Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 3, Number 1, 2020, Pages 1-19
https://doi.org/10.34198/ejms.3120.119

Anti Fuzzy Equivalence Relation on Rings with respect to *t*-conorm *C*

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Abstract

In this paper, by using t-conorms, we define the concept of anti fuzzy equivalence relation and anti fuzzy congruence relation on ring R and we investigate some of their basic properties. Also we define fuzzy ideals of ring R under t-conorms and compare this with fuzzy equivalence relation and fuzzy congruence relation on ring R such that we define new introduced ring. Next we investigate this concept under homomorphism of new introduced ring.

1. Introduction

A ring is a set equipped with two operations (usually referred to as addition and multiplication) that satisfy certain properties: there are additive and multiplicative identities and additive inverses, addition is commutative, and the operations are associative and distributive. The study of rings has its roots in algebraic number theory, via rings that are generalizations and extensions of the integers, as well as algebraic geometry, via rings of polynomials. These kinds of rings can be used to solve a variety of problems in number theory and algebra; one of the earliest such applications was the use of the Gaussian integers by Fermat, to prove his famous two-square theorem. There are many examples of rings in other areas of mathematics as well, including topology and mathematical analysis. In mathematics, an equivalence relation is a binary relation that is

Received: September 2, 2019; Accepted: October 22, 2019

2010 Mathematics Subject Classification: 13A99, 37A20, 08A30, 03E72, 03B45, 20K30.

Keywords and phrases: ring theory, equivalence relations, congruence relations, fuzzy set theory, conorms, anti fuzzy equivalence relations, anti fuzzy congruence relations, homomorphisms.

at the same time a reflexive relation, a symmetric relation and a transitive relation. A congruence relation is an equivalence relation whose domain X is also the underlying set for an algebraic structure, and which respects the additional structure. In general, congruence relations play the role of kernels of homomorphisms, and the quotient of a structure by a congruence relation can be formed. In many important cases congruence relations have an alternative representation as substructures of the structure on which they are defined. E.g. the congruence relations on groups correspond to the normal subgroups. In mathematics, fuzzy sets (aka uncertain sets) are somewhat like sets whose elements have degrees of membership. Fuzzy sets were introduced independently by Zadeh [30] and Gottwald [4] in 1965 as an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions (aka characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. The concept of a fuzzy relation was first proposed by Zadeh [30]. Subsequently, Goguen [3] and Sanchez [29] studied fuzzy relations in various contexts. In [7] Nemitz discussed fuzzy equivalence relations, fuzzy functions as fuzzy relations, and fuzzy partitions. Murali [6] developed some properties of fuzzy equivalence relations and certain lattice theoretic properties of fuzzy equivalence relations. Samhan [28] characterized the fuzzy congruences generated by fuzzy relations on a semigroup and studied the lattice of fuzzy congruences on a semigroup. T-conorms are a generalization of the usual two-valued logical conjunction, studied by classical logic, for fuzzy logics. Indeed, the classical Boolean conjunction is both commutative and associative. The monotonicity property ensures that the degree of truth of conjunction does not decrease if the truth values of conjuncts increase. The requirement that 1 be an identity element corresponds to the interpretation of 1 as true (and consequently 0 as false). Continuity, which is often required from fuzzy conjunction as well, expresses the idea that, roughly speaking, very small changes in truth values of conjuncts should not macroscopically affect the truth value of their conjunction. T-conorms are also used to construct the intersection of fuzzy sets or as a basis for aggregation operators (see fuzzy set operations). In probabilistic metric spaces, t-conorms are used to generalize triangle inequality of ordinary metric spaces. Individual *t*-conorms may of course frequently occur in further disciplines of mathematics, since the class contains many familiar functions. The author by using norms, investigated some properties of fuzzy algebraic structures [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. The purpose of this paper is to deal with the anti fuzzy congruence and equivalence relation on rings with respect to *t*-conorms, homomorphisms and isomorphisms of them. Section 2 contains some basic definitions and preliminary results which will be needed in the sequel. In Section 3, by using *t*-norms, we introduce the concept of anti fuzzy equivalence relation, anti fuzzy congruence relation and anti fuzzy ideal on rings and we obtain some results of them. Also we define addition and product of them and introduce new rings. In Section 4, we define the kernel of ring homomorphisms and we prove that it will be a congruence relation on rings. Also we show that anti characteristic function of it is an anti fuzzy congruence relation on rings. Next we obtain ring homomorphism and ring isomorphisms of it.

2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1 (see [5])**.** A ring $\langle R, +, \cdot\rangle$ consists of a nonempty set R and two binary operations + and · that satisfy the axioms:

(1) $\langle R, +, \cdot\rangle$ is an abelian group;

(2) $(ab)c = a(bc)$ (associative multiplication) for all $a, b, c \in R$;

(3) $a(b + c) = ab + ac,\ (b + c)a = ba + ca$ (distributive laws) for all $a, b, c \in R$.

