The New Results in Injective Modules

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Abstract

In this paper, we introduce and clarify a new presentation between the divisible module and the injective module. Also, we obtain some new results about them.

1 Introduction

In mathematics, a module is one of the fundamental algebraic structures used in abstract algebra. A module taking its scalars from a ring R is called an R-module. Thus, a module, like a vector space, is an additive abelian group, a product is defined between elements of the rings and elements of the module that is distributive over the addition operation of each parameter and is compatible with the ring multiplication. In mathematics, especially in the area of abstract of algebra known as module theory, an injective module is a module Q that shares certain desirable properties with the Z-module Q of all rational numbers. Injective modules have been heavily studied, and a variety of additional notions are defined in terms of them. Injective co generators are injective modules that faithfully represent the entire category of modules. Injective resolutions measure how far

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from injective a module is terms of the injective dimension and represent modules in the derived category. Injective hulls are maximal essential extensions, and turn out to be minimal injective extensions. Over a Noetherian ring, every injective modules is uniquely a direct sum of incomparable modules, and their structure is well understood. An injective module over one ring, may not be injective over another, but there well-understood methods of changing rings which handle special cases. Rings which are themselves injective modules have a number of interesting properties and include rings such as group rings of finite groups over fields. Injective modules include divisible groups and are generalized by the notion of injective objects in category theory. In mathematics, one can often define a direct Product of objects already known, giving a new one. This generalizes the Cartesian product of the underlying sets, together with a suitably defined structure on the product set. More abstractly, one talks a bout the product in category theory, which formalizes these notions. Examples are the product of sets, groups (describe below), rings and other algebraic structures. The product of topological spaces is another instance. There is also the direct sum in some areas this is used interchangeably, while in others it is a different concept.

In this paper, we show to prove the important theorems and examples of injective modules. Next, we show a strong relationship between the injective module and the divisible module, such that every injective module gives divisible but the converse needs another condition $P.I.D$. Finally, we recall the definition of n-cokernel, n-kernel, and n-exact sequence and we give an open problem about n-injective modules.

2   Preliminaries

In this section, we recall some of the fundamental concepts and definitions, which are necessary for this paper. For details we refer readers to [1,6–8]

Definition 2.1. Let R be a ring. A (left) R-module is an additive abelian group A together with a function $R \times A \longrightarrow A$ (the image of (r, a) being denoted by ra) such that for all $r, s \in R$ and $a, b \in A$:

(i) $r(a + b) = ra + rb$.

(ii) $(r + s)a = ra + sa$.

(iii) $r(sa) = (rs)a$.

If R has an identity element 1_R and

(iv) $1_R a = a$ for all $a \in A$,

then A is said to be a unitary R-module. If R is a division ring, then a unitary R-module is called a (left) vector space.

Definition 2.2. Suppose R is a ring, and $\{M_i\}_{i \in I}$. The final family are R-modules. We define the direct product of modules M_i, denoted by

$$\prod_{i \in I} M_i = \{(x_i) : x_i \in M_i, \forall i \in I\}.$$

Scalar addition and multiplication is defined by

$$(x_i) + (y_i) = (x_i + y_i),$$

$$r(x_i) = (rx_i).$$

Definition 2.3. Suppose R is a ring, and A and B are R-modules. We define the direct sum of modules A and B as

$$A \bigoplus B = \{a \bigoplus b \mid a \in A, b \in B\},$$

where all algebraic operations are defined componentwise. In particular, suppose that A and B are left R-modules, then

$$(a_1 \bigoplus b_1) + (a_2 \bigoplus b_2) = (a_1 + b_1) \bigoplus (b_1 + b_2)$$

and

$$r(a \bigoplus b) = (ra \bigoplus rb).$$

Definition 2.4. Let M and N be R-modules. Define $Hom_R(M, N)$ to be the Abelian group of R-homomorphisms $f : M \longrightarrow N$, given by $(f + g)(m) = f(m) + g(m)$ for every $m \in M$ and $f, g \in Hom_R(M, N)$.

