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Insertion of a Contra-continuous Function between Two Comparable Real-valued Functions

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Abstract

A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

1. Introduction

The concept of a C-open set in a topological space was introduced by Hatir et al. [12]. The authors define a set S to be a C-open set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a C-closed set if its complement (denoted by S^c) is a C-open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an α-open set and a C-open set or equivalently a subset of a topological space is closed if and only if it

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is an α-closed set and a *C*-closed set. This enables them to provide the following decomposition of continuity: a function is continuous if and only if it is α-continuous and *C*-continuous or equivalently a function is contra-continuous if and only if it is contra-α-continuous and contra-*C*-continuous.

Recall that a subset *A* of a topological space (X, τ) is called α-*open* if *A* is the difference of an open and a nowhere dense subset of *X*. A set *A* is called α-*closed* if its complement is α-open or equivalently if *A* is the union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called *semi-preopen* or β-*open*. A set is *semi-preclosed* or β-*closed* if its complement is semi-preopen or β-open.

In [7] it was shown that a set *A* is β-open if and only if $A \subseteq Cl(Int(Cl(A)))$. A generalized class of closed sets was considered by Maki [19]. He investigated the sets that can be represented as union of closed sets and called them *V*-sets. Complements of *V*-sets, i.e., sets that are intersection of open sets are called Λ-sets [19].

Recall that a real-valued function *f* defined on a topological space *X* is called *A*-continuous [24] if the preimage of every open subset of $\mathbb{R}$ belongs to *A*, where *A* is a collection of subsets of *X*. Most of the definitions of function used throughout this paper are consequences of the definition of *A*-continuity. However, for unknown concepts the reader may refer to [4, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev [5] introduced a new class of mappings called contra-continuity. Jafari and Noiri [13, 14] exhibited and studied among others a new weaker form of this class of mappings called contra-α-continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 23].

Hence, a real-valued function f defined on a topological space *X* is called *contra-continuous* (resp. *contra-C-continuous*, *contra-α-continuous*) if the preimage of every open subset of $\mathbb{R}$ is closed (resp. *C*-closed , α-closed) in *X* [5].

Results of Katětov [15, 16] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that Λ-sets or kernel of sets are open [19].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [17].

A property P defined relative to a real-valued function on a topological space is a *cc-property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-continuous function also has property P. If P_1 and P_2 are *cc*-properties, the following terminology is used: (i) A space X has the *weak cc-insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$. (ii) A space X has the *cc-insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f$, g has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g < h < f$. (iii) A space X has the *strong cc-insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. (iv) A space X has the *weakly cc-insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f$, g has property P_1, f has property P_2 and $f - g$ has property P_2, then there exists a contra-continuous function h such that $g < h < f$.

In this paper, for a topological space whose Λ-sets or kernel of sets are open, is given a sufficient condition for the weak *cc*-insertion property. Also for a space with the weak *cc*-insertion property, we give a necessary and sufficient condition for the space to have the *cc*-insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the insertion and strong insertion of a contra-α-continuous function between two comparable real-valued functions has also recently considered by the authors in [20, 21].

2. The Main Result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^{Λ} and A^{V} as follows:

$$A^{\Lambda} = \bigcap\{O : O \supseteq A, O \in (X, \tau)\} \text{ and } A^{V} = \bigcup\{F : F \subseteq A, F^{c} \in (X, \tau)\}.$$

In [6, 18, 22], A^{Λ} is called the *kernel* of A.

The family of all α-open, α-closed, C-open and C-closed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $CO(X, \tau)$ and $CC(X, \tau)$, respectively.