Moreover, the ring R is a commutative ring if $ab = ba$ and ring with identity if R contains an element 1_R such that $1_R a = a1_R = a$ for all $a \in R$.

Throughout this paper, R stands for the ring $(R, +, \cdot)$ with an identity element 1_R and zero element 0_R such that is commutative.

Example 2.2. (1) The ring $\mathbb{Z}$ of integers is a commutative ring with identity. So are $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_n, \mathbb{R}[x]$, etc.

(2) $5\mathbb{Z}$ is a commutative ring with no identity.

(3) The ring $\mathbb{Z}^{3\times3}$ of 3×3 matrices with integer coefficients is a noncommutative ring with identity.

(4) $(5\mathbb{Z})^{3\times3}$ is a noncommutative ring with no identity.

Theorem 2.3 (see [5]). *If* $f : R \to S$ *is a ring homomorphism, then f induces a ring isomorphism* $\frac{R}{\operatorname{Ker} f} \cong \operatorname{Im} f$.

Definition 2.4 (see [2]). A *t-conorm C* is a function $C : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:

(C1) $C(x, 0) = x$,

(C2) $C(x, y) \leq C(x, z)$ if $y \leq z$,

(C3) $C(x, y) = C(y, x)$,

(C4) $C(x, C(y, z)) = C(C(x, y), z)$,

for all $x, y, z \in [0, 1]$.

Corollary 2.5. *Let C be a C-conorm. Then for all* $x \in [0, 1]$

(1) $C(x, 1) = 1$.

(2) $C(0, 0) = 0$.

Example 2.6. (1) Standard union *t*-conorm $C_m(x, y) = \max\{x, y\}$.

(2) Bounded sum *t*-conorm $C_b(x, y) = \min\{1, x + y\}$.

(3) Algebraic sum *t*-conorm $C_p(x, y) = x + y - xy$.

(4) Drastic *T*-conorm

$$C_D(x, y) = \begin{cases} y & \text{if } x = 0, \\ x & \text{if } y = 0, \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic *T*-norm.

(5) Nilpotent maximum T-conorm, dual to the nilpotent minimum T-norm:

$$C_{nM}(x, y) = \begin{cases} \max\{x, y\}, & \text{if } x + y < 1, \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x, y) = \dfrac{x + y}{1 + xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x, y) \leq C(x, y) \leq C_D(x, y)$ for any t-conorm C and all $x, y \in [0, 1]$.

Recall that t-norm T (t-conorm C) is idempotent if for all $x \in [0, 1]$, $T(x, x) = x (C(x, x) = x)$.

Lemma 2.7 (see [1])**.** *Let C be a t-conorm. Then*

$$C(C(x, y), C(w, z)) = C(C(x, w), C(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

3. *t*-conorms over Anti Fuzzy Equivalence Relation on Rings

Definition 3.1. A fuzzy relation $\mu : R \times R \to [0, 1]$ on a ring R is an anti fuzzy equivalence relation on R with respect to t-conorm C if the following conditions are satisfied:

(1) $\mu(x, x) = 0$,

(2) $\mu(x, y) = \mu(y, x)$,

(3) $\mu(x, z) \leq C(\mu(x, y), \mu(y, z))$,

for all $x, y, z \in R$.

Definition 3.2. A fuzzy relation $\mu : R \times R \to [0, 1]$ on a ring R is an anti fuzzy congruence relation on R with respect to t-conorm C if the following conditions are satisfied:

(1) $\mu(x, x) = 0$,

(2) $\mu(x, y) = \mu(y, x)$,

(3) $\mu(x, z) \le C(\mu(x, y), \mu(y, z))$,

(4) $\mu(x + z, y + t) \le C(\mu(x, y), \mu(z, t))$,

(5) $\mu(xz, yt) \le C(\mu(x, y), \mu(z, t))$,

for all $x, y, z, t \in R$.

We denote the set of all anti fuzzy congruence relations on R by $AFC(R)$.

Example 3.3. Let $R = (\mathbb{R}, +, \cdot)$ be a ring of real numbers. Define $\mu : \mathbb{R} \times \mathbb{R} \to [0, 1]$ by

$$\mu(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 0.55 & \text{otherwise.} \end{cases}$$

Let C be an algebraic sum t-conorm as $C_p(x, y) = x + y - xy$ for all $x, y \in [0, 1]$, then $\mu \in AFC(R)$.

Proposition 3.4. *Let C be idempotent and $\mu \in AFC(R)$. Then for all $x, y, z \in R$, the following assertions hold*:

(1) $\mu(x, y) \le C(\mu(xz, yz), \mu(zx, zy))$.

