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Notes on Binomial Transform of the Generalized Narayana Sequence

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Abstract

In this paper, we define the binomial transform of the generalized Narayana sequence and as special cases, the binomial transform of the Narayana, Narayana-Lucas, Narayana-Perrin sequences will be introduced. We investigate their properties in details.

1 Introduction and Preliminaries

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1.$$

and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \; L_1 = 1.$$

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. Horadam [10] defined

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a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \quad W_0 = a, \ W_1 = b, \quad (n \geq 2)$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [9,11,12].

In this paper, we introduce the binomial transform of the generalized Narayana sequence and we investigate, in detail, three special cases which we call them the binomial transform of the Narayana, Narayana-Lucas, Narayana-Perrin sequences. We investigate their properties in the next sections. In this section, we present some properties of the generalized Tribonacci sequence which is a generalization of Fibonacci numbers. The generalized Tribonacci sequence

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \qquad (1.1)$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers. This sequence has been studied by many authors, see for example [2,3,4,5,6,16,17,19,21,23,29,31].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \qquad (1.2)$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r, s, t) = \frac{r}{3} + A + M, \\ \beta &= \beta(r, s, t) = \frac{r}{3} + \omega A + \omega^2 M, \\ \gamma &= \gamma(r, s, t) = \frac{r}{3} + \omega^2 A + \omega M \end{aligned}$$

where

$$
\begin{aligned}
A &= \left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}+\sqrt{\Delta}\right)^{1/3}, \ M=\left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}-\sqrt{\Delta}\right)^{1/3}, \\
\Delta &= \Delta(r,s,t)=\frac{r^3t}{27}-\frac{r^2s^2}{108}+\frac{rst}{6}-\frac{s^3}{27}+\frac{t^2}{4}, \quad \omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).
\end{aligned}
$$

Note that we have the following identities

$$
\begin{aligned}
\alpha+\beta+\gamma &= r, \\
\alpha\beta+\alpha\gamma+\beta\gamma &= -s, \\
\alpha\beta\gamma &= t.
\end{aligned}
$$

If $\Delta(r,s,t)>0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that the generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$
W_n=\frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.3}
$$

where

$$
p_1=W_2-(\beta+\gamma)W_1+\beta\gamma W_0, \quad p_2=W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0,
$$

$$
p_3=W_2-(\alpha+\beta)W_1+\alpha\beta W_0.
$$

(1.3) can be written in the following form:

$$
W_n=M_1\alpha^n+M_2\beta^n+M_3\gamma^n
$$

where

$$
M_1=\frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)}, \ M_2=\frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)},
$$

$$
M_3=\frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}.
$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [13]. This result of Howard and Saidak [13] is even true in the case of higher-order recurrence relations.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.1. *Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized Tribonacci sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{1.4}$$

We next find Binet's formula of the generalized Tribonacci sequence $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.2. *(Binet's formula of the generalized Tribonacci numbers) For all integers* n, *we have*

$$W_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.5}$$

where

$$\begin{aligned} q_1 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ q_2 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ q_3 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

Note that from (1.3) and (1.5) we have

$$\begin{aligned} W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

In this paper, we consider the case $r = 1$, $s = 0, t = 1$ and in this case we write $V_n = W_n$. So, the generalized Narayana sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-3} \tag{1.6}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.6) holds for all integer n.

(1.3) can be used to obtain Binet's formula of generalized Narayana numbers. Binet's formula of generalized Narayana numbers can be given as

$$V_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$\begin{aligned} p_1 &= V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0 = V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1) = q_1, \\ p_2 &= V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0 = V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1) = q_2, \\ p_3 &= V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0 = V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1) = q_3. \end{aligned}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - x^2 - 1 = 0$.

