Certain Subclass of Analytic Functions Defined by Wanas Operator

Timilehin Gideon Shaba[1,*], Abbas Kareem Wanas[2] and
Ismaila Omeiza Ibrahim[3]

[1] Department of Mathematics, University of Ilorin, P. M. B. 1515, Ilorin, Nigeria

e-mail: shabatimilehin@gmail.com

[2] Department of Mathematics, College of Science, University of Al-Qadisiyah, Al Diwaniyah,

Al-Qadisiyah, Iraq

e-mail: abbas.kareem.w@qu.edu.iq

[3] Department of Mathematical science, University of Maiduguri, Nigeria

e-mail: ibrahimismailaomeiza@gmail.com

Abstract

In present article, we introduce and study a certain family of analytic functions defined by Wanas operator in the open unit disk. We establish some important geometric properties for this family. Further we point out certain special cases for our results.

1   Introduction and Definitions

Let $\mathfrak{A}$ denote the class of function $\mathfrak{f}(z)$ which are normalized analytic in the open unit disk $\mathfrak{U} = \{z \in \mathcal{C} : |z| < 1\}$. Let $\mathfrak{L}(\mathfrak{U})$ be the space of analytic functions in the unit disc $\mathfrak{U}$. Let

$$\mathfrak{A}_e = \left\{ \mathfrak{f} \in \mathfrak{L}(\mathfrak{U}) : \mathfrak{f}(z) = z + \sigma_{e+1}z^{e+1} + \sigma_{e+2}z^{e+2} + \cdots \right\} \qquad (1.1)$$

with $\mathfrak{A}_1 = \mathfrak{A}$, $z \in \mathfrak{U}$ and

$$\mathcal{L}[\sigma, e] := \left\{ \varphi \in \mathcal{W}(\mathfrak{U}) : \varphi(z) = \sigma + \sigma_e z^e + \sigma_{e+1}z^{e+1} + \cdots \right\} \qquad (1.2)$$

*Corresponding author

for $\sigma \in \mathbb{C}$ and $e \in \mathcal{N}$.

We denote by $\mathcal{S}$ subclass of functions which are analytic, univalent in $\mathfrak{U}$ and has the normalization

$$\mathfrak{f}(z) = z + \sum_{t=2}^{\infty} \sigma_{t+1} z^{t+1}, \tag{1.3}$$

which implies that

$$\mathfrak{f}(0) = 0, \quad \mathfrak{f}'(0) = 1.$$

A function $\mathfrak{f}(z) \in \mathcal{S}$ is said to be a starlike, convex and turning bounded functions of order δ which are denoted by $S^*(\delta), \mathcal{K}(\delta), \mathcal{R}(\delta) \subset \mathcal{S}$, if the following conditions are satisfied: $\Re\left(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}\right) > \delta, \Re\left(\frac{z\mathfrak{f}''(z)}{\mathfrak{f}'(z)} + 1\right) > \delta$ and $\Re(\mathfrak{f}'(z)) > \delta$, where $0 \leqq \delta < 1$.

Next we recall the definition of subordination. For two functions $h_1, h_2 \in \mathfrak{U}$, we say that h_1 is subordinated to h_2 and symbolically written as $h_1 \prec h_2$ if there exists an analytic function w with the property $|w(z)| \leqq |z|$ such that $h_1(z) = h_2(w(z))$ for $z \in \mathbb{D}$. Further, if $h_2 \in \mathcal{S}$, then the condition becomes

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0) \text{ and } h_1(\mathfrak{U}) \subset h_2(\mathfrak{U}).$$

Wanas [16] in 2019 introduced the following operator, which can also be called (Wanas operator) $\mathfrak{W}_{\beta,n}^{\alpha,\sigma} : \mathfrak{U} \longrightarrow \mathfrak{U}$ defined by

$$\mathfrak{W}_{\beta,m}^{\mu,\gamma} \mathfrak{f}(z) = z + \sum_{t=2}^{\infty} [\amalg_t(\gamma, \mu, \beta)]^m \sigma_t z^t, \tag{1.4}$$

where

$$\amalg_t(\gamma, \mu, \beta) = \sum_{a=1}^{\gamma} \binom{\gamma}{a} (-1)^{a+1} \left(\frac{\mu^a + t\beta^a}{\mu^a + \beta^a}\right), \tag{1.5}$$

$a, m \in \mathcal{N}_0, \beta \geqq 0, \mu \in \mathcal{R}$ and $\mu + \beta > 0$.

Special cases of this operator can be found in [1, 2, 3, 6, 9, 10, 13, 14, 15]. For more details see [17].