(2) $\mu(x^{-1}, y^{-1}) = \mu(x, y)$.

(3) $\mu(x, y) \le C(\mu(x + z, y + z), C(\mu(xz, yz), \mu(zx, zy)))$.

(4) $\mu(-x, -y) = \mu(x, y)$.

Proof. Let $x, y, z \in R$. Then

(1)

$$\begin{aligned} \mu(x, y) &= \mu(xzz^{-1}, yzz^{-1}) \\ &\le C(\mu(xz, yz), \mu(z^{-1}, z^{-1})) \\ &= C(\mu(xz, yz), 0) = \mu(xz, yz). \qquad \text{(a)} \end{aligned}$$

Also

$$\begin{aligned}\mu(x, y) &= \mu(z^{-1}zx, z^{-1}zy) \\ &\leq C(\mu(z^{-1}, z^{-1}), \mu(zx, zy)) \\ &= C(0, \mu(zx, zy)) \\ &= \mu(zx, zy). \qquad \text{(b)}\end{aligned}$$

Then from (a) and (b) we get

$$\mu(x, y) = C(\mu(x, y), \mu(x, y)) \leq C(\mu(xz, yz), \mu(zx, zy)).$$

(2) From (a) and (b) in (1) we have

$$\begin{aligned}\mu(x^{-1}, y^{-1}) &\leq \mu(xx^{-1}, xy^{-1}) \\ &= \mu(1_R, xy^{-1}) \leq \mu(1_R y, xy^{-1}y) \\ &= \mu(y, x) = \mu(x, y) \leq \mu(x^{-1}x, x^{-1}y) \\ &= \mu(1_R, x^{-1}y) \leq \mu(1_R y^{-1}, x^{-1}yy^{-1}) \\ &= \mu(y^{-1}, x^{-1}1_R) = \mu(y^{-1}, x^{-1}) = \mu(x^{-1}, y^{-1}).\end{aligned}$$

Thus $\mu(x^{-1}, y^{-1}) = \mu(x, y)$.

(3)

$$\begin{aligned}\mu(x, y) &= \mu(x + z - z, y + z - z) \\ &\leq C(\mu(x + z, y + z), \mu(-z, -z)) \\ &= C(\mu(x + z, y + z), 1) = \mu(x + z, y + z). \qquad \text{(c)}\end{aligned}$$

Now by (1) and (c) we obtain that

$$\mu(x, y) = C(\mu(x, y), \mu(x, y)) \leq C(\mu(x + z, y + z), C(\mu(xz, yz), \mu(zx, zy))).$$

(4) By (c) we have

$$\mu(x, y) \leq \mu(x - y, y - y) = \mu(x - y, 0_R) \leq \mu(-x + x - y, 0_R - x)$$

$$= \mu(-y, -x) = \mu(-x, -y) \leq \mu(x - x, x - y) = \mu(0_R, x - y)$$

$$\leq \mu(y + 0_R, y + x - y) = \mu(y, x) = \mu(x, y).$$

Therefore $\mu(-x, -y) = \mu(x, y)$.

Definition 3.5. Let $\mu \in AFC(R)$ and $a \in R$. Define a fuzzy subset μ_a on R as $\mu_a(x) = \mu(a, x)$ for all $x \in R$. We denote the set of fuzzy subset μ_a on R by $\frac{R}{\mu}$.

Proposition 3.6. *Let* $\mu \in AFC(R)$ *and* $a \in R$. *Then* $\mu_a(x) = \mu_{0_R}(x - a)$ *for all* $x \in R$.

Proof.

$$\mu_{0_R}(x - a) = \mu(0_R, x - a) = \mu(a - a, x - a) \leq C(\mu(a, x), \mu(-a, -a))$$

$$= C(\mu(a, x), \mu(a, a)) = C(\mu(a, x), 0) = \mu(a, x) = \mu(a + 0_R, x - a + a)$$

$$\leq C(\mu(0_R, x - a), \mu(a, a)) = C(\mu(0_R, x - a), 0) = \mu(0_R, x - a)$$

$$= \mu_{0_R}(x - a).$$

Thus $\mu_{0_R}(x - a) = \mu(a, x) = \mu_a(x)$.

Definition 3.7. If μ is a fuzzy set of R, then μ is an anti fuzzy ideal of R with respect to t-conorm C if the following conditions are satisfied for all $x, y \in R$:

(1) $\mu(x + y) \leq C(\mu(x), \mu(y))$;

(2) $\mu(xy) \leq C(\mu(x), \mu(y))$;

(3) $\mu(-x) = \mu(x)$;

(4) $\mu(0_R) = 0$;

(5) $\mu(xy) \leq \mu(x)$ and $\mu(xy) \leq \mu(y)$.

We denote the set of all anti fuzzy ideals of R with respect to t-conorm C by $AFIC(R)$.