Definition 2.5. Let $M_{i \in \mathbb{Z}}$ be a family of R-modules, and let $f_{i_{i \in \mathbb{Z}}}$ be a family of R-homomorphisms such that $M_{i-1} \xrightarrow{f_i} M_i$ for every $i \in \mathbb{Z}$. Then the sequence

$$... \xrightarrow{f_{-1}} M_{-1} \xrightarrow{f_0} M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} ... \tag{2.1}$$

is said to be exact provided that $Im(f_{i-1}) = Ker(f_i)$ for every $i \in \mathbb{Z}$. Note that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if f is an R-monomorphism, g is an R-epimorphism, and $Im(f) = Ker(g)$. This type of sequence is called short exact.

Definition 2.6. An R-module M is *injective* provided that for every R-monomorphism $g : A \longrightarrow B$ between R-modules, any R-homomorphism $f : A \longrightarrow M$ can be extended to an R-homomorphism $h : B \longrightarrow M$ such that $hg = f$; i.e., the following diagram commutes

$$
\begin{array}{ccc}
0 \longrightarrow & A & \xrightarrow{\ g\ } B \\
 & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle h} \\
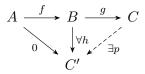 & M &
\end{array}
$$

Definition 2.7. Let M be an R-module. An element $m \in M$ is divisible provided that for any $r \in R$ that is not a right zero-divisor, there exists an $x \in M$ such that $m = rx$. We also say that M is a divisible module provided that every element of M is divisible. Note that a divisible group is a divisible $\mathbb{Z}$-module.

Definition 2.8. Let $\mathcal{C}$ be an additive category and $f : A \longrightarrow B$ a morphism in $\mathcal{C}$. A weak cokernel of f is a morphism $g : B \longrightarrow C$ such that for all $C' \in \mathcal{C}$ the sequence of abelian groups

$$\mathcal{C}(C, C') \xrightarrow{g^*} \mathcal{C}(B, C') \xrightarrow{f^*} \mathcal{C}(A, C')$$

is exact. Equivalently, g is a weak cokernel of f if $fg = 0$ and for each morphism $h : B \longrightarrow C'$ such that $fh = 0$ there exists a (not necessarily unique) morphism $p : C \longrightarrow C'$ such that $h = gp$. These properties are subsumed in the following commutative diagram:

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C$$

with 0, $\forall h$ to C', and $\exists p$.

Clearly, a weak cokernel g of f is a cokernel of f if and only if g is an epimorphism. The concept of weak kernel is defined dually.

Proposition 2.9. *Let R be a ring and let J be an R-module. The following conditions are equivalent.*

1) For any homomorphism $f : N \longrightarrow J$ and an monomorphism $g : M \longrightarrow N$ there is a homomorphism $h : M \longrightarrow J$ such that the following diagram commutes:

$$\begin{array}{ccc} & J & \\ f \uparrow & \nwarrow h & \\ M & \xrightarrow{\;g\;} & N \end{array}$$

2) Every short exact sequence $0 \longrightarrow J \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ splits.

Lemma 2.10. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a family of R-modules. Then for every $\lambda \in \Lambda$, $\pi_\lambda i_\lambda : M_\lambda \longrightarrow M_\lambda$ is the identity function on M_λ.*

Theorem 2.11. *Let $f : M \longrightarrow N$ be an R-homomorphism, let A be a submodule of M, let B be a submodule of N, and let $f(A) \subseteq B$. Define the function $\rho : M/A \longrightarrow N/B$ by $\rho(m + A) = f(m) + B$. We say that ρ is the R-homomorphism induced by f. Then ρ is an R-epimorphism if f is an R-epimorphism, and ρ is a R-monomorphism if $A = f^{-1}(B)$.*