Moreover

$$\begin{aligned} \alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3} \\ \beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3} \\ \gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3} \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 1, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 0, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

Now, we present three special cases of the generalized Narayana sequence $\{V_n\}$. Narayana sequence $\{N_n\}_{n\geq 0}$, Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$, Narayana-Perrin sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} N_{n+3} &= N_{n+2}+N_n, \quad N_0=0, N_1=1, N_2=1, & (1.7)\\ U_{n+3} &= U_{n+2}+U_n, \quad U_0=3, U_1=1, U_2=1, & (1.8)\\ H_{n+3} &= H_{n+2}+H_n, \quad H_0=3, H_1=0, H_2=2. & (1.9)\end{aligned}$$

The sequences $\{N_n\}_{n\geq 0}, \{U_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} N_{-n} &= -N_{-(n-2)}+N_{-(n-3)}\\ U_{-n} &= -U_{-(n-2)}+U_{-(n-3)}\\ H_{-n} &= -H_{-(n-2)}+H_{-(n-3)}\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.7)- (1.9) hold for all integer n.

For more details on the generalized Narayana numbers, see Soykan [28]. Note that N_n is the sequence A000930 in [20] associated with the Narayana's cows sequence and the sequence A078012 in [20] associated with the expansion of $(1-x)/(1-x-x^3)$ and U_n is the sequence A001609 in [20].

For all integers n, Narayana, Narayana-Lucas, Narayana-Perrin numbers (using initial conditions in (1.7)-(1.9)) can be expressed using Binet's formulas as

$$\begin{aligned} N_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},\\ U_n &= \alpha^n+\beta^n+\gamma^n,\\ H_n &= \frac{(3+2\alpha)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(3+2\beta)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{(3+2\gamma)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)},\end{aligned}$$

respectively, see, Soykan [28] for more details.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the generalized Narayana sequence V_n (see, Soykan [28] for more details.).

Lemma 1.3. *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized Narayana sequence* $\{V_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} V_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2}{1 - x - x^3}. \quad (1.10)$$

Proof. Take $r = 1,\ s = 0, t = 1$ in Lemma 1.1.

The previous lemma gives the following results as particular examples.

Corollary 1.4. *Generating functions of Narayana, Narayana-Lucas, Narayana-Perrin numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} N_n x^n &= \frac{x}{1 - x - x^3}, \\ \sum_{n=0}^{\infty} U_n x^n &= \frac{3 - 2x}{1 - x - x^3}, \\ \sum_{n=0}^{\infty} H_n x^n &= \frac{3 - 3x + 2x^2}{1 - x - x^3}, \end{aligned}$$

respectively.

2 Binomial Transform of the Generalized Narayana Sequence V_n

In [15, p. 137], Knuth introduced the idea of the binomial transform. Given a sequence of numbers (a_n), its binomial transform $(\hat{a}_n)$ may be defined by the rule

$$\hat{a}_n = \sum_{i=0}^{n} \binom{n}{i} a_i, \quad \text{with inversion } a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \hat{a}_i,$$

or, in the symmetric version

$$\hat{a}_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} a_i, \quad \text{with inversion } a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} \hat{a}_i.$$

For more information on binomial transform, see, for example, [7,8,18,30] and references therein.

In this section, we define the binomial transform of the generalized Narayana sequence V_n and as special cases the binomial transform of the Narayana, Narayana-Lucas, Narayana-Perrin sequences will be introduced.

Definition 2.1. *The binomial transform of the generalized Narayana sequence V_n is defined by*

$$b_n = \widehat{V}_n = \sum_{i=0}^{n} \binom{n}{i} V_i.$$

The few terms of b_n are

$$\begin{aligned} b_0 &= \sum_{i=0}^{0} \binom{0}{i} V_i = V_0, \\ b_1 &= \sum_{i=0}^{1} \binom{1}{i} V_i = V_0 + V_1, \\ b_2 &= \sum_{i=0}^{2} \binom{2}{i} V_i = V_0 + 2V_1 + V_2. \end{aligned}$$