Example 3.8. Let $R = (\mathbb{R}, +, \cdot)$ be a ring of real numbers. Define $\mu : \mathbb{R} \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0 & \text{if } x = 0_R, \\ 0.65 & \text{otherwise.} \end{cases}$$

If C is bounded sum t-conorm as $C_b(x, y) = \min\{1, x + y\}$ for all $x, y \in [0, 1]$, then $\mu \in AFIC(R)$.

Proposition 3.9. *Let* $\mu \in AFC(R)$ *and C be idempotent. Then* $\mu_{0_R} \in AFIC(R)$.

Proof. Let $\mu \in AFC(R)$ and $x, y \in R$. Then

(1) $\mu_{0_R}(x + y) = \mu(0_R, x + y) = \mu(0_R + 0_R, x + y)$

$$\leq C(\mu(0_R, x), \mu(0_R, y)) = C(\mu_{0_R}(x), \mu_{0_R}(y)).$$

(2) $\mu_{0_R}(xy) = \mu(0_R, xy) = \mu(0_R 0_R, xy) \leq C(\mu(0_R, x), \mu(0_R, y))$

$$= C(\mu_{0_R}(x), \mu_{0_R}(y)).$$

(3) $\mu_{0_R}(-x) = \mu(0_R, -x) = \mu(-0_R, -x) = \mu(0_R, x) = \mu_{0_R}(x)$ (by Proposition 3.4 (part 4)).

(4) $\mu_{0_R}(0_R) = \mu(0_R, 0_R) = 0$.

(5) $\mu_{0_R}(xy) = \mu(0_R, xy) = \mu(0_R y, xy) \leq C(\mu(0_R, x), \mu(y, y))$

$$= C(\mu(0_R, x), 0) = \mu(0_R, x) = \mu_{0_R}(x).$$

Similarly $\mu_{0_R}(xy) \leq \mu_{0_R}(y)$.

Thus $\mu_{0_R} \in AFIC(R)$.

Definition 3.10. Define fuzzy relation $C(\mu)$ on R by $C(\mu)(x, y) = \mu(x - y)$ for all $x, y \in R$.

We call that $C(\mu)$ is the fuzzy relation induced by μ.

Proposition 3.11. *If* $\mu \in AFIC(R)$, *then* $C(\mu) \in AFC(R)$.

Proof. Let $\mu \in AFIC(R)$ and $x, y, z, t \in R$. Then

(1) $C(\mu)(x, x) = \mu(x - x) = \mu(0_R) = 0$.

(2) $C(\mu)(x, y) = \mu(x-y) = \mu(-(x-y)) = \mu(y-x) = C(\mu)(y, x)$.

(3) $C(\mu)(x, y) = \mu(x-y) = \mu(x-z+z-y) \leq C(\mu(x-z), \mu(z-y))$

$$= C(C(\mu)(x, z), C(\mu)(z, y)).$$

(4) $C(\mu)(x+z, y+t) = \mu(x+z-(y+t)) = \mu(x-y+z-t)$

$$\leq C(\mu(x-y), \mu(z-t)) = C(C(\mu)(x, y), C(\mu)(z, t)).$$

(5) $C(\mu)(xz, yt) = \mu(xz-(yt)) = \mu((x-y)z + y(z-t))$

$$\leq C(\mu((x-y)z), \mu(y(z-t))) \leq C(\mu(x-y), \mu(z-t))$$

$$= C(C(\mu)(x, y), C(\mu)(z, t)).$$

Therefore $C(\mu) \in AFC(R)$.

Proposition 3.12. *Let* $\mu \in AFC(R)$ *and* $a, b \in R$. *Then* $\mu_a = \mu_b$ *if and only if* $\mu_{0_R}(a-b) = 0$.

Proof. Suppose that $\mu_a = \mu_b$ and $a, b, x \in R$. Then $\mu_a(x) = \mu_b(x)$ and so $\mu(a, x) = \mu(b, x)$. Now by Proposition 3.9 we get that $\mu_{0_R}(a-b) = \mu(b, a)$ $= \mu(a, a) = 0$. Conversely, let $\mu_{0_R}(a-b) = 0$ and we show that $\mu_a = \mu_b$. Now

$$\begin{aligned}
\mu_a(x) &= \mu(a, x) = \mu(0_R + a, x - a + a) \\
&\leq C(\mu(0_R, x-a), \mu(a, a)) = C(\mu(0_R, x-a), 0) \\
&= \mu(0_R, x-a) \leq C(\mu(0_R, b-a), \mu(b-a, x-a)) \\
&= C(0, \mu(b-a, x-a)) \\
&= \mu(b-a, x-a) = \mu(x-a, b-a) \leq C(\mu(x, b), \mu(-a, -a)) \\
&= C(\mu(x, b), 0) = \mu(x, b) = \mu_b(x)
\end{aligned}$$

and then $\mu_a \subseteq \mu_b$. Also by symmetry, we obtain that $\mu_a \supseteq \mu_b$. Therefore $\mu_a = \mu_b$.

