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Fixed Point Theorems for the Alternate Interpolative Ciric-Reich-Rus Operator

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Abstract

In [1], the authors introduced the interpolative Ciric-Reich-Rus operator in Branciari metric space and obtained some fixed point theorems; in this work we present an alternate characterization of the interpolative Ciric-Reich-Rus operator in metric space, and obtain some fixed point theorems.

1 The Alternate Characterization

Recall from [1], that if (X, d) is a Branciari metric space, then the map $T : X \mapsto X$ is called an *interpolative Ciric-Reich-Rus operator* if there exists $\lambda \in [0, 1)$ and positive reals α, β with $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)^{\beta} \cdot d(x, Tx)^{\alpha} \cdot d(y, Ty)^{1-\alpha-\beta}$$

for all $x, y \in X$, $x, y \notin Fix(T) = \{x \in X : Tx = x\}$. Since division by zero is not permissible, we keep $\lambda \in (0, 1)$, and observe the above inequality implies the following

$$\frac{d(Tx, Ty)}{\lambda} \leq d(x, y)^{\beta} \cdot d(x, Tx)^{\alpha} \cdot d(y, Ty)^{1-\alpha-\beta}$$

$\Longrightarrow$

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$$\log\left(\frac{d(Tx,Ty)}{\lambda}\right) \leq \beta \log d(x,y) + \alpha \log d(x,Tx) + (1-\alpha-\beta)\log d(y,Ty)$$

$\Longrightarrow$

$$\log\left(\frac{d(Tx,Ty)}{\lambda}\right) \leq (\beta+\alpha+1-\alpha-\beta)\max\{\log d(x,y), \log d(x,Tx), \log d(y,Ty)\}$$

$\Longrightarrow$

$$3\log\left(\frac{d(Tx,Ty)}{\lambda}\right) \leq \log d(x,y) + \log d(x,Tx) + \log d(y,Ty)$$

$\Longrightarrow$

$$3\log\left(\frac{d(Tx,Ty)}{\lambda}\right) \leq \log(d(x,y)d(x,Tx)d(y,Ty))$$

$\Longrightarrow$

$$\log\left(\frac{d(Tx,Ty)}{\lambda}\right) \leq \log(d(x,y)d(x,Tx)d(y,Ty))^{\frac{1}{3}}$$

$\Longrightarrow$

$$\frac{d(Tx,Ty)}{\lambda} \leq (d(x,y)d(x,Tx)d(y,Ty))^{\frac{1}{3}}.$$

From the inequality immediately above, we introduce the following

Definition 1.1. Let (X,d) be a metric space. A map $T : X \mapsto X$ will be called an *alternate interpolative Ciric-Reich-Rus operator* if there exists $\lambda \in (0,1)$ such that

$$d(Tx,Ty) \leq \lambda d(x,y)^{\frac{1}{3}} d(x,Tx)^{\frac{1}{3}} d(y,Ty)^{\frac{1}{3}}$$

for all $x, y \in X$, $x, y \notin Fix(T)$.

2 The Contraction Mapping Theorem

Theorem 2.1. *Let (X,d) be a metric space. Suppose $T : X \mapsto X$ is an alternate interpolative Ciric-Reich-Rus operator, that is, there exists $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda d(x,y)^{\frac{1}{3}} d(x,Tx)^{\frac{1}{3}} d(y,Ty)^{\frac{1}{3}}$$

for all $x,y \in X$, $x,y \notin Fix(T)$. If X is complete, then the fixed point exists.

Proof. Define the sequence $\{x_n\} \in X$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Now observe we have the following

$$\begin{aligned} d(x_{n+1},x_{n+2}) &= d(Tx_n,Tx_{n+1}) \\ &\leq \lambda d(x_n,x_{n+1})^{\frac{1}{3}} d(x_n,Tx_n)^{\frac{1}{3}} d(x_{n+1},Tx_{n+1})^{\frac{1}{3}} \\ &= \lambda d(x_n,x_{n+1})^{\frac{1}{3}} d(x_n,x_{n+1})^{\frac{1}{3}} d(x_{n+1},x_{n+2})^{\frac{1}{3}} \\ &= \lambda d(x_n,x_{n+1})^{\frac{2}{3}} d(x_{n+1},x_{n+2})^{\frac{1}{3}}. \end{aligned}$$

From the above, we deduce that

$$d(x_{n+1},x_{n+2}) \leq \lambda^{\frac{3}{2}} d(x_n,x_{n+1}).$$

By induction, we have the following for all $n \in \mathbb{N}$

$$d(x_n,x_{n+1}) \leq (\lambda^{\frac{3}{2}})^n d(x_0,x_1).$$

Now we show the sequence is Cauchy. For this let $n,m \in \mathbb{N}$ with $m > n$, and observe we have the following

