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On Generating Tridiagonal Matrices of Generalized (s, t)-Pell, (s, t)-Pell Lucas and (s, t)-Modified Pell Sequences

S. Uygun

Department of Mathematics, Science and Art Faculty,
Gaziantep University, Campus, Gaziantep, Turkey
e-mail: suygun@gantep.edu.tr

Abstract

In this study, we define some tridigional matrices depending on two real parameters. By using the determinant of these matrices, the elements of (s, t)-Pell, (s, t)-Pell Lucas and (s, t)-modified Pell sequences with even or odd indices are generated. Then we construct the inverse matrices of these tridigional matrices. We also investigate eigenvalues of these matrices.

1 Introduction and Preliminaries

Special integer sequences are encountered in different branches of science, art, nature, the structure of our body. So it is a popular topic in applied mathematics. Two of the special integer sequences are the Pell and Pell Lucas sequences. By changing the initial conditions but preserving the recurrence relation the Pell Lucas sequence is obtained. The authors investigated some sum formulas for Pell numbers in [2]. Gulec and Taskara generalized the Pell and Pell Lucas numbers by using two parameters in [3]. The authors generalized the modified Pell sequence similarly in [4]. By the determinant of the tridiagonal matrix, the values of the Fibonacci and Lucas numbers are demonstrated in [8]. Feng gave Fibonacci identities via determinant of the special tridiagonal

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matrix in [9]. Seibert and Trojovsk studied the factorization of the Fibonacci and Lucas numbers using determinants of tridiagonal matrices in [10]. In [11], it is demonstrated that the permanents of some special tridiagonal matrices are equal to Fibonacci numbers. Falcon displayed some equalities of k-Fibonacci numbers with the determinant of some special matrices in [12]. Catarino obtained the n-th elements of k-Pell, k-Pell-Lucas, and modified k-Pell sequences by the determinants of some tridiagonal matrices in [13]. In [14], properties of hyperbolic generalized Pell numbers are studied. The recurrence relations for Pell, Pell Lucas and modified Pell sequences are $P_n = 2P_{n-1} + P_{n-2},\ P_0 = 0,\ P_1 = 1,$ $Q_n = 2Q_{n-1} + Q_{n-2},\ \ Q_0 = 2,\ Q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2},\ \ q_0 = 1,\ q_1 = 1$ for $n \geq 2$, respectively in [1]. There are some generalizations of these integer sequences. For example, the generalizations for Pell, Pell Lucas, and modified Pell sequences called (s,t)-Pell, (s,t)-Pell Lucas, (s,t)-modified Pell sequences are defined by the aid of the following recurrence relations respectively

$$\begin{aligned}
\wp_n(s,t) &= 2s\wp_{n-1}(s,t) + t\wp_{n-2}(s,t), \quad \wp_0(s,t) = 0, \wp_1(s,t) = 1, \\
\Re_n(s,t) &= 2s\Re_{n-1}(s,t) + t\Re_{n-2}(s,t), \quad \Re_0(s,t) = 2,\ \Re_1(s,t) = 2s, \\
\aleph_n(s,t) &= 2s\aleph_{n-1}(s,t) + t\aleph_{n-2}(s,t), \quad \aleph_0(s,t) = 1,\ \aleph_1(s,t) = s,
\end{aligned}$$

for $n \geq 2$ in [3, 4].

Some elements of (s,t)-Pell and (s,t)-Pell Lucas sequences are given in the following tables:

n	$(s,t)-$ Pell numbers
1	1
2	$2s$
3	$4s^2 + t$
4	$8s^3 + 4st$
5	$16s^4 + 12s^2t + t^2$
6	$32s^5 + 32s^3t + 6st^2$
7	$64s^6 + 80s^4t + 24s^2t^2 + t^3$

n	$(s,t)-$ Pell Lucas numbers
1	$2s$
2	$4s^2+2t$
3	$8s^3+6st$
4	$16s^4+16s^2t+2t^2$
5	$32s^5+40s^3t+10st^2$
6	$64s^6+96s^4t+36s^2t^2+2t^3$
7	$128s^7+224s^5t+112s^3t^2+14st^3$

The elements of (s,t)-modified Pell sequences are obtained by taking half of the (s,t)-Pell Lucas numbers.

Special integer sequences are obtained with special numerical choices for (s,t)-Pell and (s,t)-Pell Lucas, (s,t)-modified Pell sequences. For example, if $s=t=1$, then we get well-known Pell, Pell Lucas, modified Pell sequences. If $s=1$, $t=k$, then we get k-Pell, k-Pell Lucas, and modified k-Pell sequences.

