(\mu+1)-\gamma)^2 d_2^2 = (\mathfrak{m}_1^2+\mathfrak{n}_1^2)(H_2(x))^2. \tag{2.15}$$

If we add (2.11) and (2.13), then we obtain

$$2(1+\beta)^{2k}\phi^2(s)(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))d_2^2 = H_2(x)(\mathfrak{m}_2+\mathfrak{n}_2)+H_3(x)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.16}$$

Substituting the value of $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (2.15) in (2.16), we get

$$d_2^2 = \frac{(H_2(x))^3(\mathfrak{m}_2+\mathfrak{n}_2)}{2(1+\beta)^{2k}\phi^2(s)\left[(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(h_2(x))^2-(2(\mu+1)-\gamma)^2h_3(x)\right]}, \tag{2.17}$$

which yields (2.1) on using (2.9).

After subtracting (2.13) from (2.11) and then using (2.14), we obtain

$$d_3 = \frac{(1+\beta)^{2k}\phi(s)}{(1+2\beta)^k}d_2^2 + \frac{H_2(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)}. \tag{2.18}$$

Then in view of (2.15), (2.18) becomes

$$d_3 = \frac{(H_2(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{2(1+2\beta)^k\phi(s)(2(\mu+1)-\gamma)^2} + \frac{H_2(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)},$$

which yields (2.2) on using (2.9).

From (2.17) and (2.18), for $\delta\in\mathbb{R}$, we get

$$|d_3-\delta d_2^2| = |H_2(x)|\left|\left(T(\delta,x)+\frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)}\right)\mathfrak{m}_2+\left(T(\delta,x)-\frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)}\right)\mathfrak{n}_2\right|,$$

where

$$T(\delta,x) = \frac{\left(\frac{(1+\beta)^{2k}\phi(s)}{(1+2\beta)^k}-\delta\right)(H_2(x))^2}{2(1+\beta)^{2k}\phi^2(s)\left[(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(H_2(x))^2-(2(\mu+1)-\gamma)^2H_3(x)\right]}.$$

In view of (1.3), we conclude that

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|H_2(x)|}{(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)} & ;\ 0\le|T(\delta,x)|\le\frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)} \\ 2|H_2(x)||T(\delta,x)| & ;\ |T(\delta,x)|\ge\frac{1}{2(1+2\beta)^k\phi(s)(3(2\mu+1)-\gamma)}, \end{cases}$$

which gets (2.3) with J as in (2.4). This evidently ends the proof of Theorem 2.1. □

By setting i) $\mu = 0$, ii) $\gamma = 0$, iii) $\gamma = 1$ and iv) $k = 0, \phi(s) = 1$ in Theorem 2.1, we have the following four corollaries, respectively.

Corollary 2.1. *If the function* $g \in SK_{\sum}(x, \gamma, k, \beta, \phi(s))$, *then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|(\gamma^2-3\gamma+3)(bx)^2-(2-\gamma)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{b^2x^2}{(2-\gamma)^2}+\frac{|bx|}{3-\gamma}\right]$$

and for some $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{(1+2\beta)^k\phi(s)(3-\gamma)} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J_1 \\ \frac{|bx|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|(\gamma^2-3\gamma+3)(bx)^2-(2-\gamma)^2(pbx^2+qa)|} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J_1, \end{cases}$$

where $J_1 = \frac{1}{(3-\gamma)}\left|\gamma^2-3\gamma+3-(2-\gamma)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Remark 2.1. For $\gamma = \beta = 1$, Corollary 2.1 reduce to Corollary 2.1 of Magesh et al. [25] and Corollary 2.1 further coincide with Corollary 2.1 of Abirami et al. [1], when $k = 0$ and $\phi(s) = 1$. Corollary 2.1 coincide with Theorem 2.2 of Alamoush [3], when $\gamma = k = 0$ and $\phi(s) = 1$ and also we obtain Corollary 1 and Corollary 3 of [31] for $k = 0$, $\gamma = \phi(s) = 1$.

Corollary 2.2. *If the function* $g \in SL_{\sum}(x, \gamma, k, \beta, \phi(s))$, *then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|3(2\mu+1)(bx)^2-4(\mu+1)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{b^2x^2}{4(\mu+1)^2}+\frac{|bx|}{3(2\mu+1)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu+1)(1+2\beta)^k\phi(s)} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J_2 \\ \frac{|bx|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|3(2\mu+1)(bx)^2-4(\mu+1)^2(pbx^2+qa)|} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J_2, \end{cases}$$

where $J_2 = \left|1-\frac{4(\mu+1)^2}{3(2\mu+1)}\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Remark 2.2. For $\mu = k = 0$ and $\phi(s) = 1$ Corollary 2.2 coincide with Theorem 2.2 of [3].