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \cdots + d(x_{n+1},x_n) \\ &\leq [(\lambda^{\frac{3}{2}})^{m-1} + (\lambda^{\frac{3}{2}})^{m-2} + \cdots + (\lambda^{\frac{3}{2}})^n] d(x_0,x_1) \\ &\leq [(\lambda^{\frac{3}{2}})^n + (\lambda^{\frac{3}{2}})^{n+1} + \cdots] d(x_0,x_1) \\ &\leq (\lambda^{\frac{3}{2}})^n [1 + \lambda^{\frac{3}{2}} + \cdots] d(x_0,x_1) \\ &\leq \frac{(\lambda^{\frac{3}{2}})^n}{1-\lambda^{\frac{3}{2}}} d(x_0,x_1). \end{aligned}$$

Now letting $m, n \to \infty$ in the above, it follows that $\{x_n\}$ is Cauchy. Since X is complete, there is $a \in X$ such that $\lim_{n\to\infty} x_n = a$. Now we show the fixed point exists. For this, suppose $a \neq Ta$, then observe we have the following

$$\begin{aligned}
0 &< d(a, Ta) \\
&\leq d(a, x_{n+1}) + d(x_{n+1}, Ta) \\
&\leq d(a, x_{n+1}) + d(Tx_n, Ta) \\
&\leq d(a, x_{n+1}) + \lambda d(x_n, a)^{\frac{1}{3}} d(x_n, Tx_n)^{\frac{1}{3}} d(a, Ta)^{\frac{1}{3}} \\
&= d(a, x_{n+1}) + \lambda d(x_n, a)^{\frac{1}{3}} d(x_n, x_{n+1})^{\frac{1}{3}} d(a, Ta)^{\frac{1}{3}}.
\end{aligned}$$

Take limits in the above as $n \to \infty$, we deduce $0 < d(a, Ta) \leq 0$, which implies $d(a, Ta) = 0$, thus, $a = Ta$, and the fixed point exists. □

3 A Best Proximity Point Theorem

Let W and V be two nonempty subsets of a metric space (X, d), and let $S : W \mapsto V$ be a non-self map. If $W \cap V$ is nonempty, then the equation $Sx = x$ may not have a solution. Naturally, the following question arises

Question 3.1. How far is the distance between x and Sx?

The problem of global optimization for determining the minimum value of the distance $d(x, Sx) = \min\{d(x, y) : x \in W \text{ and } y \in V\}$ is the study of best proximity point theory. In this section, we obtain a best proximity point theorem in the sense of [2].

Notation 3.2. Throughout this section

(a) W and V denote nonempty subsets of a metric space (X, d);

(b) $d(W, V) := \inf\{d(x, y) : x \in W \text{ and } y \in V\}$;

(c) $W_0 := \{x \in W : d(x, y) = d(W, V) \text{ for some } y \in V\}$;

(d) $V_0 = \{y \in V : d(x, y) = d(W, V) \text{ for some } x \in W\}$.

The notion of proximal contraction appeared in [3], now we introduce the following

Definition 3.3. Let $S : W \mapsto V$ be a non-self mapping. We say S is a *proximal alternate interpolative Ciric-Reich-Rus operator* if there exists $\lambda \in (0, 1)$ and $u_1, u_2, x, y \in W$ such that $d(u_1, Sx) = d(W, V)$ and $d(u_2, Sy) = d(W, V)$ implies

$$d(u_1, u_2) \leq \lambda d(x, y)^{\frac{1}{3}} d(x, u_1)^{\frac{1}{3}} d(y, u_2)^{\frac{1}{3}}.$$

The notion of G-proximal Kannan mapping appeared in [4], now we introduce the following

Definition 3.4. Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called a *G-proximal alternate interpolative Ciric-Reich-Rus operator* if there exists $\lambda \in (0, 1)$ such that $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$, $d(v, Sy) = d(W, V)$ impliess

$$d(u, v) \leq \lambda d(x, y)^{\frac{1}{3}} d(x, u)^{\frac{1}{3}} d(y, v)^{\frac{1}{3}},$$

where $x, y, u, v \in W$.

Definition 3.5. [4] Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. A non-self mapping $S : W \mapsto V$ is called *proximally G-edge preserving*, if for each $x, y, u, v \in W$, $(x, y) \in E(G)$, $d(u, Sx) = d(W, V)$, and $d(v, Sy) = d(W, V)$ implies $(u, v) \in E(G)$.

The main result of this section is a best proximity point theorem for a G-proximal alternate interpolative Ciric-Reich-Rus operator in complete metric space endowed with a directed graph.

Theorem 3.6. *Let (X, d) be a complete metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Let W and V be nonempty closed subsets of X with W_0 nonempty. Let $S : W \mapsto V$ be a nonself mapping satisfying the following conditions*

(a) *S is proximally G-edge preserving, continuous, and G-proximal alternate interpolative Ciric-Reich-Rus operator such that* $S(W_0) \subset V_0$.

(b) *there exists* $x_0, x_1 \in W_0$ *such that* $d(x_1, Sx_0) = d(W, V)$ *and* $(x_0, x_1) \in E(G)$.