Let us consider a tridiagonal matrix as

$$A_n=\begin{bmatrix} a & b & & & & & \\ c & d & e & & & & \\ & c & d & e & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & & c & d & e \\ & & & & & c & d \end{bmatrix}.$$

Then

$$\det A_n = d\det A_{n-1} - ce\det A_{n-2}. \tag{1}$$

The inverse of a matrix A can be obtained by the formula $A^{-1}=\frac{(cof(A))^T}{\det A}$, where $(cof(A))^T$ is the transpose of the cofactor matrix A or adjugate matrix of A [4]. Let T be a nonsingular tridiagonal matrix as

$$T = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}.$$

Usmani [6] gave a formula for the inverse of this matrix $T^{-1} = (t_{i,j})$ as

$$t_{i,j} = \begin{cases} (-1)^{i+j} \frac{1}{\theta_n} b_i ... b_{j-1} \theta_{i-1} \phi_{j+1} & \text{if } i \leq j \\ (-1)^{i+j} \frac{1}{\theta_n} c_j ... c_{i-1} \theta_{j-1} \phi_{i+1} & \text{if } i > j \end{cases} \quad (2)$$

where

- θ_i verify the recurrence relation $\theta_i = a_i \theta_{i-1} - b_{i-1} c_{i-1} \theta_{i-2}$ for $i = 2, ...n$, with the initial conditions $\theta_0 = 1$ and $\theta_1 = a_1$. Observe that $\theta_n = \det(T)$.

- ϕ_i verify the recurrence relation $\phi_i = a_i \phi_{i+1} - b_i c_i \phi_{i+2}$ for $i = n-1, ...1$, with the initial conditions $\phi_{n+1} = 1$ and $\phi_n = a_n$.

In [7], if the tridiagonal matrix is given in the following form

$$T = \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & c & a & \ddots & \\ & & \ddots & \ddots & b \\ & & & c & a \end{bmatrix},$$

then the eigenvalues of this matrix are

$$\lambda_r = a + 2\sqrt{bc} \cos(\frac{r\pi}{n+1}), \qquad r = 1, 2, ..., n. \quad (3)$$

1.1 Some properties of tridiagonal matrices $A_{p,n}$ by (s,t)-Pell sequence

Theorem 1. *Assume that $A_{p,n}$ is an $n \times n$ tridiagonal matrix defined as*

$$A_{p,n} = \begin{bmatrix} 2s & t & & & & \\ -1 & 2s & t & & & \\ & -1 & 2s & & & \\ & & & & \ddots & \\ & & & \ddots & \ddots & t \\ & & & & -1 & 2s \end{bmatrix}.$$

Then the determinant of $A_{p,n}$ is

$$\det(A_{p,n}) = \wp_{n+1}.$$

Proof. The proof is made by induction applied on n. For $n = 1$, we have $\det(A_{p,1}) = \wp_2 = 2s$. Assume that $\det(A_{p,n-1}) = \wp_n$, $\det(A_{p,n}) = \wp_{n+1}$ for $n > 2$. Then

$$\begin{aligned} \det(A_{p,n+1}) &= 2s\det(A_{p,n}) - (-t)\det(A_{p,n-1}) \\ &= 2s\wp_{n+1} + t\wp_n = \wp_{n+2}. \end{aligned}$$

■

(s,t)-Pell sequence is also obtained by using the following tridiagonal matrix with complex entries. Assume that A_n is an $n \times n$ matrix defined as

$$A_n = \begin{bmatrix} 2s & it & & & & \\ i & 2s & it & & & \\ & i & 2s & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & it \\ & & & & i & 2s \end{bmatrix}.$$

Then it is easily seen that the determinant of A_n is also $(n+1)$th element of the (s,t)-Pell sequence

$$\det(A_n) = \wp_{n+1}.$$

For the inverse of $A_{p,n}$, by using (2), it is obtained that

$$\begin{aligned} a_i &= 2s, \quad b_i = t, \quad c_i = -1, \\ \theta_i &= \wp_{i+1}, \quad \phi_j = \wp_{n-j+2}. \end{aligned}$$

Therefore the inverse of $A_{p,n}$ is displayed by

$$(A_{p,n}^{-1})_{(i,j)} = \begin{cases} (-1)^{i+j} t^{j-i} \wp_i \wp_{n-j+1} \frac{1}{\wp_{n+1}}, & \text{if} \quad i \leq j \\ \wp_j \wp_{n-i+1} \frac{1}{\wp_{n+1}}, & \text{if} \quad i > j \end{cases}.$$

The elements of the cofactor matrix are given as

$$cof(A_{p,n})_{(i,j)} = \begin{cases} \wp_i \wp_{n-j+1}, & \text{if} \quad i < j \\ (-1)^{i+j} t^{i-j} \wp_j \wp_{n-i+1}, & \text{if} \quad i \geq j \end{cases}.$$

It is well-known that $|cof(A_{p,n})| = |adj(A_{p,n})| = |A_{p,n}|^{n-1} = \wp_{n+1}^{n-1}$. By using cofactor matrix, we get some properties of (s,t)-Pell sequence. For $n = 2$, we get

$$|cof(A_{p,2})| = \begin{vmatrix} \wp_1 \wp_2 & \wp_1 \wp_1 \\ -t \wp_1 \wp_1 & \wp_2 \wp_1 \end{vmatrix}$$