Corollary 2.3. *If the function* $g \in SM_{\sum}(x, \mu, k, \beta, \phi(s))$*, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{\left[|(4\mu+1)(bx)^2 - (2\mu+1)^2(pbx^2+qa)|\right]}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{b^2x^2}{(2\mu+1)^2} + \frac{|bx|}{2(3\mu+1)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{2(3\mu+1)(1+2\beta)^k\phi(s)} & ; \left|1 - \frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J_3 \\ \frac{|bx|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)[(4\mu+1)(bx)^2-(2\mu+1)^2(pbx^2+qa)]} & ; \left|1 - \frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J_3, \end{cases}$$

where $J_3 = \frac{1}{2(3\mu+1)}\left|(4\mu+1) - (2\mu+1)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Remark 2.3. Corollary 2.3 coincide with Theorem 2.1 of Magesh et al. [26], when $k = 0$ and $\phi(s) = 1$. Also we obtain Corollary 2.1 of [25] from Corollary 2.3, when $\mu = 0$ and $\beta = 1$.

Corollary 2.4. *If the function* $g(z) \in SN_{\sum}(x, \gamma, \mu)$*, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(\gamma^2 - (2\mu+3)\gamma + 3(2\mu+1))(bx)^2 - (2(\mu+1)-\gamma)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{(bx)^2}{(2(\mu+1)-\gamma)^2} + \frac{|bx|}{(3(2\mu+1)-\gamma)}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu+1)-\gamma} & ; |1-\delta| \leq J_4 \\ \frac{|bx|^3\,|1-\delta|}{|(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(bx)^2-(2(\mu+1)-\gamma)^2(pbx^2+qa)|} & ; |1-\delta| \geq J_4, \end{cases}$$

where

$$J_4 = \frac{1}{(3(2\mu+1)-\gamma)}\left|\gamma^2 - (2\mu+3)\gamma + 3(2\mu+1) - (2(\mu+1)-\gamma)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|.$$

Remark 2.4. By choosing appropriate values for parameters γ and μ in Corollary 2.4, we obtain Theorem 2.2, Theorem 2.1 and Corollaries 1, 2 of [3], [26] and [31], respectively, as it can be seen from earlier remarks.

3 Estimates for the Function Family $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k,\ \beta, \phi(s))$

In the next theorem, we find the first two Taylor-Maclaurin coefficients and the inequality of Fekete-Szegö for functions in $S\mathfrak{B}_{\sum}(x, \gamma, \tau, k,\ \beta, \phi(s))$.

Theorem 3.1. *Let* $0 \le \gamma \le 1$, $\tau \ge 1$, $k \in \mathbb{N}_0$, $\beta \ge 0$ *and* $\phi(s)$ *the MSAF. If the function* $g \in S\mathfrak{B}_{\sum}(x,\ \gamma,\ \tau, k,\ \beta, \phi(s))$, *then*

$$|d_2| \le \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k \phi(s)\sqrt{|(\gamma^2+(\tau-\gamma)(2\tau+1))(bx)^2-(2\tau-\gamma)^2(pbx^2+qa)|}}, \tag{3.1}$$

$$|d_3| \le \frac{1}{(1+2\beta)^k \phi(s)}\left[\frac{(bx)^2}{(2\tau-\gamma)^2}+\frac{|bx|}{(3\tau-\gamma)}\right] \tag{3.2}$$

and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|b(x)|}{(1+2\beta)^k\phi(s)(3\tau-\gamma)} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \le \Omega \\ \frac{\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right||bx|^3}{(1+2\beta)^k\phi(s)|(\gamma^2+(\tau-\gamma)(2\tau+1))(bx)^2-(2\tau-\gamma)^2(pbx^2+qa)|} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \ge \Omega, \end{cases} \tag{3.3}$$

where

$$\Omega = \frac{1}{(3\tau-\gamma)}\left|(\gamma^2+(\tau-\gamma)(2\tau+1))-(2\tau-\gamma)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|.$$

Proof. Let $g \in S\mathfrak{B}_{\sum}(x, \gamma,\ \tau, k,\ \beta, \phi(s))$. Then, for some regular functions $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M}(0) = 0, |\mathfrak{M}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + ...| < 1, \mathfrak{N}(0) = 0$ and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + ...| < 1$, $z, \omega \in \mathfrak{D}$ and on account of Definition 1.3, we can write

$$\frac{z[(D_\beta^k g_\phi(z))']^\tau}{\gamma D_\beta^k g_\phi(z)+(1-\gamma)z} = 1-a+\mathcal{G}(x,\mathfrak{M}(z)),\ z \in \mathfrak{D}$$

and

$$\frac{\omega[(D_\beta^k f_\phi(\omega))']^\tau}{\gamma D_\beta^k f_\phi(\omega)+(1-\gamma)\omega} = 1-a+\mathcal{G}(x,\mathfrak{N}(\omega)),\ \omega\ \in \mathfrak{D}.$$