Translated to matrix language, b_n has the nice (lower-triangular matrix) form

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \end{pmatrix}.$$

As special cases of $b_n = \widehat{V}_n$, the binomial transforms of the Narayana, Narayana-Lucas, Narayana-Perrin sequences are defined as follows: The binomial transform of the Narayana sequence N_n is

$$\widehat{N}_n = \sum_{i=0}^{n} \binom{n}{i} N_i,$$

the binomial transform of the Narayana-Lucas sequence U_n is

$$\widehat{U}_n = \sum_{i=0}^{n} \binom{n}{i} U_i,$$

the binomial transform of the Narayana-Perrin sequence H_n is

$$\widehat{H}_n = \sum_{i=0}^{n} \binom{n}{i} H_i.$$

Lemma 2.2. *For $n \geq 0$, the binomial transform of the generalized Narayana sequence V_n satisfies the following relation:*

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}).$$

Proof. We use the following well-known identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}.$$

Note also that

$$\binom{n+1}{0} = \binom{n}{0} = 1 \text{ and } \binom{n}{n+1} = 0.$$

Then

$$\begin{aligned} b_{n+1} &= V_0 + \sum_{i=1}^{n+1} \binom{n+1}{i} V_i \\ &= V_0 + \sum_{i=1}^{n+1} \binom{n}{i} V_i + \sum_{i=1}^{n+1} \binom{n}{i-1} V_i \\ &= V_0 + \sum_{i=1}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}). \end{aligned}$$

This completes the proof. □

Remark 2.3. *From the last Lemma, we see that*

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} V_{i+1}.$$

The following theorem gives recurrent relations of the binomial transform of the generalized Narayana sequence.

Theorem 2.4. *For $n \geq 0$, the binomial transform of the generalized Narayana sequence V_n satisfies the following recurrence relation:*

$$b_{n+3} = 4b_{n+2} - 5b_{n+1} + 3b_n. \tag{2.1}$$

Proof. To show (2.1), by writing

$$b_{n+3} = r_1 \times b_{n+2} + s_1 \times b_{n+1} + t_1 \times b_n$$

and taking the values $n = 0, 1, 2$ and then solving the system of equations

$$\begin{aligned} b_3 &= r_1 \times b_2 + s_1 \times b_1 + t_1 \times b_0 \\ b_4 &= r_1 \times b_3 + s_1 \times b_2 + t_1 \times b_1 \\ b_5 &= r_1 \times b_4 + s_1 \times b_3 + t_1 \times b_2 \end{aligned}$$

we find that $r_1 = 4, s_1 = -5, t_1 = 3$. □

The sequence $\{b_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$b_{-n} = \frac{5}{3} b_{-n+1} - \frac{4}{3} b_{-n+2} + \frac{1}{3} b_{-n+3}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n.

Note that the recurence relation (2.1) is independent from initial values. So,

$$\begin{aligned} \widehat{N}_{n+3} &= 4\widehat{N}_{n+2} - 5\widehat{N}_{n+1} + 3\widehat{N}_n, \\ \widehat{U}_{n+3} &= 4\widehat{U}_{n+2} - 5\widehat{U}_{n+1} + 3\widehat{U}_n, \\ \widehat{H}_{n+3} &= 4\widehat{H}_{n+2} - 5\widehat{H}_{n+1} + 3\widehat{H}_n. \end{aligned}$$

and

$$\begin{aligned}\widehat{N}_{-n} &= \frac{5}{3}\widehat{N}_{-n+1} - \frac{4}{3}\widehat{N}_{-n+2} + \frac{1}{3}\widehat{N}_{-n+3},\\ \widehat{U}_{-n} &= \frac{5}{3}\widehat{U}_{-n+1} - \frac{4}{3}\widehat{U}_{-n+2} + \frac{1}{3}\widehat{U}_{-n+3},\\ \widehat{H}_{-n} &= \frac{5}{3}\widehat{H}_{-n+1} - \frac{4}{3}\widehat{H}_{-n+2} + \frac{1}{3}\widehat{H}_{-n+3}.\end{aligned}$$

The first few terms of the binomial transform of the generalized Narayana sequence with positive subscript and negative subscript are given in the following Table 1.