Corollary 3.13. *Let* $\mu \in AFC(R)$ *and* $a, b \in R$. *Then* $\mu_a = \mu_b$ *if and only if* $\mu(a, b) = 0$.

Proof. From Proposition 3.12 we have $\mu_a = \mu_b$ if and only if $\mu_{0_R}(a-b) = 0$ if and only if $\mu_b(a) = 0$ if and only if $\mu(a, b) = 0$.

Definition 3.14. Let $\mu \in AFC(R)$ and $a, b \in R$. Define addition $\mu_a \oplus \mu_b : R \to [0, 1]$ by

$$(\mu_a \oplus \mu_b)(x) = \begin{cases} \inf_{x=y+z} C(\mu_a(y), \mu_b(z)) & \text{if } x = y + z, \\ 0 & \text{otherwise} \end{cases}$$

and product $\mu_a \odot \mu_b : R \to [0, 1]$ by

$$(\mu_a \odot \mu_b)(x) = \begin{cases} \inf_{x=yz} C(\mu_a(y), \mu_b(z)) & \text{if } x = yz, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.15. *If* $\mu \in AFC(R)$, *then* $\mu_a \oplus \mu_b = \mu_{a+b}$.

Proof. First, we prove the binary operation $\oplus$ is well-defined. Let $\mu_a \oplus \mu_b = \mu_c \oplus \mu_d$. Then $\mu_a = \mu_c$ and $\mu_b = \mu_d$. Now by Corollary 3.13 we get that $\mu(a, c) = 0$ and $\mu(b, d) = 0$. Then $\mu(a+b, c+d) \leq C(\mu(a, c), \mu(b, d)) = C(0, 0) = 0$ and so $\mu(a+b, c+d) = 0$. Therefore by Corollary 3.13 we obtain that $\mu_{a+b} = \mu_{c+d}$.

Now we prove $\mu_a \oplus \mu_b = \mu_{a+b}$. Let $x \in R$ such that $x = y + z$. Then

$$\begin{aligned}(\mu_a \oplus \mu_b)(x) &= \inf_{x=y+z} C(\mu_a(y), \mu_b(z)) \\ &\leq \inf_{x=y+z,\, \mu_a(y)\neq\mu_b(z)} C(\mu_a(y), \mu_b(z)) \\ &= \inf_{x=y+z,\, \mu_a(y)\neq\mu_b(z)} C(\mu_{0_R}(y-a), \mu_{0_R}(z-b)) \\ &\leq \mu_{0_R}((y+z)-(a+b)) = \mu_{0_R}((x)-(a+b)) \\ &= \mu(a+b, x) = \mu_{a+b}(x),\end{aligned}$$

then $\mu_a \oplus \mu_b \subseteq \mu_{a+b}$. Conversely, for each $y, z \in R$ satisfies $x = y + z$ we get that

$$\begin{aligned}\mu_{a+b}(x) &= \mu(a+b, x) = \mu_{0_R}(x-(a+b)) = \mu_{0_R}(y+z-(a+b)) \\ &= \mu_{0_R}((y-a)+(z-b)) \leq C(\mu_{0_R}(y-a), \mu_{0_R}(z-b))\end{aligned}$$

$$= C(\mu_a(y), \mu_b(z)).$$

Thus $\mu_{a+b}(x) \leq \inf_{x=y+z} C(\mu_a(y), \mu_b(z))$ and so $\mu_{a+b} \subseteq \mu_a \oplus \mu_b$. Then $\mu_a \oplus \mu_b = \mu_{a+b}$.

Proposition 3.16. *If* $\mu \in AFC(R)$, *then* $\mu_a \odot \mu_b = \mu_{ab}$.

Proof. Let $x \in R$ such that $x = yz$. Then

$$\mu_{ab}(x) = \mu(ab, x) = \mu(ab, yz) \leq C(\mu(a, y), \mu(b, z)) = C(\mu_a(y), \mu_b(z))$$

$$= \inf_{x=yz, \mu_a(y)\neq\mu_b(z)} C(\mu_a(y), \mu_b(z)) = (\mu_a \odot \mu_b)(x)$$

$$= \inf_{x=yz} C(\mu_a(y), \mu_b(z)) \leq \inf_{x=yz, \mu_a(y)\neq\mu_b(z)} C(\mu_a(y), \mu_b(z))$$

$$= \inf_{x=yz, \mu_a(y)\neq\mu_b(z)} C(\mu_{0_R}(y-a), \mu_{0_R}(z-b))$$

$$= \mu_{0_R}(yz - ab) = \mu_{0_R}(x - ab) = \mu(ab, x) = \mu_{ab}(x).$$

Thus $\mu_a \odot \mu_b = \mu_{ab}$.