We define the subsets $\alpha(A^{\Lambda})$, $\alpha(A^{V})$, $C(A^{\Lambda})$ and $C(A^{V})$ as follows:

$\alpha(A^{\Lambda}) = \bigcap\{O : O \supseteq A, O \in \alpha O(X, \tau)\}$,

$\alpha(A^{V}) = \bigcup\{F : F \subseteq A, F \in \alpha C(X, \tau)\}$,

$C(A^{\Lambda}) = \bigcap\{O : O \supseteq A, O \in CO(X, \tau)\}$ and

$C(A^{V}) = \bigcup\{F : F \subseteq A, F \in CC(X, \tau)\}$.

$\alpha(A^{\Lambda})$ (resp. $C(A^{\Lambda})$) is called the α-*kernel* (resp. C-*kernel*) of A.

The following first two definitions are modifications of conditions considered in [15, 16].

Definition 2.2. If ρ is a binary relation in a set S, then $\overline{\rho}$ is defined as follows: $x \,\overline{\rho}\, y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

(1) If $A_i \rho B_j$ for any $i \in \{1, ..., m\}$ and for any $j \in \{1, ..., n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, ..., m\}$ and any $j \in \{1, ..., n\}$.

(2) If $A \subseteq B$, then $A \,\overline{\rho}\, B$.

(3) If $A \rho B$, then $A^{\Lambda} \subseteq B$ and $A \subseteq B^{V}$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. *Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1)\ \rho\ A(g, t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.*

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1)\ \rho\ A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2)$, $G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 in [16] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2)$, $H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$, then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) =$

$H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1, t_2)$ is closed in *X*, i.e., *h* is a contra-continuous function on *X*.

The above proof used the technique of Theorem 1 in [15].

Theorem 2.2. *Let P_1 and P_2 be cc-property and X be a space that satisfies the weak cc-insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g < f$, g has property P_1 and f has property P_2. The space X has the cc-insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each n, $X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by contra-continuous functions.*

Proof. Assume that *X* has the weak *cc*-insertion property for (P_1, P_2). Let *g* and *f* be functions such that $g < f$, *g* has property P_1 and *f* has property P_2. By hypothesis there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a sequence (D_n) such that $\bigcap_{n=1}^{\infty} D_n = \varnothing$ and such that for each *n*, $X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by contra-continuous functions. Let k_n be a contra-continuous function such that $k_n = 0$ on $A(f - g, 3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function *k* on *X* be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the properties contra-continuous functions, the function *k* is a contra-continuous function. Since $\bigcap_{n=1}^{\infty} D_n = \varnothing$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if *x* is in $A(f - g, 3^{-n+1})$, then $k(x) \le 1/4(3^{-n})$. If *x* is any point in *X*, then $x \notin (f - g, 1)$ or for some *n*,

$$x \in A(f - g, 3^{-n+1}) - A(f - g, 3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \le 1/2(3^{-n}) < f(x) - g(x)$. Thus if

$f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since P_1 and P_2 are E-properties, then g_1 has property P_1 and f_1 has property P_2. Since X has the weak cc-insertion property for (P_1, P_2), then there exists a contra-continuous function h such that $g_1 \le h \le f_1$. Thus $g < h < f$, it follows that X satisfies the cc-insertion property for (P_1, P_2). (The technique of this proof is by Katětov [15]).

Conversely, let g and f be functions on X such that g has property P_1, f has property P_2 and $g < f$. By hypothesis, there exists a contra-continuous function h such that $g < h < f$. We follow an idea contained in Lane [17]. Since the constant function 0 has property P_1, since $f - h$ has property P_2, and since X has the cc-insertion property for (P_1, P_2), then there exists a contra-continuous function k such that $0 < k < f - h$. Let $A(f - g, 3^{-n+1})$ be any lower cut set for $f - g$ and let $D_n = \{x \in X : k(x) < 3^{-n+2}\}$. Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n = \varnothing$. Since

$$A(f - g, 3^{-n+1}) \subseteq \{x \in X : (f - g)(x) \le 3^{-n+1}\} \subseteq \{x \in X : k(x) \le 3^{-n+1}\}$$

and since $\{x \in X : k(x) \le 3^{-n+1}\}$ and $\{x \in X : k(x) \ge 3^{-n+2}\} = X \setminus D_n$ are completely separated by contra-continuous functions $\sup\{3^{-n+1}, \inf\{k, 3^{-n+2}\}\}$, it follows that for each n, $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ are completely separated by contra-continuous functions.