Then S has a best proximity point in W, that is, there exists an element $w \in W$ *such that* $d(w, Sw) = d(W, V)$. *Further the sequence* $\{x_n\}$ *defined by* $d(x_n, Sx_{n-1}) = d(W, V)$ *for all* $n \in \mathbb{N}$ *converges to the element w.*

Proof. From condition (b), there exists $x_0, x_1 \in W_0$ such that $d(x_1, Sx_0) = d(W, V)$ and $(x_0, x_1) \in E(G)$. Since $S(W_0) \subseteq V_0$, we have $Sx_1 \in V_0$ and hence there exists $x_2 \in W_0$ such that $d(x_2, Sx_1) = d(W, V)$. By the proximally G-edge preserving of S and using $d(x_1, Sx_0) = d(W, V)$ and $d(x_2, Sx_1) = d(W, V)$, we get $(x_1, x_2) \in E(G)$. By continuing this process, we form the sequence $\{x_n\}$ in W_0 such that $d(x_n, Sx_{n-1}) = d(W, V)$ with $(x_{n-1}, x_n) \in E(G)$, for all $n \in \mathbb{N}$. Next we show that S has a best proximity point in W. Suppose there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$. Since $d(x_n, Sx_{n-1}) = d(W, V)$, we obtain that $d(x_{n_0}, Sx_{n_0}) = d(x_{n_0+1}, Sx_{n_0}) = d(W, V)$, and so x_{n_0} is a best proximity point of S. Now we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. We show that $\{x_n\}$ is a Cauchy sequence in W. As S is a G-proximal alternate interpolative Ciric-Reich-Rus operator, and for each $n \in \mathbb{N}$, $(x_{n-1}, x_n) \in E(G)$, $d(x_n, Sx_{n-1}) = d(W, V)$, and $d(x_{n+1}, Sx_n) = d(W, V)$, then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_n, x_{n+1})^{\frac{1}{3}} \\ &\leq \lambda d(x_{n-1}, x_n)^{\frac{2}{3}} d(x_n, x_{n+1})^{\frac{1}{3}}. \end{aligned}$$

By the above inequality, we deduce

$$d(x_n, x_{n+1}) \leq \lambda^{\frac{3}{2}} d(x_{n-1}, x_n).$$

By induction, we have

$$d(x_n, x_{n+1}) \leq (\lambda^{\frac{3}{2}})^n d(x_0, x_1).$$

Now for each $m, n \in \mathbb{N}$ with $m > n$, we deduce the following

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (\lambda^{\frac{3}{2}})^n d(x_0, x_1) + (\lambda^{\frac{3}{2}})^{n+1} d(x_0, x_1) + \cdots + (\lambda^{\frac{3}{2}})^{m-1} d(x_0, x_1) \\ &= d(x_0, x_1) \sum_{i=n}^{m-1} (\lambda^{\frac{3}{2}})^i \\ &\leq \frac{(\lambda^{\frac{3}{2}})^n}{1 - \lambda^{\frac{3}{2}}} d(x_0, x_1). \end{aligned}$$

Since $\lambda^{\frac{3}{2}} \in (0, 1)$, it follows that $\{x_n\}$ is a Cauchy sequence in W. Since W is closed, there exist $w \in W$ such that $x_n \to w$. By continuity of S, we have $Sx_n \to Sw$ as $n \to \infty$. As the metric function is continuous, we obtain $d(x_{n+1}, Sx_n) \to d(w, Sw)$ as $n \to \infty$. Since $d(x_n, Sx_{n-1}) = d(W, V)$, we also obtain $d(w, Sw) = d(W, V)$. It follows that $w \in W$ is a best proximity point of S. Moreoever, the sequence $\{x_n\}$ defined by $d(x_{n+1}, Sx_n) = d(W, V)$, $n \in \mathbb{N}$, converges to an element w, and the proof is completed. □

4 A Weakly Contractive Mapping Theorem

Generalizations of the Banach mapping theorem have appeared in the literature. A weaker contraction appeared in Hilbert spaces [5] with the following definition in metric spaces

Definition 4.1. A mapping $T : X \mapsto X$ is said to be *weakly contractive* if

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)),$$

where $x, y \in X$, $\psi : [0, \infty) \mapsto [0, \infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$, and $\lim_{x \to \infty} \psi(x) = \infty$.

If $\psi(x) = kx$, where $0 < k < 1$, then weakly contractive mapping in the above definition reduces to Banach mapping. A number of works concerning weakly contractive mappings have appeared in the literature, and the reader is referred to [6]-[9], for some of them. In this section we introduce a generalization of the alternate interpolative Ciric-Reich-Rus operator, and establish in complete metric spaces that the weak alternate interpolative Ciric-Reich-Rus contractions have a fixed point. This section is partially inspired by [10].