It is readily confirmed from (1.4) that

$$z(\mathfrak{W}_{\beta,m}^{\mu,\gamma} \mathfrak{f}(z))' = \left[\sum_{a=1}^{\gamma} \binom{\gamma}{a} (-1)^{a+1} \left(\left(\frac{\mu}{\beta}\right)^a + 1\right)\right] \mathfrak{W}_{\beta,m+1}^{\mu,\gamma} \mathfrak{f}(z)$$

$$- \left[\sum_{a=1}^{\gamma} \binom{\gamma}{a} (-1)^{a+1} \left(\frac{\mu}{\beta}\right)^a\right] \mathfrak{W}_{\beta,m}^{\mu,\gamma} \mathfrak{f}(z). \tag{1.6}$$

Lemma 1.1. *[8] Let ϕ be holomorphic in $\mathfrak{U}$ with $\phi(0) = 1$. If*

$$\Re\left(1 + \frac{z\phi'(z)}{\phi(z)}\right) > \frac{3\delta - 1}{2\delta},$$

then $\Re(\phi(z)) > \delta$ in $\mathfrak{U}$, $z \in \mathfrak{U}$ and $\frac{1}{2} \leqq \delta < 1$.

2 Main Result

Definition 2.1. We say that a function $\mathfrak{f}(z) \in \mathfrak{A}$ is in the class $\mathcal{G}^m_{\lambda,\delta}(\gamma, \mu, \beta)$ if

$$\left|\frac{\mathfrak{W}^{\mu,\gamma}_{\beta,m+1}\mathfrak{f}(z)}{z}\left(\frac{z}{\mathfrak{W}^{\mu,\gamma}_{\beta,m}\mathfrak{f}(z)}\right)^{\lambda} - 1\right| < 1 - \delta \quad (z \in \mathfrak{U}),$$

where $\beta \geqq 0$, $\mu \in \mathcal{R}$, $\mu + \beta > 0$, $\lambda \geq 0$, $a, m \in \mathfrak{N}_0 = \{0, 1, 2, 3\cdots\}$ and $0 \leqq \delta < 1$.

Remark 2.2. The family $\mathcal{G}^m_{\lambda,\delta}(\gamma, \mu, \beta)$ is a new comprehensive class of holomorphic functions which includes numerous new classes of holomorphic univalent functions as well as some very well-known ones. In place of "equivalence" we are going to take "contained in" as it was discussed in [4], also see [11, 12] for more details. For example,

1. For $m = \mu = 0$ and $\gamma = \beta = \lambda = 1$, we have the class $\mathcal{G}^0_{1,\delta}(1, 0, 1)$ contained in $S^*(\delta)$.

2. For $\mu = 0$ and $m = \gamma = \beta = \lambda = 1$, we have the class $\mathcal{G}^1_{1,\delta}(1, 0, 1)$ contained in $\mathcal{K}(\delta)$.

3. For $m = \mu = \lambda = 0$ and $\gamma = \beta = 1$, we have the class $\mathcal{G}^0_{0,\delta}(1, 0, 1)$ contained in $\mathcal{R}(\delta)$.

4. For $\mu = 1 - \beta$ and $\gamma = 1$, we have the class

$$\mathcal{G}^m_{\lambda,\delta}(1, 1 - \beta, \beta) = \mathcal{G}^m_{\lambda,\delta}(\beta).$$

introduced by Cătaş and Lupaş [5].

5. For $m = 0$, the class

$$\mathcal{B}(\lambda, \delta) = \left\{ \mathfrak{f}(z) \in \mathfrak{A} : \left| \mathfrak{f}'(z) \left(\frac{z}{\mathfrak{f}(z)} \right)^{\lambda} - 1 \right| < 1 - \delta; \lambda \geq 0, 0 \leqq \delta < 1, \ z \in \mathfrak{U} \right\}$$

intoduced by Frasin and Jahangiri [8].

6. For $m = 0$ and $\lambda = 2$, the class

$$\mathcal{B}(\delta) = \left\{ \mathfrak{f}(z) \in \mathfrak{A} : \left| \frac{z^z \mathfrak{f}'(z)}{\mathfrak{f}^2(z)} - 1 \right| < 1 - \delta; 0 \leqq \delta < 1, \ z \in \mathfrak{U} \right\}$$

intoduced by Frasin and Darus [7].