Corollary 2.12. *Let $f : M \longrightarrow N$ be an R-epimorphism. Then $M/Ker(f) \approx N$.*

Proposition 2.13. *Let $0 \longrightarrow A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \longrightarrow 0$ be a short exact sequence of R-homomorphisms. Then the following statements are equivalent:*
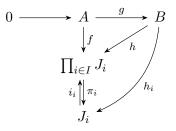
  i. *The sequence is split;*

  ii. *There is an R-homomorphism $h : A_2 \longrightarrow B$ with $gh = 1_{A_2}$;*

  iii. *There is an R-homomorphism $k : B \longrightarrow A_1$ with $kf = 1_{A_2}$;*

3   Main Results

Definition 3.1. An R-module J is an injective module if J satisfies one of the equivalent conditions of Proposition (2.9)
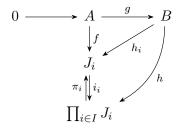
Proposition 3.2. *A direct product of R-modules $\prod_{i\in\mathbb{I}} J_i$ is injective iff J_i is injective for every $i \in \mathbb{I}$.*

Proof. Suppose that J_i is injective $\forall i \in \mathbb{I}$. Now in this diagram

we have to find h. Since J_i is injective $\exists h_i : B \longrightarrow J_i$ such that $h_i g = \pi_i f$. Define $h : B \longrightarrow \prod_{i\in\mathbb{I}} J_i$ to be $h(b) := (h_i(b))_{i\in\mathbb{I}} = (h_1(b); h_2(b); ...)$. Now it is very easy to check that, $hg = f$.

  Conversely, suppose that $\prod_{i\in\mathbb{I}} J_i$ is injective. To show that, J_i is injective for each $i \in \mathbb{I}$. Now in this diagram
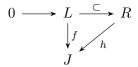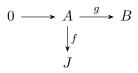

we have $hg = i_i f$.

  We have to find h_i. Define $h_i : B \longrightarrow J_i$ to be $h_i = \pi_i h$. Now it is very easy to check that, $h_i g = f \ \forall i \in \mathbb{I}$. Here, $h_i g = \pi_i h g = \pi_i i_i f = I J_i f = f \ \forall i \in \mathbb{I}$. $\square$

Lemma 3.3. *Let R be a ring with identity. A unitary R-module J is injective if and only if for every left ideal L of R, any R-module homomorphism $L \longrightarrow J$ may be extended to an R-module homomorphism $R \longrightarrow J$.*

Proof. To say that $f : L \longrightarrow J$ may be extended to R means there is a homomorphism $h : R \longrightarrow J$ such that the diagram

$$
\begin{array}{ccc}
0 \longrightarrow & L & \xrightarrow{\subset} R \\
 & \downarrow{f} & \diagdown{h} \\
 & J &
\end{array}
$$

is commutative. Clearly, such an h always exists if J is injective. Conversely, suppose J has the stated extension property and suppose we are given a diagram of module homomorphisms

$$
\begin{array}{ccc}
0 \longrightarrow & A & \xrightarrow{g} B \\
 & \downarrow{f} & \\
 & J &
\end{array}
$$

with top row exact. To show that J is injective we must find a homomorphism $h : B \longrightarrow J$ with $hg = f$. Let S be the set of all R-module homomorphisms $h : C \longrightarrow J$, where

$$Im\ g \subset C \subset B \subset S$$

is nonempty since $fg^{-1} : Im\ g \longrightarrow J$ is an element of S (g is a monomorphism). Partially order S by extension : $h_1 \leqslant h_2$ iff $Dom\ h_1 \subset Dom\ h_2$ and $h_2|\ Dom\ h_1 = h_1$. We can verify that the hypotheses of *Zorn's Lemma* are satisfied and conclude that S contains a maximal element $h : H \longrightarrow J$ with $hg = f$. We shall complete the proof by showing $H = B$.