Table 1: A few binomial transform (terms) of the generalized Narayana sequence.

n	b_n	b_{-n}
0	V_0	...
1	V_0+V_1	$\frac{1}{3}(2V_0-2V_1+V_2)$
2	$V_0+2V_1+V_2$	$\frac{1}{9}(V_0-7V_1+5V_2)$
3	$2V_0+3V_1+4V_2$	$-\frac{1}{27}(10V_0+11V_1-13V_2)$
4	$6V_0+5V_1+11V_2$	$-\frac{1}{81}(44V_0-11V_1-14V_2)$
5	$17V_0+11V_1+27V_2$	$-\frac{1}{243}(91V_0-124V_1+41V_2)$
6	$44V_0+28V_1+65V_2$	$-\frac{1}{729}(17V_0-389V_1+256V_2)$
7	$109V_0+72V_1+158V_2$	$\frac{1}{2187}(611V_0+556V_1-662V_2)$
8	$267V_0+181V_1+388V_2$	$\frac{1}{6561}(2440V_0-772V_1-607V_2)$
9	$655V_0+448V_1+957V_2$	$\frac{1}{19\,683}(4715V_0-7031V_1+2605V_2)$
10	$1612V_0+1103V_1+2362V_2$	$-\frac{1}{59\,049}(206V_0+20\,887V_1-14\,351V_2)$
11	$3974V_0+2715V_1+5827V_2$	$-\frac{1}{177\,147}(35\,650V_0+27\,011V_1-35\,032V_2)$
12	$9801V_0+6689V_1+14369V_2$	$-\frac{1}{531\,441}(133\,343V_0-52\,310V_1-26\,393V_2)$
13	$24170V_0+16490V_1+35427V_2$	$-\frac{1}{1594\,323}(240\,769V_0-397\,699V_1+159\,260V_2)$

The first few terms of the binomial transform numbers of the Narayana , Narayana-Lucas, Narayana-Perrin sequences with positive subscript and negative subscript are given in the following Table 2.

Table 2: A few binomial transform (terms).

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\widehat{N}_n$	0	1	3	7	16	38	93	230	569	1405	3465	8542	21058
$\widehat{N}_{-n}$		$-\frac{1}{3}$	$-\frac{2}{9}$	$\frac{2}{27}$	$\frac{25}{81}$	$\frac{83}{243}$	$\frac{133}{729}$	$-\frac{106}{2187}$	$-\frac{1379}{6561}$	$-\frac{4426}{19683}$	$-\frac{6536}{59049}$	$\frac{8021}{177147}$	$\frac{78703}{531441}$
$\widehat{U}_n$	3	4	6	13	34	89	225	557	1370	3370	8301	20464	50461
$\widehat{U}_{-n}$		$\frac{5}{3}$	$\frac{1}{9}$	$-\frac{28}{27}$	$-\frac{107}{81}$	$-\frac{190}{243}$	$\frac{82}{729}$	$\frac{1727}{2187}$	$\frac{5941}{6561}$	$\frac{9719}{19683}$	$-\frac{7154}{59049}$	$-\frac{98929}{177147}$	$-\frac{321326}{531441}$
$\widehat{H}_n$	3	3	5	14	40	105	262	643	1577	3879	9560	23576	58141
$\widehat{H}_{-n}$		$\frac{8}{3}$	$\frac{13}{9}$	$-\frac{4}{27}$	$-\frac{104}{81}$	$-\frac{355}{243}$	$-\frac{563}{729}$	$\frac{509}{2187}$	$\frac{6106}{6561}$	$\frac{19355}{19683}$	$\frac{28084}{59049}$	$-\frac{36886}{177147}$	$-\frac{347243}{531441}$