Theorem 2.3. *If for all function $\mathfrak{f}(z) \in \mathfrak{A}$, $\beta \geqq 0$, $\mu \in \mathcal{R}$, $\mu + \beta > 0$, $\lambda \geq 0$, $a, m \in \mathfrak{N}_0 = \{0, 1, 2, 3 \cdots\}$ and $0 \leqq \delta < 1$, we have*

$$\frac{\left[\sum_{a=1}^{\gamma} \binom{\gamma}{a} (-1)^{a+1} \left(\left(\frac{\mu}{\beta} \right)^a + 1 \right) \right] \mathfrak{W}_{\beta,m+2}^{\mu,\gamma} \mathfrak{f}(z)}{\mathfrak{W}_{\beta,m+1}^{\mu,\gamma} \mathfrak{f}(z)}$$

$$- \frac{\lambda \left[\sum_{a=1}^{\gamma} \binom{\gamma}{a} (-1)^{a+1} \left(\left(\frac{\mu}{\beta} \right)^a + 1 \right) \right] \mathfrak{W}_{\beta,m+1}^{\mu,\gamma} \mathfrak{f}(z)}{\mathfrak{W}_{\beta,m}^{\mu,\gamma} \mathfrak{f}(z)}$$

$$- (1 - \lambda) \left(\sum_{a=1}^{\gamma} \binom{\gamma}{a} (-1)^{a+1} \left(\frac{\mu}{\beta} \right)^a + 1 \right) + 1 \prec \varphi z + 1, \quad z \in \mathfrak{U}, \quad (2.1)$$

where $\varphi = \frac{-1+3\delta}{2\delta}$, then $\mathfrak{f} \in \mathcal{G}_{\lambda,\delta}^m(\gamma, \mu, \beta)$.

Proof. Now taking

$$\phi(z) = \frac{\mathfrak{W}_{\beta,m+1}^{\mu,\gamma} \mathfrak{f}(z)}{z} \left(\frac{z}{\mathfrak{W}_{\beta,m}^{\mu,\gamma} \mathfrak{f}(z)} \right)^{\lambda},$$

then $\phi(z)$ is holomorphic in $\mathfrak{U}$ with $\phi(0) = 1$. With the knowledge of differentiation we have

$$\log(\phi(z)) = \log(\mathfrak{W}_{\beta,m+1}^{\mu,\gamma} \mathfrak{f}(z)) - \log(z) + \lambda \log(z) - \lambda \log(\mathfrak{W}_{\beta,m}^{\mu,\gamma} \mathfrak{f}(z))$$

$$\frac{\phi'(z)}{\phi(z)} = \frac{(\mathfrak{W}^{\mu,\gamma}_{\beta,m+1}\mathfrak{f}(z))'}{\mathfrak{W}^{\mu,\gamma}_{\beta,m+1}\mathfrak{f}(z)} - \frac{\phi(\mathfrak{W}^{\mu,\gamma}_{\beta,m}\mathfrak{f}(z))'}{\mathfrak{W}^{\mu,\gamma}_{\beta,m}\mathfrak{f}(z)} - \left(\frac{1-\phi}{z}\right)$$

$$\frac{\phi'(z)}{\phi(z)} = \frac{\left[\sum_{a=1}^{\gamma}\binom{\gamma}{a}(-1)^{a+1}\left(\left(\frac{\mu}{\beta}\right)^a + 1\right)\right]\mathfrak{W}^{\mu,\gamma}_{\beta,m+2}\mathfrak{f}(z)}{\mathfrak{W}^{\mu,\gamma}_{\beta,m+1}\mathfrak{f}(z)}$$

$$- \frac{\sum_{a=1}^{\gamma}\binom{\gamma}{a}(-1)^{a+1}\left(\frac{\mu}{\beta}\right)^a}{z} + \frac{\lambda\sum_{a=1}^{\gamma}\binom{\gamma}{a}(-1)^{a+1}\left(\frac{\mu}{\beta}\right)^a}{z}$$

$$- \frac{\lambda\left[\sum_{a=1}^{\gamma}\binom{\gamma}{a}(-1)^{a+1}\left(\left(\frac{\mu}{\beta}\right)^a + 1\right)\right]\mathfrak{W}^{\mu,\gamma}_{\beta,m+1}\mathfrak{f}(z)}{z\mathfrak{W}^{\mu,\gamma}_{\beta,m}\mathfrak{f}(z)} - \left(\frac{1-\phi}{z}\right)$$

multiplying throughout by z, yields

$$\frac{z\phi'(z)}{\phi(z)} = \frac{\left[\sum_{a=1}^{\gamma}\binom{\gamma}{a}(-1)^{a+1}\left(\left(\frac{\mu}{\beta}\right)^a + 1\right)\right]\mathfrak{W}^{\mu,\gamma}_{\beta,m+2}\mathfrak{f}(z)}{\mathfrak{W}^{\mu,\gamma}_{\beta,m+1}\mathfrak{f}(z)}$$