If $H \neq B$ and $b \in B - H$, then $L = \{r \in |rb \in H\}$ is left ideal of R. The map $L \longrightarrow J$ given by $r \longmapsto h(rb)$ is a well-defined R-module homomorphism. By the hypothesis there is a R-module homomorphism $k : R \longrightarrow J$ such that $k(r) = h(rb)$ for all $r \in L$. Let $c \in k(1_R)$ and define a map $\bar{h} : H + Rb \longrightarrow J$ by $a + rb \longmapsto h(a) + rc$. We claim that $\bar{h}$ is well-defined. For if $a_1 + r_1b =$

$a_2 + r_2 b \in H + Rb$, then $a_1 - a_2 = (r_2 - r_1)b \in H \cap Rb$. Hence $r_2 - r_1 \in L$ and $h(a_1) - h(a_2) = h(a_1 - a_2) = h((r_2 - r_1)b) = k(r_2 - r_1) = (r_2 - r_1)k(1_R) = (r_2 - r_1)c$. Therefore, $\bar{h} : H + Rb \longrightarrow J$ is an R-module homomorphism that is an element of the set S. This contradicts the maximality of h since $b \notin H$ and hence $H \subsetneq H + Rb$. Therefore, $H = B$ and J is injective.

$\square$

Example 3.4.

1. Q is an injective Z-module by Lemma (3.3) since for every Z-homomorphism $f : nZ \longrightarrow Q$, where nZ is an ideal of Z for $0 \neq n \in Z$, there exists a Z-homomorphism $g : Z \longrightarrow Q$ defined by $g(z) = \frac{zf(n)}{n}$, so $g(nz) = \frac{(nz)f(n)}{n} = zf(n) = f(nz)$ for every $nz \in Z$.

2. Note that Z is not an injective Z-module since using the Z-homomorphism $f : 2Z \longrightarrow Z$ given by $f(2z) = z$, there is no Z-homomorphism $g : Z \longrightarrow Z$ such that $g(2z) = f(2z)$ for every $2z \in 2Z$. Otherwise, $1 = f(2) = g(2) = 2g(1)$, implying that $g(1) = \frac{1}{2}$. However, since $g(1) \in Z$, this is impossible.

Theorem 3.5. *Let M be an R-module. Then M is injective if and only if for every short exact sequence $0 \longrightarrow A \xrightarrow{\theta} B \xrightarrow{\psi} C \longrightarrow 0$ of R-modules,*

$$0 \longrightarrow Hom_R(C, M) \xrightarrow{\Psi} Hom_R(B, M) \xrightarrow{\Theta} Hom_R(A, M) \longrightarrow 0$$

is also a short exact sequence, where $\Psi(f) = f\psi$ and $\Theta(f) = f\theta$.

Proof. Suppose that M is injective.

We will first show that Ψ is an R-monomorphism. Suppose that $\Psi(f) = \Psi(g)$ for some $f, g \in Hom_R(C, M)$. Then for every $b \in B$, $f(\psi(b)) = g(\psi(b))$. We want to see that $f = g$. Note that for every $c \in C$, there exists $b \in B$ such that $\psi(b) = c$ since ψ is an R-epimorphism. Then for every $c \in C$, $f(c) = f(\psi(b)) = g(\psi(b)) = g(c)$. Thus, $f = g$ and Ψ is an R-monomorphism.

Secondly, we will see that Θ is an R-epimorphism. Let $g \in Hom_R(A, M)$. Since M is injective, there exists an $f \in Hom_R(B, M)$ such that $g = f\theta = \Theta(f)$. Thus, Θ is an R-epimorphism.