$\wp_2^2 + t \wp_1^2 = \wp_3$. For $n = 3$, we get

$$|cof(A_{p,3})| = \begin{vmatrix} \wp_1 \wp_3 & \wp_1 \wp_2 & \wp_1 \wp_1 \\ -t \wp_1 \wp_2 & \wp_2 \wp_2 & \wp_2 \wp_1 \\ t^2 \wp_1 \wp_1 & -t \wp_2 \wp_1 & \wp_3 \wp_1 \end{vmatrix}$$

$$\begin{aligned} |cof(A_{p,3})| &= \wp_1 \wp_3 (\wp_2^2 \wp_3 \wp_1 + t \wp_2^2 \wp_1^2) \\ &\quad + t \wp_1 \wp_2 (\wp_1^2 \wp_3 \wp_2 + t \wp_1^3 \wp_2) + t^2 \wp_1^2 (\wp_1^2 \wp_2^2 - \wp_1^2 \wp_2^2) \\ &= \wp_1^2 \wp_2^2 \wp_3 (\wp_3 + t \wp_1) + t \wp_1^3 \wp_2^2 (\wp_3 + t \wp_1) \\ &= (\wp_3 + t \wp_1)^2 \wp_1^2 \wp_2^2 \\ &= \wp_1^2 (\wp_3 + t \wp_1)^2 (\frac{\wp_3 - t \wp_1}{2s})^2 \\ \wp_4^2 &= (\frac{\wp_3^2 - t^2 \wp_1^2}{s})^2. \end{aligned}$$

For $n = 4$, we get

$$|cof(A_{p,4})| = \begin{vmatrix} \wp_1\wp_4 & \wp_1\wp_3 & \wp_1\wp_2 & \wp_1\wp_1 \\ -t\wp_1\wp_3 & \wp_2\wp_3 & \wp_2\wp_2 & \wp_2\wp_1 \\ t^2\wp_1\wp_2 & -t\wp_2\wp_2 & \wp_3\wp_2 & \wp_3\wp_1 \\ -t^3\wp_1\wp_1 & t^2\wp_2\wp_1 & -t\wp_3\wp_1 & \wp_4\wp_1 \end{vmatrix}$$

$$\wp_5^3 = (t\wp_2^2 + \wp_3^2)(t\wp_1\wp_3 + \wp_2\wp_4)^2.$$

Eigenvalues of the matrices $A_{p,n}$ construct the spectra of the $A_{p,n}$. By using the property (3), the sequence of the spectra of $A_{p,n}$ for $n = 1, 2, 3, 5$ is

$n = 1 \Rightarrow \lambda_r = \{2s\}$

$n = 2 \Rightarrow \lambda_r = \{2s + i\sqrt{t},\ 2s - i\sqrt{t}\}$

$n = 3 \Rightarrow \lambda_r = \{2s + i\sqrt{2t},\ 2s - i\sqrt{2t}, 2s\}$

$n = 5 \Rightarrow \lambda_r = \{2s + i2\sqrt{t},\ 2s + i\sqrt{3t}, 2s,\ 2s - i2\sqrt{t},\ 2s - i\sqrt{3t}\}$.

If $s = t = 1$, the sequence of the spectra of the matrices $A_{p,n}$ for $n = 2, 3, 4, 5, 6$ is computed by using the Matlab Program as

$S_2 = \{2 + i,\ 2 - i\}$,

$S_3 = \{2 + \sqrt{2}i,\ 2,\ 2 - \sqrt{2}i\}$,

$$S_4 = \left\{ \begin{array}{c} 2 + 1.618033988749896i,\ 2 + 0.618033988749895i, \\ 2 - 0.618033988749895i,\ 2 - 1.618033988749896i \end{array} \right\},$$

$$S_5 = \left\{ \begin{array}{c} 2 + 1.732050807568879i,\ 2 - 1.732050807568879i \\ 2,\ 2 + i,\ 2 - i \end{array} \right\},$$

$$S_6 = \left\{ \begin{array}{c} 2 + 1.801937735804838i,\ 2 - 1.801937735804838i, \\ 2 + 0.445041867912629i,\ 2 - 0.445041867912629i, \\ 2 + 1.246979603717468i,\ 2 - 1.246979603717468i \end{array} \right\}.$$

Evidently, the product of eigenvalues is the determinant of the matrix and the sum of eigenvalues is the trace of the matrix. Therefore

$$\sum_{i=1}^{n} \lambda_i = tr(A_{p,n}) = 2sn$$

and

$$\prod \lambda_i = \det(A_{p,n}) = \wp_{n+1} = \prod_{j=1}^{n} (2s + \sqrt{2}i\cos(\frac{\pi j}{n+1})).$$