Proposition 3.17. *If* $\mu \in AFC(R)$, *then* $\left(\frac{R}{\mu}, \oplus, \odot\right)$ *is a ring.*

Proof. Let $a, b, c \in R$. As Propositions 3.15 and 3.16 we get that $\mu_a \oplus \mu_b \in \frac{R}{\mu}$ and $\mu_a \odot \mu_b \in \frac{R}{\mu}$. It is easy to prove that $\left(\frac{R}{\mu}, \oplus\right)$ is an abelian group. Now

(1) $\mu_a \odot (\mu_b \odot \mu_c) = \mu_a \odot \mu_{bc} = \mu_{a(bc)} = \mu_{(ab)c} = (\mu_a \odot \mu_b) \odot \mu_c$ (associative multiplication).

(2) $\mu_a \odot (\mu_b \oplus \mu_c) = \mu_a \odot \mu_{b+c} = \mu_{a(b+c)} = \mu_{ab+ac} = \mu_{ab} \oplus \mu_{ac} = (\mu_a \odot \mu_b) \oplus (\mu_a \odot \mu_c)$ (distributive laws).

(3) $(\mu_b \oplus \mu_c) \odot \mu_a = \mu_{b+c} \odot \mu_a = \mu_{(b+c)a} = \mu_{ba+ca} = \mu_{ba} \oplus \mu_{ca} = (\mu_b \odot \mu_a) \oplus (\mu_c \odot \mu_a)$ (distributive laws). Then $\left(\frac{R}{\mu}, \oplus, \odot\right)$ will be a ring.

If R be a commutative ring, then $\mu_a \odot \mu_b = \mu_{ab} = \mu_{ba} = \mu_b \odot \mu_a$ and so $\left(\frac{R}{\mu}, \oplus, \odot\right)$ will be a commutative ring.

If R be a ring with identity 1_R, then $\mu_a \odot \mu_{1_R} = \mu_{a1_R} = \mu_a$ and thus $\left(\frac{R}{\mu}, \oplus, \odot\right)$ will be a ring with identity μ_{1_R}.

Proposition 3.18. *Let* $\mu \in AFC(R)$ *and define* $\mu^{-1}(0) = \{(x, y) | \mu(x, y) = 0\}$ *for all* $x, y \in R$. *Then*

(1) $\mu^{-1}(0)$ *will be an anti fuzzy equivalence relation on R under t-conorm C*:

(2) $\mu^{-1}(0)$ *is a congruence relation on R.*

Proof. Let $\mu \in AFC(R)$ and $x, y, z, t \in R$. Then

(1) Since $\mu(x, x) = 0$ so $(x, x) \in \mu^{-1}(0)$ and then

$$\mu^{-1}(0)(x, x) = \mu(x, x) = 0.$$

If $(x, y), (y, x) \in \mu^{-1}(0)$, then $\mu^{-1}(0)(x, y) = \mu(x, y) = 0$ and $\mu^{-1}(0)(y, x) = \mu(y, x) = 0$ so by $\mu(x, y) = \mu(y, x)$ we have

$$\mu^{-1}(0)(x, y) = \mu^{-1}(0)(y, x).$$

Also if $(x, y), (x, z), (z, y) \in \mu^{-1}(0)$, then

$$\mu^{-1}(0)(x, y) = \mu(x, y) = 0 \leq 0 = C(0, 0)$$

$$= C(\mu(x, z), \mu(z, y)) = C(\mu^{-1}(0)(x, y), \mu^{-1}(0)(z, y)).$$

Thus $\mu^{-1}(0)$ will be an anti fuzzy equivalence relation on R under t-conorm C:

(2) (a) Since $\mu(x, x) = 0$ so $(x, x) \in \mu^{-1}(0)$.

(b) Let $(x, y) = \mu^{-1}(0)$, then $\mu(y, x) = \mu(x, y) = 0$ and so $(y, x) \in \mu^{-1}(0)$.

(c) Let $(x, y), (y, z) \in \mu^{-1}(0)$. Then $\mu(x, y) = 0 = \mu(y, z)$ and from

$$\mu(x, z) \leq C(\mu(x, y), \mu(y, z)) = C(0, 0) = 0$$

we get that $\mu(x, z) = 0$ which implies $(x, z) \in \mu^{-1}(0)$.

(d) Let $(x, y), (z, t) \in \mu^{-1}(0)$ so $\mu(x, y) = 0 = \mu(z, t)$. Now

$$\mu(x + z, y + t) \leq C(\mu(x, y), \mu(z, t)) = C(0, 0) = 0$$

and

$$\mu(xz, yt) \leq C(\mu(x, y), \mu(z, t)) = C(0, 0) = 0$$

which gives us $(x + z, y + t), (xz, yt) \in \mu^{-1}(0)$.