Finally, we will show that $Ker(\Theta) = Im(\Psi)$ Let $f \in Im(\Psi)$, so there exists $g \in Hom_R(C, M)$ such that $\Theta(g) = g\theta = f$. Let $a \in A$. So, $\theta(a) \in Im(\theta) = Ker(\psi)$. Then $\psi(\theta(a)) = 0$, so $g\psi(\theta(a)) = g(0) = 0$. Thus, for every $a \in A$, $f(\theta(a)) = 0$. Therefore, $f \in Ker(\theta)$ so that $Im(\Psi) \subseteq Ker(\Theta)$. Let $f \in Ker(\Theta)$ be arbitrary. Then $\Theta(f) = f\theta = 0$ so that $0 = f(Im\theta) = f(Ker\psi)$. Since $Ker(\psi) \subseteq B$ and $f(Ker\psi) \subseteq 0$, there is an induced R-homomorphism $\rho : B/Ker(\psi) \longrightarrow M$ given by $\rho(m + Ker\psi) = f(m)$ by Theorem (2.11). Also, since $Ker(\psi) \subseteq B$ and $\psi(Ker\psi) \subseteq 0$, there is an induced R-isomorphism $\varsigma : B/Ker\psi \longrightarrow C$ given by $\varsigma(b + Ker\psi) = \psi(b)$ by Corollary (2.12). Consider $\rho\varsigma^{-1} : C \longrightarrow M$. Notice that $\rho\varsigma^{-1}$ is an R-homomorphism since ρ and ς^{-1} are. Since $\phi(b + Ker\psi) = \psi(b)$ implies that $b + Ker\psi = \varsigma^{-1}\psi(b)$, for every $b \in B$, $\rho\varsigma^{-1}\psi(b) = \rho(b + Ker\psi) = f(b)$, so $f = \rho\varsigma^{-1}\psi = \Psi(\rho\varsigma^{-1})$. Thus, $f \in Im(\Psi)$ and $Ker(\Theta) \subseteq Im(\Psi)$. Therefore, $Ker(\Theta) = Im(\Psi)$, and hence, $0 \longrightarrow Hom_R(C, M) \xrightarrow{\Psi} Hom_R(B, M) \xrightarrow{\Theta} Hom_R(A, M) \longrightarrow 0$ is a short exact sequence.

Conversely, suppose that whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence between R-modules, $0 \longrightarrow Hom_R(C, M) \longrightarrow Hom_R(B, M) \longrightarrow Hom_R(A, M) \longrightarrow 0$ is also exact. Let I be a left ideal of a ring R, so I is a submodule of an R-module R, where $0 \longrightarrow I \xrightarrow{i} R$ is a sequence with the inclusion map i and $f : I \longrightarrow M$ is an R-homomorphism. Consider the following diagram

$$0 \longrightarrow I \xrightarrow{i} R \xrightarrow{\pi} R/I \longrightarrow 0$$
$$\downarrow f \quad \swarrow g$$
$$M$$

where $\pi : R \longrightarrow R/I$ is the projection R-homomorphism. Since the inclusion map i is an R-monomorphism, the projection map π is an R-epimorphism, and $Ker(\pi) = Im(i)$, the top row is exact. Then $0 \longrightarrow Hom_R(R/I, M) \xrightarrow{\Pi} Hom_R(R, M) \xrightarrow{I} Hom_R(I, M) \longrightarrow 0$ is also exact. Since $f \in Hom_R(I, M)$ and I is an R-epimorphism, there exists $g \in Hom_R(R, M)$ such that $I(g) = gi = g|_I = f$. Thus, M is injective $\qquad\square$

Corollary 3.6. *Let R be an integral domain and let K be the field of fractions of R. Then K is an injective R-module.*

Proof. Let I be an ideal of R and let $f : I \longrightarrow K$ be a homomorphism of R-modules. For $0 \neq r, s \in I$ we have $rf(s) = f(rs) = sf(r)$. As consequence in K we have $f(r)/r = f(s)/s$ for any $0 \neq r, s \in I$. Denote this element by a. Define

$$\bar{f} : R \longrightarrow K, \ \bar{f}(r) := ra$$

Check: $\bar{f}$ is a homomorphism of R-modules and $\bar{f}|_I = f$. By Lemma (3.3) it follows that K is an injective R-module. $\square$