3. Applications

The abbreviations $c\alpha c$ and cCc are used for contra-α-continuous and contra-C-continuous, respectively.

Before stating the consequences of Theorems 2.1, 2.2, we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. *If for each pair of disjoint* α*-open* (*resp. C-open*) *sets* G_1, G_2 *of X, there exist closed sets* F_1 *and* F_2 *of X such that* $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then X has the weak cc-insertion property for* $(c\alpha c, c\alpha c)$ (*resp.* (cCc, cCc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are $c\alpha c$

(resp. cCc), and $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $\alpha(A^{\Lambda}) \subseteq \alpha(B^{V})$ (resp. $C(A^{\Lambda}) \subseteq C(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an α-open (resp. C-open) set and since $\{x \in X : g(x) < t_2\}$ is an α-closed (resp. C-closed) set, it follows that $\alpha(A(f, t_1)^{\Lambda}) \subseteq \alpha(A(g, t_2)^{V})$ (resp. $C(A(f, t_1)^{\Lambda}) \subseteq C(A(g, t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1. □

Corollary 3.2. *If for each pair of disjoint* α*-open* (*resp. C-open*) *sets* G_1, G_2 *there exist closed sets* F_1 *and* F_2 *such that* $G_1 \subseteq F_1, G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then every contra-*$\alpha$*-continuous* (*resp. contra-C-continuous*) *function is contra-continuous.*

Proof. Let f be a real-valued contra-α-continuous (resp. contra-C-continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function h such that $g = h = f$. □

Corollary 3.3. *If for each pair of disjoint* α*-open* (*resp. C-open*) sets G_1, G_2 *of X, there exist closed sets* F_1 *and* F_2 *of X such that* $G_1 \subseteq F_1, G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then X has the strong cc-insertion property for* (cαc, cαc) (*resp.* (*cCc*, *cCc*)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are $c\alpha c$ (resp. cCc), and $g \leq f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are contra-continuous functions hence h is a contra-continuous function. □

Corollary 3.4. *If for each pair of disjoint subsets* G_1, G_2 *of X, such that* G_1 *is* α*-open and* G_2 *is C-open, there exist closed subsets* F_1 *and* F_2 *of X such that* $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then X have the weak cc-insertion property for* ($c\alpha c$, cCc) *and* (cCc, $c\alpha c$).

Proof. Let g and f be real-valued functions defined on X, such that g is $c\alpha c$ (resp.

cCc) and *f* is *cCc* (resp. *cαc*), with $g \le f$. If a binary relation ρ is defined by $A \,\rho\, B$ in case $C(A^{\Lambda}) \subseteq \alpha(B^{V})$ (resp. $\alpha(A^{\Lambda}) \subseteq C(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of *X*. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 \le t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \le t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \le t_1\}$ is a *C*-open (resp. α-open) set and since $\{x \in X : g(x) \le t_2\}$ is an α-closed (resp. *C*-closed) set, it follows that $C(A(f, t_1)^{\Lambda}) \subseteq \alpha(A(g, t_2)^{V})$ (resp. $\alpha(A(f, t_1)^{\Lambda}) \subseteq C(A(g, t_2)^{V})$). Hence $t_1 \le t_2$ implies that $A(f, t_1) \,\rho\, A(g, t_2)$. The proof follows from Theorem 2.1. □

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.1. *The following conditions on the space X are equivalent*:

(i) *For each pair of disjoint subsets* G_1, G_2 *of X, such that* G_1 *is* α*-open and* G_2 *is C-open, there exist closed subsets* F_1, F_2 *of X such that* $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$.