Therefore from (a)-(d) we obtain that $\mu^{-1}(0)$ is a congruence relation on R.

4. Ring Homomorphisms over AFC(R)

Definition 4.1. Let R and S be two rings and $f : R \to S$ be a ring homomorphism. Define $\text{Ker}(f) = \{(x, y) | f(x) = f(y)\}$ for all $x, y \in R$.

Lemma 4.2. *Let $f : R \to S$ be a ring homomorphism. Then $\text{Ker}(f)$ will be a congruence relation on R.*

Proof. Let $x, y, z, t \in R$. Then

(a) As $f(x) = f(x)$ so $(x, x) \in \text{Ker}(f)$.

(b) Let $(x, y) \in \text{Ker}(f)$, then $f(x) = f(y)$ and then $f(y) = f(x)$ and so $(y, x) \in \text{Ker}(f)$.

(c) If $(x, y), (y, z) \in \text{Ker}(f)$, then $f(x) = f(y) = f(z)$ which implies $f(x) = f(z)$ and then $(x, z) \in \text{Ker}(f)$.

(d) Let $(x, y), (z, t) \in \text{Ker}(f)$, then $f(x) = f(y)$ and $f(z) = f(t)$. Now $f(x + z) = f(x) + f(z) = f(y) + f(t) = f(y + t)$ and $f(xz) = f(x)f(z) = f(y)f(t)$ $= f(yt)$ and then $(x + z, y + t), (xz, yt) \in \text{Ker}(f)$.

Thus (a)-(d) give us that $\text{Ker}(f)$ is a congruence relation on R.

Proposition 4.3. *Let* $\chi_{\text{Ker}(f)} : R \times R \to \{0, 1\}$ *be an anti characteristic function of* $\text{Ker}(f)$ *such that defined by*

$$\chi_{\text{Ker}(f)}(x, y) = \begin{cases} 0 & if\ (x, y) \in \text{Ker}(f) \\ 1 & otherwise \end{cases} = \begin{cases} 0 & if\ f(x) = f(y) \\ 1 & otherwise. \end{cases}$$

Then $\chi_{\text{Ker}(f)} \in AFC(R)$.

Proof. Let $x, y, z, t \in R$. Then by using Lemma 4.2 we have:

(a) $(x, x) \in \text{Ker}(f)$ so $\chi_{\text{Ker}(f)}(x, x) = 0$.

(b) If $(x, y) \in \text{Ker}(f)$, then $(y, x) \in \text{Ker}(f)$ and so $\chi_{\text{Ker}(f)}(x, y) = 0$ $= \chi_{\text{Ker}(f)}(y, x)$.

(c) If $(x, y), (y, z) \in \text{Ker}(f)$, then $(x, z) \in \text{Ker}(f)$. Now

$$\chi_{\text{Ker}(f)}(x, z) = 0 \le 0 = C(0, 0) = C(\chi_{\text{Ker}(f)}(x, y), \chi_{\text{Ker}(f)}(y, z)).$$

If $(x, y), (y, z) \notin \text{Ker}(f)$, then $(x, z) \notin \text{Ker}(f)$. Then

$$\chi_{\text{Ker}(f)}(x, z) = 1 \le 1 = C(1, 1) = C(\chi_{\text{Ker}(f)}(x, y), \chi_{\text{Ker}(f)}(y, z)).$$

If $(x, y) \in \text{Ker}(f)$ and $(y, z) \notin \text{Ker}(f)$, then $(x, z) \notin \text{Ker}(f)$. So

$$\chi_{\text{Ker}(f)}(x, z) = 1 \le 1 = C(0, 1) = C(\chi_{\text{Ker}(f)}(x, y), \chi_{\text{Ker}(f)}(y, z)).$$

Thus

$$\chi_{\text{Ker}(f)}(x, z) \le C(\chi_{\text{Ker}(f)}(x, y), \chi_{\text{Ker}(f)}(y, z)).$$

(d) Let $(x, y), (z, t) \in \text{Ker}(f)$, then $(x + z, y + t), (xz, yt) \in \text{Ker}(f)$. So

$$\chi_{\text{Ker}(f)}(x + z, y + t) = 0 \le 0 = C(0, 0) = C(\chi_{\text{Ker}(f)}(x, y), \chi_{\text{Ker}(f)}(z, t))$$

and

$$\chi_{\text{Ker}(f)}(xz, yt) = 0 \le 0 = C(0, 0) = C(\chi_{\text{Ker}(f)}(x, y), \chi_{\text{Ker}(f)}(z, t)).$$

Then from (a)-(d) we obtain that $\chi_{\text{Ker}(f)} \in AFC(R)$.