1.2 Some properties of tridiagonal matrices $E_{p,n}$ by even (s,t)-Pell sequence

Assume that $E_{p,n}(p)$ is an $n \times n$ tridiagonal matrix defined as

$$E_{p,n} = \begin{bmatrix} 2s & 0 & & & & \\ t & 4s^2+2t & t & & & \\ & t & 4s^2+2t & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & t \\ & & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $E_{p,n}$ is computed by (1) as

$$\det E_{p,n} = \wp_{2n}.$$

For the inverse of $E_{p,n}$, the values in (2) are computed as

$$\begin{aligned} a_1 &= 2s, \\ a_i &= 4s^2+2t, \quad i \geq 2 \\ b_i &= t, \quad i > 1 \\ b_1 &= 0, \\ c_i &= t, \\ \theta_i &= \wp_{2i}, \quad i \geq 1 \\ \phi_j &= \frac{1}{2s}\wp_{2(n-j+2)}, \quad j \geq 1. \end{aligned}$$

Therefore the inverse of $E_{p,n}$ is displayed as

$$(E_{p,n}^{-1})_{(i,j)} = \begin{cases} 0, & \text{if} \quad i=1 \\ (-1)^{i+j} t^{j-i} \wp_{2i-2} \wp_{2(n-j+1)} \frac{1}{2s\wp_{2n}}, & \text{if} \quad i \leq j \\ (-1)^{i+j} t^{i-j} \wp_{2j-2} \wp_{2(n-i+1)} \frac{1}{2s\wp_{2n}}, & \text{if} \quad i > j \end{cases}.$$

If all entries of the matrix are real and nonnegative, then the matrix is called positive. All eigenvalues are real if the matrix positive and tridiagonal [4].

Therefore all eigenvalues of $E_{p,n}$ are real if $s,t \geq 0$. If we choose $s = t = 1$, then the sequence of the spectra of the matrix $E_{p,n}$ for $n = 2,3,4,5,6$ is given in the following result with the help of the Matlab program

$S_2 = \{2, 6\}$

$S_3 = \{2,\ 5,\ 7\}$

$S_4 = \{2,\ 4.5857864376269042,\ 6,\ 7.414213562373092\}$

$$S_5 = \left\{ \begin{array}{c} 2, 4.3819660112501022,\ 5.381966011250107, \\ 6.618033988749892,\ 7.618033988749891 \end{array} \right\}$$

$S_6 = \{2,\ 4.267949192431121,\ 5,\ 6,\ 7,\ 7.732050807568871\}$.

Evidently

$\sum \lambda_i = tr(E_{p,n}) = (n-1)(4s^2 + 2t) + 2s$ and $\prod \lambda_i = \det(E_{p,n}) = \wp_{2n}$.

If we take care of the spectra, one of the eigenvalues is $2 = 2s$ if $s = t = 1$ for all positive integer n. And the minimum eigenvalue of spectra converges to $2 = 2s$, the maximum eigenvalue of spectra converges to $4s^2 + 4t$.

1.3 The properties of tridiagonal matrices $O_{p,n}$ by odd (s,t)-Pell sequence

Assume that $O_{p,n}$ is an $n \times n$ tridiagonal matrix defined as

$$O_{p,n} = \begin{bmatrix} 4s^2+t & t & & & \\ t & 4s^2+2t & t & & \\ & t & 4s^2+2t & \ddots & \\ & & \ddots & \ddots & t \\ & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $O_{p,n}$ is given by (1) as

$$\det O_{p,n} = \wp_{2n+1}.$$

For the inverse of $O_{p,n}$, the values are computed by (2) as

$$\begin{aligned} a_1 &= 4s^2 + t, \\ a_i &= 4s^2 + 2t, \quad i \geq 2 \\ b_i &= c_i = t, \quad i \geq 1 \\ \theta_i &= \wp_{2i+1}, \quad i \geq 1 \\ \phi_j &= \frac{1}{2s}\wp_{2(n-j+2)}, \quad j \geq 1. \end{aligned}$$

Therefore the inverse of $O_{p,n}$ is given by

$$(O_{p,n}^{-1})_{(i,j)} = \begin{cases} (-1)^{i+j} t^{j-i} \wp_{2i-1} \wp_{2(n-j+1)} \frac{1}{2s\wp_{2n+1}}, & \text{if } \quad i \leq j \\ (-1)^{i+j} t^{i-j} \wp_{2j-1} \wp_{2(n-i+1)} \frac{1}{2s\wp_{2n+1}}, & \text{if } \quad i > j \end{cases}.$$

Matrices $O_{p,n}$ are symmetric so the eigenvalues are real. If $s = t = 1$, the sequence of the spectra of the matrices $O_{p,n}$ for $n = 2, 3, 4, 5, 6$ is given in the following:

$S_2 = \{4.381966011250105,\ 6.618033988749895\}$

$S_3 = \{4.198062264195162,\ 5.554958132087372,\ 7.246979603717467\}$

$S_4 = \{4.120614758428182,\ 5,\ 6.347296355333861,\ 7.532088886237956\}$

$$S_5 = \left\{ \begin{array}{c} 4.081014052771005,\ 4.690278532109430,\ 5.715370323453431, \\ 6.830830026003771,\ 7.682507065662362 \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{c} 4.058116365147897,\ 4.502978503657798,\ 5.290790225914926, \\ 6.241073360510647,\ 7.136129493462313,\ 7.770912051306419 \end{array} \right\}.$$

Then

$\sum \lambda_i = tr(O_{p,n}) = (n-1)(4s^2 + 2t) + (4s^2 + t) = n(4s^2 + 2t) - t$

$\prod \lambda_i = \det(O_{p,n}) = \wp_{2n+1}$.

If we take care of the spectra, minimum eigenvalue converges to $4s^2$. The maximum eigenvalue of spectra converges to $4s^2 + 4t$.

Theorem 2. *If λ_i is an eigenvalue of the matrix $O_{p,n}$, then $\lambda_i + 8s + 4$ is an eigenvalue of $O_{p,n}(s+1)$.*

Proof. λ_i is an eigenvalue of $O_{p,n}$, then

$$\begin{aligned}
&|O_{p,n} - \lambda_i I| \\
= &\begin{bmatrix} 4(s+1)^2 + t - (\lambda_i + 8s + 4) & t & & & \\ t & \ddots & & & \\ & \ddots & & & \\ & & & & t \\ & & & t & 4(s+1)^2 + 2t - (\lambda_i + 8s + 4) \end{bmatrix} \\
= &\left| O_{p,n}(s+1) - (\lambda_i + 8s + 4)I \right|.
\end{aligned}$$

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1.4 Some properties of tridiagonal matrices $A_{Q,n}$ by (s,t)-Pell Lucas sequence

Assume that $A_{Q,n}$ is an $n \times n$ tridiagonal matrix defined as

$$A_{Q,n} = \begin{bmatrix} 2s & 2t & & & \\ -1 & 2s & t & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & & t \\ & & & -1 & 2s \end{bmatrix}.$$

Then the determinant of $A_{Q,n}$

$$\det A_{Q,n} = \Re_n.$$

(s,t)-Pell Lucas sequence is also obtained by using the following symmetric matrix with complex entries. Assume that A_n is an $n \times n$ tridiagonal matrix defined as

$$A_n = \begin{bmatrix} 2s & 2it & & & & \\ i & 2s & it & & & \\ & i & 2s & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & it \\ & & & & i & 2s \end{bmatrix}.$$

Then the determinant of A_n is given by (1) as

$$\det A_n = \Re_n.$$

The sequence of the spectra of the matrices $A_{Q,n}$ for $n = 2,3,4,5,6$ is given in the following:

$S_2 = \{2 + 1.414213562373095i,\ 2 - 1.414213562373095i\}$

$S_3 = \{2,\ 2 + 1.732050807568877i,\ 2 + 1.732050807568877i\}$

$$S_4 = \left\{ \begin{array}{c} 2 + 1.847759065022573i,\ 2 - 1.847759065022573i \\ 2 + 0.765366864730179i,\ 2 - 0.765366864730179i \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 2 + 1.902113032590306i,\ 2 - 1.902113032590306i \\ ,2,\ 2 + 1.175570504584946i,\ 2 - 1.175570504584946i \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{cc} 2 + 1.931851652578137i, & 2 - 1.931851652578137i, \\ 2 + 1.414213562373096i, & 2 - 1.414213562373096i, \\ 2 + 0.517638090205041i, & 2 - 0.517638090205041i \end{array} \right\}.$$

Evidently, $\sum \lambda_i = tr(A_{Q,n}) = 2sn$ and $\prod \lambda_i = \det(A_{Q,n}) = \Re_n$.

For the inverse of $A_{Q,n}$, by using (2), it is obtained that

$$\begin{aligned} a_i &= 2s, \quad i > 0 \\ b_1 &= 2t, \quad b_i = t, \qquad i > 1 \\ c_i &= -1, \quad i > 0 \\ \theta_0 &= 1, \quad \theta_i = \Re_i, \qquad i > 1 \\ \phi_j &= \wp_{n-j+2}, \quad j > 0. \end{aligned}$$