Definition 4.2. A mapping $T : X \mapsto X$, where (X, d) is a complete metric space, will be called *weakly alternate interpolative Ciric-Reich-Rus contractive* if

$$d(Tx, Ty) \leq d(x, y)^{\frac{1}{3}} d(x, Tx)^{\frac{1}{3}} d(y, Ty)^{\frac{1}{3}} - \psi(d(x, y)^{\frac{1}{3}} d(x, Tx)^{\frac{1}{3}} d(y, Ty)^{\frac{1}{3}}),$$

where $x, y \in X$, $x, y \notin Fix(T)$, $\psi : [0, \infty) \mapsto [0, \infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$, and $\lim_{x\to\infty} \psi(x) = \infty$.

The main result of this section is the following

Theorem 4.3. *Let $T : X \mapsto X$, where (X, d) is a complete metric space, be a weak alternate interpolative Ciric-Reich-Rus contraction. Then T possesses a fixed point.*

Proof. Let $x_0 \in X$, and for all $n \geq 1$, $x_{n+1} = Tx_n$. If $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point of T. So we assume $x_n \neq x_{n+1}$. Now observe we have the following

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_{n-1}, Tx_{n-1})^{\frac{1}{3}} d(x_n, Tx_n)^{\frac{1}{3}} \\
&\quad - \psi(d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_{n-1}, Tx_{n-1})^{\frac{1}{3}} d(x_n, Tx_n)^{\frac{1}{3}}) \\
&= d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_n, x_{n+1})^{\frac{1}{3}} \\
&\quad - \psi(d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_{n-1}, x_n)^{\frac{1}{3}} d(x_n, x_{n+1})^{\frac{1}{3}}) \\
&= d(x_{n-1}, x_n)^{\frac{2}{3}} d(x_n, x_{n+1})^{\frac{1}{3}} - \psi(d(x_{n-1}, x_n)^{\frac{2}{3}} d(x_n, x_{n+1})^{\frac{1}{3}}) \\
&\leq d(x_{n-1}, x_n)^{\frac{2}{3}} d(x_n, x_{n+1})^{\frac{1}{3}}.
\end{aligned}$$

From the above inequality, we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

It now follows that $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers, and hence is convergent. Let $d(x_n, x_{n+1}) \to r$ as $n \to \infty$. We show that $r = 0$. Since

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)^{\frac{2}{3}} d(x_n, x_{n+1})^{\frac{1}{3}} - \psi(d(x_{n-1}, x_n)^{\frac{2}{3}} d(x_n, x_{n+1})^{\frac{1}{3}}).$$

If we take limits in the above as $n \to \infty$ and use the continuity of ψ, we deduce that

$$r \leq r^{\frac{2}{3}} r^{\frac{1}{3}} - \psi(r^{\frac{2}{3}} r^{\frac{1}{3}})$$

or

$$r \leq r - \psi(r)$$

or

$$\psi(r) \leq 0$$

which is a contradiction unless $r = 0$. It follows that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Next we show that $\{x_n\}$ is a Cauchy sequence. If otherwise, then there exists $\epsilon > 0$ and increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$ such that for all integers k, $n(k) > m(k) > k$, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, and $d(x_{m(k)}, x_{n(k)-1}) < \epsilon$. Now observe we have the following

$$\begin{aligned}
\epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\
&= d(Tx_{m(k)-1}, Tx_{n(k)-1}) \\
&\leq d(x_{m(k)-1}, x_{n(k)-1})^{\frac{1}{3}} d(x_{m(k)-1}, Tx_{m(k)-1})^{\frac{1}{3}} d(x_{n(k)-1}, Tx_{n(k)-1})^{\frac{1}{3}} \\
&\quad - \psi(d(x_{m(k)-1}, x_{n(k)-1})^{\frac{1}{3}} d(x_{m(k)-1}, Tx_{m(k)-1})^{\frac{1}{3}} d(x_{n(k)-1}, Tx_{n(k)-1})^{\frac{1}{3}}) \\
&= d(x_{m(k)-1}, x_{n(k)-1})^{\frac{1}{3}} d(x_{m(k)-1}, x_{m(k)})^{\frac{1}{3}} d(x_{n(k)-1}, x_{n(k)})^{\frac{1}{3}} \\
&\quad - \psi(d(x_{m(k)-1}, x_{n(k)-1})^{\frac{1}{3}} d(x_{m(k)-1}, x_{m(k)})^{\frac{1}{3}} d(x_{n(k)-1}, x_{n(k)})^{\frac{1}{3}}).
\end{aligned}$$

Again

$$\begin{aligned}
\epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\
&\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\
&\leq \epsilon + d(x_{n(k)-1}, x_{n(k)}).
\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, if we take limits in the inequality immediately above as $k \to \infty$, we deduce the following

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon.$$

By using the triangle inequality, we can also show the following limits:

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon$$

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{m(k)}) = \epsilon$$

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{n(k)}) = \epsilon.$$

Since

$$\begin{aligned}
\epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\
&= d(Tx_{m(k)-1}, Tx_{n(k)-1}) \\
&\leq d(x_{m(k)-1}, x_{n(k)-1})^{\frac{1}{3}} d(x_{m(k)-1}, Tx_{m(k)-1})^{\frac{1}{3}} d(x_{n(k)-1}, Tx_{n(k)-1})^{\frac{1}{3}} \\
&\quad - \psi(d(x_{m(k)-1}, x_{n(k)-1})^{\frac{1}{3}} d(x_{m(k)-1}, Tx_{m(k)-1})^{\frac{1}{3}} d(x_{n(k)-1}, Tx_{n(k)-1})^{\frac{1}{3}}) \\
&= d(x_{m(k)-1}, x_{n(k)-1})^{\frac{1}{3}} d(x_{m(k)-1}, x_{m(k)})^{\frac{1}{3}} d(x_{n(k)-1}, x_{n(k)})^{\frac{1}{3}} \\
&\quad - \psi(d(x_{m(k)-1}, x_{n(k)-1})^{\frac{1}{3}} d(x_{m(k)-1}, x_{m(k)})^{\frac{1}{3}} d(x_{n(k)-1}, x_{n(k)})^{\frac{1}{3}})
\end{aligned}$$

if we take limits in the inequality immediately above as $k \to \infty$, we deduce that

$$\epsilon \leq \epsilon^{\frac{1}{3}} \epsilon^{\frac{1}{3}} \epsilon^{\frac{1}{3}} - \psi(\epsilon^{\frac{1}{3}} \epsilon^{\frac{1}{3}} \epsilon^{\frac{1}{3}})$$

or

$$\epsilon \leq \epsilon - \psi(\epsilon)$$

or

$$\psi(\epsilon) \leq 0$$

which is a contradiction unless $\epsilon = 0$. Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Finally, we show the fixed point exists. Observe we have the following

$$\begin{aligned}
d(z, Tz) &\leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \\
&\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\
&\leq d(z, x_{n+1}) + d(x_n, z)^{\frac{1}{3}} d(x_n, Tx_n)^{\frac{1}{3}} d(z, Tz)^{\frac{1}{3}} \\
&\quad - \psi(d(x_n, z)^{\frac{1}{3}} d(x_n, Tx_n)^{\frac{1}{3}} d(z, Tz)^{\frac{1}{3}}) \\
&= d(z, x_{n+1}) + d(x_n, z)^{\frac{1}{3}} d(x_n, x_{n+1})^{\frac{1}{3}} d(z, Tz)^{\frac{1}{3}} \\
&\quad - \psi(d(x_n, z)^{\frac{1}{3}} d(x_n, x_{n+1})^{\frac{1}{3}} d(z, Tz)^{\frac{1}{3}}) \\
&\leq d(z, x_{n+1}) + d(x_n, z)^{\frac{1}{3}} d(x_n, x_{n+1})^{\frac{1}{3}} d(z, Tz)^{\frac{1}{3}}.
\end{aligned}$$

Taking limits in the above as $n \to \infty$, we get that $d(z, Tz) \leq 0$, which implies $d(z, Tz) = 0$. Hence $z = Tz$, and the fixed point exists. □

5 A Common Point of Coincidence

In this section we obtain points of coincidence for three self-mappings satisfying Jungck type contractive conditions without the assumption of normality in cone metric spaces. First we collect some notions and notations that will be useful in the sequel. This section takes inspiration from [11].

Definition 5.1. A subset P of a real Banach space E is called a *cone* if it has the following properties:

(a) P is nonempty closed and $P \neq \{\theta\}$;

(b) $0 \leq a, b \in \mathbb{R}$, and $x, y \in P \Longrightarrow ax + by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

Notation 5.2. For a given cone $P \subseteq E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 5.3. The cone P is called *normal* if there is a number $k > 0$ such that for all $x, y \in E$

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|.$$

The least positive number k satisfying the above implication is called the normal constant of P.

Remark 5.4. In this section we always assume E is a real Banach space, P is a cone in E with $int(P) \neq \emptyset$, and $\leq$ is a partial ordering with respect to P.

Definition 5.5. Let X be a nonempty set. Suppose that the mapping $d : X \times X \mapsto E$ satisfies

(a) $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is a *cone metric space*.

Definition 5.6. Let $\{x_n\}$ be a sequence in X, and $x \in X$. If for every $c \in E$, with $0 \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We write $x_n \to x$ as $n \to \infty$.

Definition 5.7. If for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X.

Definition 5.8. If every Cauchy sequence is convergent in X, then X is called a *complete cone metric space.*

Remark 5.9. [12] If P is a normal cone, then $\{x_n\} \in X$ converges to $x \in X$, if and only if $d(x_n, x) \to 0$ as $n \to \infty$. Furthermore, $\{x_n\} \in X$ is Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 5.10. A point $y \in X$ is called *point of coincidence* of $T, f : X \mapsto X$, if there exists a point $x \in X$ such that $y = fx = Tx$.