Lemma 4.4. *Let* $\mu \in AFC(R)$ *and* $\left(\frac{R}{\mu}, \oplus, \odot\right)$ *be a ring. Define* $\mu^* : R \to \frac{R}{\mu}$ *as* $\mu^*(a) = \mu_a$ *for all* $a \in R$. *Then* μ^* *is a ring homomorphism.*

Proof. Let $a, b \in R$. Then $\mu^*(a+b) = \mu_{a+b} = \mu_a \oplus \mu_b$ and $\mu^*(ab) = \mu_{ab}$ $= \mu_a \odot \mu_b$. Thus μ^* is a ring homomorphism.

Proposition 4.5. *Let R and S be two rings and* $f : R \to S$ *be a ring homomorphism. Then there is a homomorphism* $g : \frac{R}{\chi_{\mathrm{Ker}(f)}} \to S$ *such that* $f = go(\chi_{\mathrm{Ker}(f)})^*$.

Proof. Define $g : \frac{R}{\chi_{\mathrm{Ker}(f)}} \to S$ by $g((\chi_{\mathrm{Ker}(f)})_a) = f(a)$ for all $a \in R$. At first we show that g is well-defined. Let $a, b \in R$ and $(\chi_{\mathrm{Ker}(f)})_a = (\chi_{\mathrm{Ker}(f)})_b$. Then $\chi_{\mathrm{Ker}(f)}(a, b) = 0$ and so $(a, b) \in \mathrm{Ker}(f)$ and $f(a) = f(b)$ as wanted. Now

$$g((\chi_{\mathrm{Ker}(f)})_a) \oplus g((\chi_{\mathrm{Ker}(f)})_b) = g((\chi_{\mathrm{Ker}(f)})_{a+b})$$

$$= f(a+b) = f(a) + f(b) = g((\chi_{\mathrm{Ker}(f)})_a) + g((\chi_{\mathrm{Ker}(f)})_b).$$

Also

$$g((\chi_{\mathrm{Ker}(f)})_a) \odot g((\chi_{\mathrm{Ker}(f)})_b) = g((\chi_{\mathrm{Ker}(f)})_{ab})$$

$$= f(ab) = f(a)f(b) = g((\chi_{\mathrm{Ker}(f)})_a) \odot g((\chi_{\mathrm{Ker}(f)})_b).$$

Thus g will be a homomorphism.

Since $g((\chi_{\mathrm{Ker}(f)})^*)(a) = g((\chi_{\mathrm{Ker}(f)})_a) = f(a)$ so $f = go(\chi_{\mathrm{Ker}(f)})^*$.

Proposition 4.6. *Let* $\mu, \nu \in AFC(R)$ *such that* $\mu \supseteq \nu$. *Then there is an unique homomorphism* $g : \frac{R}{\mu} \to \frac{R}{\nu}$ *such that* $go\mu^* = \nu^*$ *and* $\frac{\mu}{\chi_{\mathrm{Ker}(g)}}$ *is isomorphic to* $\frac{R}{\nu}$.

Proof. Let $a, b \in R$. Define $g : \frac{R}{\mu} \to \frac{R}{\nu}$ by $g(\mu_a) = \nu_a$. If $\mu_a = \mu_b$, then $\mu(a, b) = 0$ and since $\nu(a, b) \le \mu(a, b) = 0$ so $\nu(a, b) = 0$ and so $\nu_a = \nu_b$ and this

means that g is well-defined. Also we get that $go\mu^*(a) = g(\mu_\alpha) = \nu_a = \nu^*(a)$ and so $go\mu^* = \nu^*$. Now we have $\text{Ker}(g) = \{(\mu_a, \mu_b) | g(\mu_a) = g(\mu_b)\} = \{(\mu_a, \mu_b) | \nu_a = \nu_b\}$.
Define $\chi_{\text{Ker}(g)} : \frac{R}{\mu} \times \frac{R}{\mu} \to \{0, 1\}$ by setting

$$\chi_{\text{Ker}(g)}(\mu_a, \mu_b) = \begin{cases} 0 & \text{if } (\mu_a, \mu_b) \in \text{Ker}(g) \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0 & \text{if } g(\mu_a) = g(\mu_b) \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0 & \text{if } \nu_a = \nu_b \\ 1 & \text{otherwise.} \end{cases}$$

Then we have that $\text{Ker}(g) = \chi_{\text{Ker}(g)}$ and by Theorem 2.3 we obtain that

$$\frac{\frac{R}{\mu}}{\chi_{\text{Ker}(g)}} \cong \frac{R}{\nu}.$$

Acknowledgment

We would like to thank the reviewers for carefully reading the manuscript and making several helpful comments to increase the quality of the paper.

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