Therefore the inverse of $A_{Q,n}$ is the following matrix

$$(A_{Q,n}^{-1})_{(i,j)} = \begin{cases} (-1)^{i+j} t^{j-i+1} \Re_{i-1} \wp_{n-j+1}(x) \frac{1}{\Re_n}, & \text{if} \quad i \le j \\ (-1)^{1+j} t^j \frac{\wp_{n-j+1}}{\Re_n}, & \text{if} \ \ i=1, \ j>1 \\ \frac{\wp_n}{\Re_n} & \text{if} \quad i=j=1 \\ \frac{\wp_{n-i+1}}{\Re_n}, & \text{if} \quad j=1, \ i>1 \\ \Re_{j-1} \wp_{n-i+1} \frac{1}{\Re_n}, & \text{if} \quad i>j \end{cases}.$$

1.5 Some properties of tridiagonal matrices $E_{Q,n}$ by even (s,t)-Pell Lucas sequence

Assume that $E_{Q,n}$ is a $n \times n$ tridiagonal matrix defined as

$$E_{Q,n} = \begin{bmatrix} 4s^2+2t & 2t & & & \\ t & 4s^2+2t & t & & \\ & t & \ddots & \ddots & \\ & & \ddots & & t \\ & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $E_{Q,n}$ is given by (1)

$$\det E_{Q,n} = \Re_{2n}.$$

For the inverse of $E_{Q,n}$, the values are computed as,

$$\begin{aligned} a_i &= 4s^2+2t, \quad i>0 \\ b_1 &= 2t, \ b_i = t, \ i>1 \\ c_i &= t, \quad i \ge 1 \\ \theta_0 &= 1, \quad \theta_i = \Re_{2i}, \\ \phi_j &= \frac{\wp_{2(n-j+2)}}{2s}, \qquad j \ge 1 \end{aligned}$$

where

$$(E_{Q,n}^{-1}) = \begin{cases} (-1)^{i+j} \frac{1}{2s\Re_{2n}} t^{j-i} \Re_{2i-2} \wp_{2(n-j+1)} & \text{if} \ \ i \le j \\ (-1)^{i+j} \frac{1}{2s\Re_{2n}} t^{j-i} \Re_{2j-2} \wp_{2(n-i+1)} & \text{if} \ \ i>j \end{cases}.$$

The sequence of the spectra of the matrices $E_{Q,n}$ for $n = 2, 3, 4, 5, 6$, are given in the following

$S_2 = \{4.585786437626905,\ 7.414213562373095\}$

$S_3 = \{4.267949192431122, 5.999999999999997, 7.732050807568876\}$

$$S_4 = \left\{ \begin{array}{c} 4.152240934977428,\ 5.234633135269817, \\ 7.847759065022574,\ 6.765366864730179 \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 4.097886967409691,\ 4.824429495415054,\ 6, \\ 7.175570504584945,\ 7.902113032590311 \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{c} 4.068148347421865,\ 4.585786437626904, \\ 5.482361909794957,\ 6.517638090205042, \\ 7.414213562373094,\ 7.931851652578145 \end{array} \right\}.$$

Then, $\sum \lambda_i = tr(E_{Q,n}) = n(4s^2 + 2t)$ and $\prod \lambda_i = \det(E_{Q,n}) = \Re_{2n}$.

If we take care of the spectra, minimum eigenvalue converges to $4s^2$. The maximum eigenvalue of spectra converges to $4s^2 + 4t$.

1.6 Some properties of tridiagonal matrices $O_{Q,n}$ by odd (s,t)-Pell Lucas sequence

Assume that $O_{Q,n}$ is an $n \times n$ tridiagonal matrix defined as

$$O_{Q,n} = \begin{bmatrix} 2s & -2s & & & \\ t & 4s^2+2t & t & & \\ & t & 4s^2+2t & \ddots & \\ & & \ddots & \ddots & t \\ & & & t & 4s^2+2t \end{bmatrix},$$

then the determinant of $O_{Q,n}$

$$\det O_{Q,n} = \Re_{2n-1}.$$

For the inverse of $O_{Q,n}$, the values are computed by (2)

$$\begin{aligned} a_1 &= 2s, \\ a_i &= 4s^2+2t,\ i\geq 2 \\ c_1 &= -2s,\quad b_1=t \\ b_i &= c_i=t,\quad i>1 \\ \theta_0 &= 1,\quad \theta_i=\Re_{2i-1},\quad i\geq 1 \\ \phi_j &= \frac{\wp_{2(n-j+2)}}{2s},\qquad j\geq 1. \end{aligned}$$

If $s=t=1$, then the sequence of the spectra of the matrices $O_{Q,n}$ for $n=2,3,4,5,6$ is given in the following

$S_2=\{2.585786437626905,\ 5.414213562373095\}$

$S_3=\{2.657076917222828,\ 4.529316580128842,\ 6.813606502648331\}$

$$S_4=\left\{\begin{array}{c} 2.665585781661023,\ 4.258036215697276, \\ 5.741963784302725,\ 7.334414218338978 \end{array}\right\}$$

$$S_5=\left\{\begin{array}{c} 2.666546286933180,\ 4.149626499380563,\ 5.132659714556408 \\ 6.474094390850837,\ 7.577073108279015 \end{array}\right\}$$

$$S_6=\left\{\begin{array}{c} 2.666653286578264,\ 4.096987803756087, \\ 4.780438656766319,\ 5.834296887585696, \\ 6.913255181694661,\ 7.708368183618974 \end{array}\right\}.$$

The sequence of maximum eigenvalue is increasing and converges to $4s^2+4t=8$. So we can say $\lim_{n\to\infty}\max(\lambda(O_n(k))=4s^2+4t.$

Evidently, $\sum\lambda_i=tr(O_{Q,n})=(n-1)(4s^2+2t)+2s$ and $\prod\lambda_i=\det(O_{Q,n})=\Re_{2n-1}$.