(ii) *If G is a C-open* (*resp.* α*-open*) *subset of X which is contained in an* α*-closed* (*resp. C-closed*) *subset F of X, then there exists a closed subset H of X such that* $G \subseteq H \subseteq H^{\Lambda} \subseteq F$.

Proof. (i) ⇒ (ii) Suppose that $G \subseteq F$, where *G* and *F* are *C*-open (resp. α-open) and α-closed (resp. *C*-closed) subsets of *X*, respectively. Hence, F^c is an α-open (resp. *C*-open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint closed subsets F_1, F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an open subset containing F_1, we conclude that $F_1^{\Lambda} \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^{\Lambda} \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint subsets of X, such that G_1 is α-open and G_2 is C-open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is an α-closed subset of X. Hence by (ii) there exists a closed set H such that $G_2 \subseteq H \subseteq H^{\Lambda} \subseteq G_1^c$.

But

$$H \subseteq H^{\Lambda} \Rightarrow H \cap (H^{\Lambda})^c = \varnothing$$

and

$$H^{\Lambda} \subseteq G_1^c \Rightarrow G_1 \subseteq (H^{\Lambda})^c.$$

Furthermore, $(H^{\Lambda})^c$ is a closed subset of X. Hence $G_2 \subseteq H$, $G_1 \subseteq (H^{\Lambda})^c$ and $H \cap (H^{\Lambda})^c = \varnothing$. This means that condition (i) holds. □

Lemma 3.2. *Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is α-open and G_2 is C-open, can be separated by closed subsets of X, then there exists a contra-continuous function $h : X \to [0, 1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.*

Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is α-open and G_2 is C-open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is an α-closed subset of X containing the C-open subset G_2 of X, by Lemma 3.1, there exists a closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq G_1^c.$$

Note that $H_{1/2}$ is also an α-closed subset of *X* and contains G_2, and G_1^c is an α-closed subset of *X* and contains the *C*-open subset $H_{1/2}^{\Lambda}$ of *X*. Hence, by Lemma 3.1, there exists closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq H_{1/4}^{\Lambda} \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq H_{3/4} \subseteq H_{3/4}^{\Lambda} \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function *h* on *X* by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X$, $0 \leq h(x) \leq 1$, i.e., *h* maps *X* into $[0, 1]$. Also, we note that for any $t \in D$, $G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that *h* is a contra-continuous function on *X*. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$, then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$, then $\{x \in X : h(x) < \alpha\} = \bigcup\{H_t : t < \alpha\}$, hence, they are closed subsets of *X*. Similarly, if $\alpha < 0$, then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$, then $\{x \in X : h(x) > \alpha\} = \bigcup \{(H_t^{\Lambda})^c : t > \alpha\}$ hence, every of them is a closed subset. Consequently *h* is a contra-continuous function. □

Lemma 3.3. *Suppose that X is a topological space such that every two disjoint C-open and* α*-open subsets of X can be separated by closed subsets of X. The following conditions are equivalent*:

(i) *Every countable covering of C-closed* (*resp.* α*-closed*) *subsets of X has a refinement consisting of* α*-closed* (*resp. C-closed*) *subsets of X such that for every* $x \in X$, *there exists a closed subset of X containing x such that it intersects only finitely many members of the refinement.*

(ii) *Corresponding to every decreasing sequence* $\{G_n\}$ *of C-open* (*resp.* α*-open*) *subsets of X with empty intersection there exists a decreasing sequence* $\{F_n\}$ *of* α*-closed* (*resp. C-closed*) *subsets of X such that* $\bigcap_{n=1}^{\infty} F_n = \emptyset$ *and for every* $n \in \mathbb{N}$, $G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of C-open (resp. α-open) subsets of X with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of C-closed (resp. α-closed) subsets of X. By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is a closed subset of X and $V_n^{\Lambda} \subseteq G_n^c$. By setting $F_n = (V_n^{\Lambda})^c$, we obtain a decreasing sequence of closed subsets of X with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of C-closed (resp. α-closed) subsets of X, we set for $n \in \mathbb{N}$, $G_n = \left(\bigcup_{i=1}^{n} H_i\right)^c$. Then $\{G_n\}$ is a decreasing sequence of C-open (resp. α-open) subsets of X with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of α-closed (resp. C-closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}$, $G_n \subseteq F_n$. Now we define the subsets W_n of X in the following manner:

W_1 is a closed subset of X such that $F_1^c \subseteq W_1$ and $W_1^{\Lambda} \cap G_1 = \varnothing$.

W_2 is a closed subset of X such that $W_1^{\Lambda} \cup F_2^c \subseteq W_2$ and $W_2^{\Lambda} \cap G_2 = \varnothing$, and so on. (By Lemma 3.1, W_n exists).

Then since $\{F_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of closed sets. Moreover, we have

(i) $W_n^{\Lambda} \subseteq W_{n+1}$,

(ii) $F_n^c \subseteq W_n$,

(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.

Now setting $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \backslash W_{n-1}^{\Lambda}$.

Then since $W_{n-1}^{\Lambda} \subseteq W_n$ and $S_n \supseteq W_{n+1} \backslash W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of closed sets and covers X. Furthermore, $S_i \cap S_j \neq \varnothing$ if and only if $|i - j| \leq 1$. Finally, consider the following sets:

$S_1 \cap H_1, \quad S_1 \cap H_2$

$S_2 \cap H_1, \quad S_2 \cap H_2, \quad S_2 \cap H_3$

$S_3 \cap H_1, \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4$

$\vdots$

$S_i \cap H_1, \quad S_i \cap H_2, \quad S_i \cap H_3, \quad S_i \cap H_4, \quad \cdots, \quad S_i \cap H_{i+1}$

These sets are closed sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a closed set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N},\ j = 1, ..., i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are closed sets, and for every point in X we can find a closed set containing the point that intersects only finitely many elements of that refinement. □

Corollary 3.5. *If every two disjoint C-open and* α*-open subsets of X can be separated by closed subsets of X, and in addition, every countable covering of C-closed* (*resp.* α*-closed*) *subsets of X has a refinement that consists of* α*-closed* (*resp. C-closed*) *subsets of X such that for every point of X we can find a closed subset containing that point such that it intersects only a finite number of refining members then X has the weakly cc-insertion property for* $(c\alpha c,\ cCc)$ (*resp.* $(cCc,\ c\alpha c)$).

Proof. Since every two disjoint C-open and α-open sets can be separated by closed subsets of X, therefore by Corollary 3.4, X has the weak cc-insertion property for $(c\alpha c,\ cCc)$ and $(cCc,\ c\alpha c)$. Now suppose that f and g are real-valued functions on X with $g < f$, such that g is $c\alpha c$ (resp. cCc), f is cCc (resp. $c\alpha c$) and $f - g$ is cCc (resp. $c\alpha c$). For every $n \in \mathbb{N}$, set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) \leq 3^{-n+1}\}$$

Since $f - g$ is cCc (resp. $c\alpha c$), hence $A(f - g, 3^{-n+1})$ is a C-open (resp. α-open) subset of X. Consequently, $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of C-open (resp.

α-open) subsets of X and furthermore since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \varnothing$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of α-closed (resp. C-closed) subsets of X such that $A(f-g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \varnothing$. But by Lemma 3.2, the pair $A(f-g, 3^{-n+1})$ and $X \backslash D_n$ of C-open (resp. α-open) and α-open (resp. C-open) subsets of X can be completely separated by contra-continuous functions. Hence by Theorem 2.2, there exists a contra-continuous function h defined on X such that $g < h < f$, i.e., X has the weakly cc-insertion property for $(c\alpha c, cCc)$ (resp. $(cCc, c\alpha c)$). □

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