$$- \frac{\lambda\left[\sum_{a=1}^{\gamma}\binom{\gamma}{a}(-1)^{a+1}\left(\left(\frac{\mu}{\beta}\right)^a + 1\right)\right]\mathfrak{W}^{\mu,\gamma}_{\beta,m+1}\mathfrak{f}(z)}{\mathfrak{W}^{\mu,\gamma}_{\beta,m}\mathfrak{f}(z)}$$

$$- \left(1-\lambda\right)\left(\sum_{a=1}^{\gamma}\binom{\gamma}{a}(-1)^{a+1}\left(\frac{\mu}{\beta}\right)^a + 1\right).$$

Applying (2.1), we have

$$\Re\left(1 + \frac{z\phi'(z)}{\phi(z)}\right) > \frac{-1+3\delta}{2\delta}.$$

Thus, by the application of Lemma 1.1, we have

$$\Re\left(\frac{\mathfrak{W}^{\mu,\gamma}_{\beta,m+1}\mathfrak{f}(z)}{z}\left(\frac{z}{\mathfrak{W}^{\mu,\gamma}_{\beta,m}\mathfrak{f}(z)}\right)^{\lambda}\right) > \delta.$$

Hence, $\mathfrak{f} \in \mathcal{G}^m_{\lambda,\delta}(\gamma,\mu,\beta)$, by the reason of Definition 2.1. $\qquad\square$

Varying the parameters of the above theorem gives the following corollaries.

Corollary 2.4. *Suppose* $\mathfrak{f}(z) \in \mathfrak{A}$ *and*

$$\Re\left(\frac{z\mathfrak{f}''(z)}{\mathfrak{f}'(z)} + 1\right) > \frac{1}{2}, \quad z \in \mathfrak{U},$$

then

$$\Re(\mathfrak{f}'(z)) > \frac{1}{2}, \quad z \in \mathfrak{U}.$$

We can as well say that, if the function $\mathfrak{f}(z)$ *is convex of order* $1/2$*, then* $\mathfrak{f}(z) \in \mathcal{G}_{0,\delta}^{0}(1,0,1)$ *contained in* $\mathcal{R}(1/2)$*.*

Corollary 2.5. *Suppose* $\mathfrak{f}(z) \in \mathfrak{A}$ *and*

$$\Re\left(-\frac{z\mathfrak{f}''(z)}{\mathfrak{f}'(z)} + \frac{z(z\mathfrak{f}'''(z) + \mathfrak{f}''(z))}{z\mathfrak{f}''(z) + \mathfrak{f}'(z)}\right) > -\frac{1}{2} \quad z \in \mathfrak{U},$$

then $\mathfrak{f}(z) \in \mathcal{G}_{1,1/2}^{1}(1,0,1)$*, therefore*

$$\Re\left(\frac{z\mathfrak{f}''(z)}{\mathfrak{f}'(z)} + 1\right) > \frac{1}{2}, \quad z \in \mathfrak{U}.$$

Which implies that $\mathfrak{f}(z)$ *is convex of order* $1/2$*.*

Corollary 2.6. *Suppose* $\mathfrak{f}(z) \in \mathfrak{A}$ *and*

$$\Re\left(-\frac{z\mathfrak{f}'(z)}{\mathfrak{f}'(z)} + \frac{\mathfrak{f}(z) + 3z\mathfrak{f}'(z) + z^2\mathfrak{f}''(z)}{z\mathfrak{f}'(z) + \mathfrak{f}'(z)}\right) > \frac{1}{2}, \quad z \in \mathfrak{U},$$

then

$$\Re\left(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)}\right) > 0, \quad z \in \mathfrak{U}.$$

Which implies that $\mathfrak{f}(z)$ *is a starlike function.*

Corollary 2.7. *[5] Suppose* $\mathfrak{f}(z) \in \mathfrak{A}$ *and*

$$\Re\left(\frac{5z\mathfrak{f}'(z) - \mathfrak{f}(z) + z^2\mathfrak{f}''(z)}{z\mathfrak{f}(z) + \mathfrak{f}(z)}\right) > -\frac{1}{2}, \quad z \in \mathfrak{U},$$

then

$$\Re\left(-2 + \frac{\mathfrak{f}(z)}{z} + \mathfrak{f}'(z)\right) > 2, \quad z \in \mathfrak{U}.$$

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