1.7 Some properties of tridiagonal matrices $A_{q,n}$ by (s,t)-Modified Pell sequence

Assume that $A_{q,n}$ is an $n\times n$ tridiagonal matrix defined as

$$A_{q,n} = \begin{bmatrix} s & t & & & \\ -1 & 2s & t & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & & t \\ & & & -1 & 2s \end{bmatrix},$$

then the determinant of $A_{q,n}$ is

$$\det A_{q,n} = \aleph_n.$$

(s,t)-modified Pell sequence is also obtained by using the following symmetric matrix with complex entries. Assume that A_n is an $n \times n$ tridiagonal matrix defined as

$$A_n = \begin{bmatrix} s & it & & & & \\ i & 2s & it & & & \\ & i & 2s & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & it \\ & & & & i & 2s \end{bmatrix}.$$

Then the determinant of A_n is given by (1) as

$$\det A_n = \aleph_n.$$

The sequence of the spectra of the matrices $A_{q,n}$ for $n = 2, 3, 4, 5, 6$ is given in the following:

$S_2 = \{1.5 + 0.866025403784438i,\ 1.5 - 0.866025403784438i\}$

$$S_3 = \left\{ \begin{array}{c} 1.430159709001947,\ 1.784920145499028 + 1.307141278682045i, \\ 1.784920145499028 - 1.307141278682045i \end{array} \right\}$$

$$S_4 = \left\{ \begin{array}{c} 1.895123382259650 + 1.552491820061880i, \\ 1.895123382259650 - 1.552491820061880i, \\ 1.604876617740350 + 0.506843901805983i, \\ 1.604876617740350 - 0.506843901805983i \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 1.941818622615235 + 1.691279149514195i, \\ 1.941818622615235 - 1.691279149514195i \\ 1.583716458826265, \\ 1.766323147971634 + 0.885556760232166i, \\ 1.766323147971634 - 0.885556760232166i \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{c} 1.964532008416013 + 1.775303570695105i, \\ 1.964532008416013 - 1.775303570695105i, \\ 1.678391681959849 + 0.359079022670333i, \\ 1.678391681959849 - 0.359079022670333i, \\ 1.857076309624141 + 1.159515656334833i, \\ 1.857076309624141 - 1.159515656334833i \end{array} \right\}.$$

Evidently, $\sum \lambda_i = tr(A_{q,n}) = 2sn - s$ and $\prod \lambda_i = \det(A_{q,n}) = \aleph_n$.

For the inverse of $A_{q,n}$, by using (2), it is obtained that

$$\begin{aligned} a_1 &= s, \quad a_i = 2s, \quad i > 1 \\ b_i &= t, \qquad i > 0 \\ c_i &= -1, \quad i > 0 \\ \theta_0 &= 1, \quad \theta_i = \aleph_i, \qquad i > 1 \\ \phi_j &= \wp_{n-j+2}, \quad j > 0. \end{aligned}$$

Therefore, the inverse of $A_{q,n}$ is the following matrix

$$(A_{q,n}^{-1})_{(i,j)} = \begin{cases} (-1)^{i+j} t^{j-i+1} \aleph_{i-1} \wp_{n-j+1}(x) \frac{1}{\aleph_n}, & \text{if} \quad i \le j \\ (-1)^{1+j} t^j \frac{\wp_{n-j+1}}{\aleph_n}, & \text{if} \ \ i = 1, \ j > 1 \\ \frac{\wp_n}{\aleph_n} & \text{if} \quad i = j = 1 \\ \frac{\wp_{n-i+1}}{\aleph_n}, & \text{if} \ \ j = 1, \ i > 1 \\ \aleph_{j-1} \wp_{n-i+1} \frac{1}{\aleph_n}, & \text{if} \quad i > j \end{cases}.$$