Proposition 3.7. *Let $M \neq 0$ and N be R-modules, and let $\theta : M \hookrightarrow N$ be a monomorphism. Then the following are equivalent:*

1. *Every nonzero submodule of N has a nonzero intersection with $\theta(M)$.*

2. *Every nonzero element of N has a nonzero multiple in $\theta(M)$.*

3. *If $\phi o\theta$ is injective for a homomorphism $\phi : N \longrightarrow Q$, then ϕ is injective.*

Proof. $(1) \Longrightarrow (2)$ If n is a nonzero element of N, then the cyclic module Rn has a nonzero intersection with $\theta(M)$.

$(2) \Longrightarrow (3)$ If not, then $\ker\phi$ has a nonzero intersection with $\theta(M)$. contradicting the assumption that $\phi o\theta$ is injective.

$(3) \Longrightarrow (1)$ Let N' be a nonzero submodule of N, and consider the canonical surjection $\phi : N \longrightarrow N/N'$. Then ϕ is not injective, hence the composition $\phi o\theta : M \longrightarrow N/N'$ is not injective, i.e., N' contains a nonzero element of $\theta(M)$.
                                                                              $\square$

Definition 3.8. An abelian group D is said to be *divisible* if given any $y \in D$ and $0 \neq n \in \mathbb{Z}$, there exists $x \in D$ such that $nx = y$.

Example 3.9.

1. If R is a division ring, then every R-module is injective by Lemma (3.3) since the only ideals of R are 0 and R.

2. Note that Q is a divisible Z-module since for every $q \in Q$, where $q = \frac{a}{b}$ for integers $a, b \in Z$ with $b \neq 0$, and for every $0 \neq z \in Z$, there exists $x \in Q$ such that $x = \frac{a}{zb}$ so that $q = zx$.

3. Note that Z is not a divisible Z-module since there is no $x \in Z$ with $3 = 2x$.

Theorem 3.10. *Every injective R-module is divisible.*

Proof. Suppose that M be an injective R-module. Let $m \in M$ and let $r \in R$ be such that r is not a right zero-divisor. Note that Rr is a left ideal in R. Define the well-defined function $f : Rr \longrightarrow M$ by $f(tr) = tm$ since r is not a right zero-divisor. Note that for every $s, t \in R$

$$f(tr + sr) = f((t+s)r) = (t+s)m = tm + sm = f(tr) + f(sr) \qquad (3.1)$$

and

$$f(t(sr)) = f((ts)r) = (ts)m = t(sm) = tf(sm), \qquad (3.2)$$

so f is an R-homomorphism. Since M is injective, there is an R-homomorphism $g : R \longrightarrow M$ such that $g|_R I = f$. Thus, $m = 1_R m = f(1_R r) = g(1_R r) = rg(1_R)$, where $g(1_R) \in M$. Hence, M is divisible. $\qquad \square$

Lemma 3.11. *An abelian group D is divisible if and only if D is an injective (unitary) $\mathbb{Z}$-module.*

Proof. If D is injective, $y \in D$ and $0 \neq n \in \mathbb{Z}$, let $f : \langle n \rangle \longrightarrow D$ be the unique homomorphism determined by $n \mapsto y$; ($\langle n \rangle$ is a free $\mathbb{Z}$ module). Since D is injective, there is a homomorphism $h : \mathbb{Z} \longrightarrow D$ such that the diagram

$$
\begin{array}{ccc}
0 \longrightarrow \langle n \rangle & \overset{\subset}{\longrightarrow} & \mathbb{Z} \\
\downarrow^{f} & \swarrow^{h} & \\
D & &
\end{array}
$$

is commutative. If $x = h(1)$, then $nx = nh(1) = h(n) = f(n) = y$. Therefore, D is divisible. To prove the converse note that the only left ideals of $\mathbb{Z}$ are the cyclic groups $\langle n \rangle, n \in \mathbb{Z}$. If D is divisible and $f : \langle n \rangle \longrightarrow D$ is a homomorphism,

then there exists $x \in D$ with $nx = f(n)$. Define $h : \mathbb{Z} \longrightarrow D$ by $1 \mapsto x$ and verify that h is a homomorphism that extends f. Therefore, D is injective by Lemma (3.3). $\qquad \square$