The main result of this section is as follows

Theorem 5.11. *Let (X, d) be a cone metric space and the mapping $S, T, f : X \mapsto X$ satisfy*

$$d(Tx, Sy) \leq \lambda d(fx, fy)^{\frac{1}{3}} d(fx, Tx)^{\frac{1}{3}} d(fy, Sy)^{\frac{1}{3}}$$

for all $x, y \in X$, $x, y \notin Fix(T)$, where $0 < \lambda < 1$. If

$$S(X) \cup T(X) \subseteq f(X)$$

and $f(X)$ is a complete subspace of X, then S, T and f have a point of coincidence.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $fx_1 = Sx_0$, This can be done since $S(X) \subseteq f(X)$. Similarly, choose a point x_2 in X such that $fx_2 = Tx_1$. Continuing this process and having chosen x_n in X, we obtain x_{n+1} in X such that

$$fx_{2k+1} = Sx_{2k}$$

and

$$fx_{2k+2} = Tx_{2k+1}, \ k = 0, 1, 2, \cdots .$$

Now observe we have the following

$$\begin{aligned} d(fx_{2k+2}, fx_{2k+1}) &= d(Tx_{2k+1}, Sx_{2k}) \\ &\leq \lambda d(fx_{2k+1}, fx_{2k})^{\frac{1}{3}} d(fx_{2k+1}, Tx_{2k+1})^{\frac{1}{3}} d(fx_{2k}, Sx_{2k})^{\frac{1}{3}} \\ &= \lambda d(fx_{2k+1}, fx_{2k})^{\frac{1}{3}} d(fx_{2k+1}, fx_{2k+2})^{\frac{1}{3}} d(fx_{2k}, fx_{2k+1})^{\frac{1}{3}} \\ &= \lambda d(fx_{2k+1}, fx_{2k})^{\frac{2}{3}} d(fx_{2k+1}, fx_{2k+2})^{\frac{1}{3}} \end{aligned}$$

which implies $d(fx_{2k+2}, fx_{2k+1}) \leq \lambda^{\frac{3}{2}} d(fx_{2k+1}, fx_{2k})$. By induction we obtain $d(fx_n, fx_{n+1}) \leq (\lambda^{\frac{3}{2}})^n d(fx_1, fx_0)$. Let $y_n = fx_n$, then we have $d(y_n, y_{n+1}) \leq (\lambda^{\frac{3}{2}})^n d(y_1, y_0)$. Now for $m > n$, we have

$$\begin{aligned} d(y_m, y_n) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\leq [(\lambda^{\frac{3}{2}})^n + (\lambda^{\frac{3}{2}})^{n+1} + \cdots + (\lambda^{\frac{3}{2}})^{m-1}] d(y_0, y_1) \\ &\leq \frac{(\lambda^{\frac{3}{2}})^n}{1 - \lambda^{\frac{3}{2}}} \ d(y_0, y_1). \end{aligned}$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + \{x \in E : \|x\| < \delta\} \subseteq P$. Also choose a natural number N_1 such that $\frac{(\lambda^{\frac{3}{2}})^n}{1-\lambda^{\frac{3}{2}}}\ d(y_0, y_1) \in \{x \in E : \|x\| < \delta\}$, for all $n \geq N_1$ Then $\frac{(\lambda^{\frac{3}{2}})^n}{1-\lambda^{\frac{3}{2}}}\ d(y_0, y_1) \ll c$ for all $n \geq N_1$. Thus, $m > n \geq N_1$ implies $d(y_m, y_n) \leq \frac{(\lambda^{\frac{3}{2}})^n}{1-\lambda^{\frac{3}{2}}}\ d(y_0, y_1) \ll c$ which implies that $\{y_n\}$ is a Cauchy sequence. Since $f(X)$ is complete, there exists $u, v \in X$ such that $y_n \to v = fu$. Choose a natural number N_2 such that $d(y_{2n+1}, v) \ll \frac{c}{2}$, $d(y_{2n+1}, y_{2n+2}) \ll \frac{c}{2}$, and $d(v, Su) \ll \frac{c}{2}$ for all $n \geq N_2$. Hence for all $n \geq N_2$, we deduce the following

$$\begin{aligned} d(fu, Su) &\leq d(fu, y_{2n+2}) + d(y_{2n+2}, Su) \\ &\leq d(v, y_{2n+2}) + d(Tx_{2n+1}, Su) \\ &\leq d(v, y_{2n+2}) + \lambda d(fx_{2n+1}, fu)^{\frac{1}{3}} d(fx_{2n+1}, Tx_{2n+1})^{\frac{1}{3}} d(fu, Su)^{\frac{1}{3}} \\ &\leq d(v, y_{2n+2}) + \lambda d(y_{2n+1}, v)^{\frac{1}{3}} d(y_{2n+1}, y_{2n+2})^{\frac{1}{3}} d(v, Su)^{\frac{1}{3}} \\ &\leq \frac{c}{2} + \left(\frac{c}{2}\right)^{\frac{1}{3}} \left(\frac{c}{2}\right)^{\frac{1}{3}} \left(\frac{c}{2}\right)^{\frac{1}{3}} \\ &= \frac{c}{2} + \frac{c}{2} \\ &= c. \end{aligned}$$