1.8 Some properties of tridiagonal matrices $E_{q,n}$ by even (s,t)-Modified Pell sequence

Assume that $E_{q,n}$ is an $n \times n$ tridiagonal matrix defined as

$$E_{q,n} = \begin{bmatrix} 2s^2+t & t & & & \\ t & 4s^2+2t & t & & \\ & t & \ddots & \ddots & \\ & & \ddots & & t \\ & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $E_{q,n}$ is given by (1)

$$\det E_{q,n} = \aleph_{2n}.$$

For the inverse of $E_{q,n}$, the values are computed by (2)

$$\begin{aligned} a_1 &= 2s^2+t,\ a_i = 4s^2+2t, \quad i>1 \\ b_i &= t,\ i>0 \\ c_i &= t, \quad i \geq 1 \\ \theta_0 &= 1, \quad \theta_i = \aleph_{2i}, \\ \phi_j &= \frac{\wp_{2(n-j+2)}}{2s}, \quad j \geq 1. \end{aligned}$$

Therefore, we get

$$(E_{q,n})^{-1}{}_{i,j} = \begin{cases} (-1)^{i+j} \frac{1}{2s\aleph_{2n}} t^{j-i} \aleph_{2i-2} \wp_{2(n-j+1)} & \text{if } i \leq j \\ (-1)^{i+j} \frac{1}{2s\aleph_{2n}} t^{j-i} \aleph_{2j-2} \wp_{2(n-i+1)} & \text{if } i > j \end{cases}.$$

The sequence of the spectra of the matrices $E_{q,n}$ for $n = 2,3,4,5,6$, are given in the following

$S_2 = \{2.697224362268005,\ 6.302775637731995\}$

$S_3 = \{2.669941260432018,\ 5.201639675723405, 7.128419063844577\}$

$$S_4 = \left\{ \begin{array}{c} 2.667028328506892,\ 4.700596159221179, \\ 6.156443643217495,\ 7.475931869054436 \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 2.666706814990114,\ 4.449448536406551, \\ 5.511997050564387,\ 6.720107682566076, \\ 7.651739915472869 \end{array} \right\},$$

$$S_6 = \left\{ \begin{array}{c} 2.666671127024368,\ 4.309895427899775, \\ 5.098757535396342,\ 6.105336258757300, \\ 7.067037793312944,\ 7.752301857609272 \end{array} \right\}.$$

Clearly, $\sum \lambda_i = tr(E_{q,n}) = n(4s^2 + 2t) - 2s^2 - t$ and $\prod \lambda_i = \det(E_{q,n}) = \aleph_{2n}$.

If we take care of the spectra, minimum eigenvalue converges to $2s^2$. The maximum eigenvalue of spectra converges to $4s^2 + 4t$.

1.9 Some properties of tridiagonal matrices $O_{q,n}$ by odd (s,t)-Modified Pell sequence

Assume that $O_{q,n}$ is an $n \times n$ tridiagonal matrix defined as

$$O_{q,n} = \begin{bmatrix} s & -s & & & \\ t & 4s^2+2t & t & & \\ & t & 4s^2+2t & \ddots & \\ & & \ddots & \ddots & t \\ & & & t & 4s^2+2t \end{bmatrix}.$$

Then the determinant of $O_{q,n}$

$$\det O_{q,n} = \aleph_{2n-1}.$$

For the inverse of $O_{q,n}$, the values are computed as

$$\begin{aligned} a_1 &= s \\ a_i &= 4s^2 + 2t,\ i \geq 2 \\ c_1 &= -s,\quad b_1 = t \\ b_i &= c_i = t, \quad i > 1 \\ \theta_0 &= 1,\quad \theta_i = \aleph_{2i-1}, \quad i \geq 1 \\ \phi_j &= \frac{\wp_{2(n-j+2)}}{2s}, \qquad j \geq 1. \end{aligned}$$

If $s = t = 1$, then the sequence of the spectra of the matrices $O_{q,n}$ for $n = 2, 3, 4, 5, 6$ is given in the following

$S_2 = \{1.208712152522080,\ 5.791287847477920\}$

$S_3 = \{1.218716204021644,\ 4.862194798386097,\ 6.919088997592260\}$

$$S_4 = \left\{ \begin{array}{c} 1.219199198376559,\ 4.504733902179790 \\ 5.898455994015420,\ 7.377610905428227 \end{array} \right\}$$

$$S_5 = \left\{ \begin{array}{c} 1.219222421149124,\ 4.332385534359584, \\ 5.293995934620866,\ 6.555626057939926, \\ 7.598770051930496 \end{array} \right\}$$

$$S_6 = \left\{ \begin{array}{c} 1.219223537248886,\ 4.235965700260784, \\ 4.930440201174632,\ 5.932935651316058, \\ 6.960679940971054,\ 7.720754969028587 \end{array} \right\}.$$

The sequence of maximum eigenvalue is increasing and converges to $4s^2 + 4t = 8$. So we can say $\lim_{n\to\infty} \max(\lambda(O_n(k)) = 4s^2 + 4t$.

Evidently, $\sum \lambda_i = tr(O_{q,n}) = (n-1)(4s^2 + 2t) + s$ and $\prod \lambda_i = \det(O_{q,n}) = \aleph_{2n-1}$.

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