Remark 3.12. The rationals $\mathbb{Q}$ (with addition) form an injective abelian group (i.e. an injective $\mathbb{Z}$-module). The factor group $\mathbb{Q}/\mathbb{Z}$ is also injective $\mathbb{Z}$-module.

Lemma 3.13. *Every abelian group A may be embedded in a divisible abelian group.*

Proof. There is a free $\mathbb{Z}$-module F and an epimorphism $F \longrightarrow A$ with $ker K$ so that $F/K \cong A$. Since F is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{Q}$, F may be embedded in a direct sum D of copies of the rationals $\mathbb{Q}$. But D is a divisible group by Proposition (3.2) and Lemma (3.11). If $f : F \longrightarrow D$ is the embedding monomorphism, then f induces an isomorphism $F/K \cong f(F)/f(K)$. Thus the composition $A \cong F/K \cong f(F)/f(K) \subset D/f(K)$ is a monomorphism. But $D/f(K)$ is divisible since it is the homomorphic image of a divisible group.
$\qquad \square$

Lemma 3.14. *If J is a divisible abelian group and R is a ring with identity, then $Hom_{\mathbb{Z}}(R, J)$ is an injective left R-module.*

Proof. By Lemma (3.3) it suffices to show that for each left ideal L of R, every R-module homomorphism $f : L \longrightarrow Hom_{\mathbb{Z}}(R, J)$ may be extended to an R-module homomorphism $h : R \longrightarrow Hom_{\mathbb{Z}}(R, J)$. The map $g : L \longrightarrow J$ given by $g(a) = [f(a)](1_R)$ is a group homomorphism. Since J is an injective $\mathbb{Z}$-module by Lemma (3.11) and we have the diagram

$$0 \longrightarrow L \overset{\subset}{\longrightarrow} R$$
$$\downarrow{\scriptstyle g}$$
$$J$$

there is group homomorphism $\bar{g} : R \longrightarrow J$ such that $\bar{g}|L = g$. Define $h : R \longrightarrow Hom_{\mathbb{Z}}(R, J)$ by $n \mapsto y$, where $h(r) : R \longrightarrow J$ is the map given by $[h(r)](x) = \bar{g}(xr)(x \in R)$. h is well-defined function (that is, each $h(r)$ is a group

homomorphism $R \longrightarrow J$) and h is group homomorphism $R \longrightarrow Hom_{\mathbb{Z}}(R, J)$. If $s, r, x \in R$, then

$$h(sr)(x) = \bar{g}(x(sr)) = \bar{g}((xs)r) = h(r)(xs). \tag{3.3}$$

By the definition of the R-module structure of $Hom_{\mathbb{Z}}(R, J), h(r)(xs) = [sh(r)](x)$, whence $h(sr) = sh(r)$ and h is an R-module homomorphism. Finally suppose $r \in L$ and $x \in R$. Then $xr \in L$ and

$$h(r)(x) = \bar{g}(xr) = g(xr) = [f(xr)](1_R). \tag{3.4}$$

Therefore, $h(r) = f(r)$ for $r \in L$ and h is an extension of f. $\quad\square$

Proposition 3.15. *Every unitary module A over a ring R with identity maybe embedded in an injective R-module*