Thus, $d(fu, Su) \ll \frac{c}{m}$ for all $m \geq 1$. So $\frac{c}{m} - d(fu, Su) \in P$ for all $m \geq 1$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, and P is closed, $-d(fu, Su) \in P$, but $P \cap (-P) = \{0\}$. Therefore $d(fu, Su) = 0$, hence $fu = Su$. Similarly by using $d(fu, Tu) \leq d(fu, y_{2n+1}) + d(y_{2n+1}, Tu)$, we can show that $fu = Tu$, it implies that v is a common point of coincidence of S, T, f, that is, $v = fu = Su = Tu$. □

6 An Expanding Mapping Theorem

In this section which is inspired by [13], we define expanding mappings in the setting of partial metric spaces analogous to expanding mappings in complete metric spaces. In particular, we extend the notion of alternate interpolative Ciric-Reich-Rus operator to the setting of partial metric spaces. First let us collect some notions and notations that will be useful in the sequel.

Definition 6.1. Given a nonempty set X, a function $\rho : X \times X \mapsto \mathbb{R}^+$ is called a *partial metric* if and only if for all $x, y, z \in X$

(*a*) $x = y$ if and only if $\rho(x,x) = \rho(x,y) = \rho(y,y)$;

(*b*) $\rho(x,x) \leq \rho(x,y)$;

(*c*) $\rho(x,y) = \rho(y,x)$;

(*d*) $\rho(x,z) \leq \rho(x,y) + \rho(y,z) - \rho(y,y)$.

A partial metric space is a pair (X, ρ) such that X is a nonempty set and ρ is a partial metric on X.

Remark 6.2. If $\rho(x,y) = 0$, then from (a) and (b), $x = y$. However, if $x = y$, then $\rho(x,y)$ may not be zero.

Example 6.3. Define $\rho(x,y) = \max\{x,y\}$, then $(\mathbb{R}^+, \rho)$ is a partial metric space.

Remark 6.4. Each partial metric ρ on X generates a t_0 topology $t(\rho)$ on X which has as a base the family of open balls $\{B_p(x;\epsilon) : x \in X; \epsilon > 0\}$, where $B_p(x;\epsilon) = \{y \in X : \rho(x,y) < \rho(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definition 6.5. A sequence $\{x_n\}$ in a partial metric space (X, ρ) converges to a point $x \in X$ if and only if $\rho(x,x) = \lim_{n\to\infty} \rho(x, x_n)$.

Definition 6.6. A sequence $\{x_n\}$ in a partial metric space (X, ρ) is called a *Cauchy sequence* if there exists $\lim_{n,m\to\infty} \rho(x_n, x_m)$.

Definition 6.7. A partial metric space (X, ρ) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges with respect to $t(\rho)$, to a point $x \in X$ such that $\rho(x,x) = \lim_{n,m\to\infty} \rho(x_n, x_m)$.

Example 6.8. Every closed subset of a complete partial metric space is complete.

Definition 6.9. If $f : X \mapsto X$, where (X, ρ) is a partial metric space, then f is continuous at the point $a \in X$ if for every sequence $\{x_n\}$ in X which converges in the partial metric ρ to a, the sequence $\{fx_n\}$ converges to fa, that is,

$$\rho(a,a) = \lim_{n\to\infty} \rho(x_n, a) \Longrightarrow \rho(fa, fa) = \lim_{n\to\infty} \rho(fx_n, fa).$$

Remark 6.10. If ρ is a partial metric on X, then the function $\rho^s : X \times X \mapsto \mathbb{R}^+$ given by

$$\rho^s(x,y) = 2\rho(x,y) - \rho(x,x) - \rho(y,y)$$

is a metric on X.

Definition 6.11. Let (X, ρ) be a partial metric space, and $T : X \mapsto X$. We say T is *interpolative Ciric-Reich-Rus expanding*, if for every $x, y \in X$, $x, y \notin Fix(T)$, there exists $\lambda > 1$ such that $\rho(Tx, Ty) \geq \lambda \rho(x,y)^{\frac{1}{3}} \rho(x, Tx)^{\frac{1}{3}} \rho(y, Ty)^{\frac{1}{3}}$.