Following the procedure similar to the proof of Theorem 2.1, one gets

$$(1+\beta)^k(2\tau-\gamma)\phi(s)d_2 = H_2(x)\mathfrak{m}_1 \tag{3.4}$$

$$(1+\beta)^{2k}\phi^2(s)(\gamma^2-2\tau\gamma+2\tau(\tau-1))d_2^2+(1+2\beta)^k\phi(s)(3\tau-\gamma)d_3 = H_2(x)\mathfrak{m}_2+H_3(x)\mathfrak{m}_1^2 \tag{3.5}$$

$$-(1+\beta)^k(2\tau-\gamma)\phi(s)d_2 = H_2(x)\mathfrak{n}_1 \tag{3.6}$$

$$\begin{aligned}&(1+\beta)^{2k}\phi^2(s)(\gamma^2-2(\tau+1)\gamma+2\tau(\tau+2))d_2^2-(1+2\beta)^k\phi(s)(3\tau-\gamma)d_3\\ &= H_2(x)\mathfrak{n}_2+H_3(x)\mathfrak{n}_1^2.\end{aligned} \tag{3.7}$$

The results (3.1)-(3.3) now follow from (3.4)-(3.7) by adopting the procedure as in Theorem 2.1. □

By setting i) $\gamma = 0$, ii) $\gamma = 1$ and iii) $k = 0$, $\phi(s) = 1$ in Theorem 3.1, we have the following three corollaries.

Corollary 3.1. *If the function $g \in SP_{\sum}(x, \tau, k, \beta, \phi(s))$, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|\tau(2\tau+1)(bx)^2-4\tau^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{(bx)^2}{4\tau^2}+\frac{|bx|}{3\tau}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leq \begin{cases}\frac{|b(x)|}{3\tau(1+2\beta)^k\phi(s)} & ;\ \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq \Omega_1\\ \frac{\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right||bx|^3}{(1+2\beta)^k\phi(s)|\tau(2\tau+1)(bx)^2-4\tau^2(pbx^2+qa)|} & ;\ \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq \Omega_1,\end{cases}$$

where $\Omega_1 = \frac{1}{3}\left|(2\tau+1)-4\tau\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Remark 3.1. Corollary 3.1 coincides with Theorem 2.1 of [3], when $k = 0$ and $\tau = \phi(s) = 1$.

Corollary 3.2. *If the function $g \in S\mathfrak{N}_{\sum}(x, \tau, k, \beta, \phi(s))$, then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{(1+\beta)^k\phi(s)\sqrt{|(\tau(2\tau-1))(bx)^2-(2\tau-1)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{(bx)^2}{(2\tau-1)^2}+\frac{|bx|}{(3\tau-1)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{(1+2\beta)^k \phi(s)(3\tau-1)} & ; \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \leq \Omega_2 \\ \frac{\left|1-\frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| |bx|^3}{(1+2\beta)^k \phi(s)|(\tau(2\tau-1))(bx)^2-(2\tau-1)^2(pbx^2+qa)|} & ; \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \geq \Omega_2, \end{cases}$$

where $\Omega_2 = \frac{1}{(3\tau-1)} \left|(\tau(2\tau-1)) - (2\tau-1)^2 \left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$.

Remark 3.2. Corollary 3.2 reduces to Theorem 2.1 of [25], when $\beta = 1$ and also the results of Corollary 3.2 coincide with Theorem 2.1 of Abirami et al. [1], when $k = 0$ and $\phi(s) = 1$.

Corollary 3.3. *If the function* $g \in SQ_{\sum}(x, \gamma, \tau)$, *then*

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(\gamma^2 + (\tau-\gamma)(2\tau+1))(bx)^2 - (2\tau-\gamma)^2(pbx^2+qa)|}},$$

$$|d_3| \leq \frac{(bx)^2}{(2\tau-\gamma)^2} + \frac{|bx|}{(3\tau-\gamma)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3\tau-\gamma} & ; |1-\delta| \leq \Omega_3 \\ \frac{|1-\delta||bx|^3}{|(\gamma^2+(\tau-\gamma)(2\tau+1))(bx)^2-(2\tau-\gamma)^2(pbx^2+qa)|} & ; |1-\delta| \geq \Omega_3, \end{cases}$$

where

$$\Omega_3 = \frac{1}{(3\tau-\gamma)} \left|(\gamma^2 + (\tau-\gamma)(2\tau+1)) - (2\tau-\gamma)^2 \left(\frac{pbx^2+qa}{b^2x^2}\right)\right|.$$

Remark 3.3. Corollary 3.3 reduces to Theorem 2.1 of [1], when $\gamma = 1$.

4 Conclusion

Two special families of holomorphic and bi-univalent (or bi-schlicht) functions are introduced by using Al-Oboudi type operator involving a modified sigmoid activation function associated with Horadam polynomials. Bounds of the first two coefficients $|d_2|$, $|d_3|$ and the celebrated Fekete-Szegö functional have been fixed for each of the two families. Through corollaries of our main results, we have highlighted many interesting new consequences.

The special families examined in this research paper using Al-Oboudi type operator could inspire further research related to other aspects such as families using q-derivative operator [22], [35], meromorphic bi-univalent function families associated with Al-Oboudi differential operator [30] and families using integro-differential operators [27].

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