Proof. Since A is an abelian group, there is a divisible group J and a group monomorphism $f : A \longrightarrow J$ by Lemma (3.13). The map $f : Hom_{\mathbb{Z}}(R, A) \longrightarrow Hom_{\mathbb{Z}}(R, J)$ given on $g \in Hom_{\mathbb{Z}}(R, A)$ by $\bar{f}(g) = fg \in Hom_{\mathbb{Z}}(R, J)$ is easily seen to be an R-module monomorphism. Since every R-module homomorphism is a $\mathbb{Z}$-module homomorphism, we have $Hom_R(R, A) \subset Hom_{\mathbb{Z}}(R, A)$. In fact, it is easy to see that $Hom_R(R, A)$ is an R-submodule of $Hom_{\mathbb{Z}}(R, A)$. Finally, the map $A \longrightarrow Hom_R(R, A)$ given by $a \mapsto f_a$, where $f_a(r) = ra$, is an R-module monomorphism (in fact it is an isomorphism). Composing these maps yields an R-module monomorphism

$$A \longrightarrow Hom_R(R, A) \xhookrightarrow{} Hom_{\mathbb{Z}}(R, A) \xrightarrow{\bar{f}} Hom_{\mathbb{Z}}(R, J). \tag{3.5}$$

Since $Hom_{\mathbb{Z}}(R, J)$ is an injective R-module by Lemma (3.14), we have embedded A in an injective. $\quad\square$

Lemma 3.16. *Let R be a P.I.D. If M is a divisible R-module, then M is injective*

Proof. Suppose that M is an injective R-module. Then by Theorem (3.10) M is divisible. Conversely, suppose that M is a divisible R-module. Let $f : I \longrightarrow M$ be an R-homomorphism, where I is a left ideal of R. Since R is a principle ideal ring, $I = Rt$ for some $t \in R$. Since M is divisible, there exists an $m \in M$ such that $f(t) = tm$. Define $g : R \longrightarrow M$ by $g(r) = rm$. Then g is an R-homomorphism. Note that for every $st \in Rt$, $g(st) = (st)m = s(tm) = sf(t) = f(st)$. Thus, M is injective.   □

Example 3.17. If Z is $P.I.D$ injective Z-module, then Z is divisible module over Z.

Corollary 3.18. *If R is a P.I.D, J is an injective R-module and K is a submodule of J, then J/K is injective.*

Proof. J is divisible so J/K is divisible with R $P.I.D$ imply J/K is injective   □

4   One Open Problem

Firstly we recall following definitions from [5].

Definition 4.1. Let $\mathcal{C}$ be an additive category and $d^0 : X^0 \longrightarrow X^1$ be a morphism in $\mathcal{C}$. An *n-coker* of d^0 is a sequence

$$(d^1, ..., d^n) : X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} ... \xrightarrow{d^n} X^{n+1}$$

such that, for all $Y \in \mathcal{C}$ the induced sequence of abelian groups

$$0 \longrightarrow \mathcal{C}(X^{n+1}, Y) \xrightarrow{d^{n*}} \mathcal{C}(X^n, Y) \xrightarrow{d^{n-1*}} ... \xrightarrow{d^{1*}} \mathcal{C}(X^1, Y) \xrightarrow{d^{0*}} \mathcal{C}(X^0, Y)$$

is exact. Equivalently, the sequence $(d^1, ..., d^n)$ is an n-coker of d^0 if, for all $1 \leq k \leq n - 1$ the morphism d^k is a weak cokernel of d^{k-1}, and d^n is moreover a cokernel of d^{n-1}. The concept of n-ker of morphism is defined dually.

Definition 4.2. Let $\mathcal{C}$ be an additive category. An *n-exact sequence* in $\mathcal{C}$ is a complex

$$X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1}$$

in $Ch^n(\mathcal{C})$ such that $(d^0, ..., d^{n-1})$ is an *n*-ker of d^n, and $(d^1, ..., d^n)$ is an *n*-coker of d^0.

Now one can investigate an *n*-injective module *n*-divisible modules. Next one can obtain all of the result of them as we obtained in this paper, and it is an open problem.

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