Lemma 6.12. *Let (X, ρ) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, ρ) if and only if it is a Cauchy sequence in the metric space (X, ρ^s);*

(b) *A partial metric space (X, ρ) is complete if and only if the metric space (X, ρ^s) is complete. Further, $\lim_{n\to\infty} \rho^s(a, x_n) = 0$ if and only if $\rho(a,a) = \lim_{n\to\infty} \rho(a, x_n) = \lim_{n,m\to\infty} \rho(x_n, x_m)$.*

Lemma 6.13. *Let (X, ρ) be a partial metric space, and $\{x_n\}$ be a sequence in X. If there exists $k \in (0,1)$ such that $\rho(x_n, x_{n+1}) \leq k\rho(x_{n-1}, x_n)$, $n = 1, 2, \cdots$, then $\{x_n\}$ is a Cauchy sequence in X.*

Theorem 6.14. *Let (X, ρ) be a complete partial metric space, and $T : X \mapsto X$ be a surjection. Suppose that there exists $\lambda > 1$ such that for all $x, y \in X$, $x, y \notin Fix(T)$, $x \neq y$*

$$\rho(Tx, Ty) \geq \lambda \rho(x,y)^{\frac{1}{3}} \rho(x, Tx)^{\frac{1}{3}} \rho(y, Ty)^{\frac{1}{3}}.$$

Then T has a fixed point in X.

Proof. Let $x_0 \in X$. Since T is surjective, choose $x_1 \in X$ such that $Tx_1 = x_0$. Inductively we can define a sequence $\{x_n\}$ in X such that $x_{n-1} = Tx_n$, $n = 1, 2, \cdots$. Without loss of generality, we may assume $x_{n-1} \neq x_n$ for all $n = 1, 2, \cdots$ (otherwise, if there exists some n_0 such that $x_{n_0-1} = x_{n_0}$, then x_{n_0} is a fixed point of T). From the expanding condition of the theorem, we have

$$\begin{aligned}\rho(x_{n-1}, x_n) &= \rho(Tx_n, Tx_{n+1})\\ &\geq \lambda\rho(x_n, x_{n+1})^{\frac{1}{3}}\rho(x_n, Tx_n)^{\frac{1}{3}}\rho(x_{n+1}, Tx_{n+1})^{\frac{1}{3}}\\ &\geq \lambda\rho(x_n, x_{n+1})^{\frac{1}{3}}\rho(x_n, x_{n-1})^{\frac{1}{3}}\rho(x_{n+1}, x_n)^{\frac{1}{3}}.\end{aligned}$$

The above implies $\rho(x_n, x_{n+1}) \leq \frac{1}{\lambda^{\frac{3}{2}}}\rho(x_{n-1}, x_n)$. Since $\frac{1}{\lambda^{\frac{3}{2}}} < 1$, by Lemma 6.13, $\{x_n\}$ is a Cauchy sequence in X. Since (X, ρ) is complete, by Lemma 6.12, (X, ρ^s) is complete, and so the sequence $\{x_n\}$ converges in the metric space (X, ρ^s), that is, there exists a point $z \in X$ such that $\lim_{n\to\infty} \rho^s(x_n, z) = 0$. Consequently, we can find $u \in X$ such that $z = Tu$. Again by Lemma 6.12, we have $\rho(z, z) = \lim_{n\to\infty} \rho(x_n, z) = \lim_{n,m\to\infty} \rho(x_n, x_m)$. Moroever, since $\{x_n\}$ is Cauchy in the metric space (X, ρ^s), we have $\lim_{n,m\to\infty} \rho^s(x_n, x_m) = 0$. On the other hand, since

$$\max\{\rho(x_n, x_n), \rho(x_{n+1}, x_{n+1})\} \leq \rho(x_n, x_{n+1})$$

and $\rho(x_n, x_{n+1}) \leq \frac{1}{\lambda^{\frac{3}{2}}}\rho(x_{n-1}, x_n)$, by simple induction we have

$$\max\{\rho(x_n, x_n), \rho(x_{n+1}, x_{n+1})\} \leq \left(\frac{1}{\lambda^{\frac{3}{2}}}\right)^n \rho(x_1, x_0).$$

Hence we have $\lim_{n\to\infty} \rho(x_n, x_n) = 0$. Thus from the definition of ρ^s, we have $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$. Since $\rho(z, z) = \lim_{n\to\infty} \rho(x_n, z) = \lim_{n,m\to\infty} \rho(x_n, x_m)$, we have $\rho(z, z) = \lim_{n\to\infty} \rho(x_n, z) = \lim_{n,m\to\infty} \rho(x_n, x_m) = 0$. Now we show that $u = z$. From the expanding condition of the theorem, we have

$$\begin{aligned}\rho(x_n, z) &= \rho(Tx_{n+1}, Tu)\\ &\geq \lambda\rho(x_{n+1}, u)^{\frac{1}{3}}\rho(x_{n+1}, x_n)^{\frac{1}{3}}\rho(u, Tu)^{\frac{1}{3}}\\ &\geq \lambda\rho(x_{n+1}, u)^{\frac{1}{3}}\rho(u, z)^{\frac{1}{3}}.\end{aligned}$$

Taking limits in the above as $n \to \infty$, we have

$$
\begin{aligned}
0 &= \rho(z, z) \\
&\geq \lambda \rho(u, z)^{\frac{2}{3}}.
\end{aligned}
$$

Hence $\rho(u, z) = 0$, that is, $u = z = Tu$ and the proof is finished. □

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