(1.3) can be used to obtain Binet's formula of the binomial transform of generalized Narayana numbers. Binet's formula of the binomial transform of generalized Narayana numbers can be given as

$$b_n = \frac{c_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{c_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{c_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \tag{2.2}$$

where

$$\begin{aligned}
c_1 &= b_2-(\theta_2+\theta_3)b_1+\theta_2\theta_3 b_0 = (V_0+2V_1+V_2)-(\theta_2+\theta_3)(V_0+V_1)+\theta_2\theta_3V_0,\\
c_2 &= b_2-(\theta_1+\theta_3)b_1+\theta_1\theta_3 b_0 = (V_0+2V_1+V_2)-(\theta_1+\theta_3)(V_0+V_1)+\theta_1\theta_3V_0,\\
c_3 &= b_2-(\theta_1+\theta_2)b_1+\theta_1\theta_2 b_0 = (V_0+2V_1+V_2)-(\theta_1+\theta_2)(V_0+V_1)+\theta_1\theta_2V_0.
\end{aligned}$$

Here, θ_1, θ_2 and θ_3 are the roots of the cubic equation $x^3-4x^2+5x-3=0$. Moreover,

$$\begin{aligned}
\theta_1 &= \frac{4}{3}+\frac{1}{6}\sqrt[3]{4(29+3\sqrt{93})}+\frac{1}{6}\sqrt[3]{4(29-3\sqrt{93})}\\
\theta_2 &= \frac{4}{3}+\frac{\omega}{6}\sqrt[3]{4(29+3\sqrt{93})}+\frac{\omega^2}{6}\sqrt[3]{4(29-3\sqrt{93})}\\
\theta_3 &= \frac{4}{3}+\frac{\omega^2}{6}\sqrt[3]{4(29+3\sqrt{93})}+\frac{\omega}{6}\sqrt[3]{4(29-3\sqrt{93})}
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned} \theta_1+\theta_2+\theta_3 &= 4, \\ \theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3 &= 5, \\ \theta_1\theta_2\theta_3 &= 3. \end{aligned}$$

For all integers n, (Binet's formulas of) binomial transforms of Narayana , Narayana-Lucas, Narayana-Perrin numbers (using initial conditions in (2.2)) can be expressed using Binet's formulas as

$$\begin{aligned} \widehat{N}_n &= \frac{(-1+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{(-1+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{(-1+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)}, \\ \widehat{U}_n &= \theta_1^n+\theta_2^n+\theta_3^n, \\ \widehat{H}_n &= \frac{(3\theta_1^2-7\theta_1+9)\theta_1^{n-1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{(3\theta_2^2-7\theta_2+9)\theta_2^{n-1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{(3\theta_3^2-7\theta_3+9)\theta_3^{n-1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)}, \end{aligned}$$

respectively.

3 Generating Functions and Obtaining Binet Formula of Binomial Transform From Generating Function

The generating function of the binomial transform of the generalized Narayana sequence V_n is a power series centered at the origin whose coefficients are the binomial transform of the generalized Narayana sequence.

Next, we give the ordinary generating function $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ of the sequence b_n.

Lemma 3.1. *Suppose that $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ is the ordinary generating function of the binomial transform of the generalized Narayana sequence $\{V_n\}_{n\geq 0}$. Then, $f_{b_n}(x)$ is given by*

$$f_{b_n}(x) = \frac{V_0+(V_1-3V_0)x+(2V_0-2V_1+V_2)x^2}{1-4x+5x^2-3x^3}. \tag{3.1}$$

Proof. Using Lemma 1.1, we obtain

$$\begin{aligned} f_{b_n}(x) &= \frac{b_0 + (b_1 - r_1 b_0)x + (b_2 - r_1 b_1 - s_1 b_0)x^2}{1 - r_1 x - s_1 x^2 - t_1 x^3} \\ &= \frac{V_0 + ((V_0 + V_1) - 4V_0)x + ((V_0 + 2V_1 + V_2) - 4(V_0 + V_1) - (-5)V_0)x^2}{1 - 4x - (-5)x^2 - 3x^3} \\ &= \frac{V_0 + (V_1 - 3V_0)x + (2V_0 - 2V_1 + V_2)x^2}{1 - 4x + 5x^2 - 3x^3} \end{aligned}$$

where

$$\begin{aligned} b_0 &= V_0, \\ b_1 &= V_0 + V_1, \\ b_2 &= V_0 + 2V_1 + V_2. \end{aligned}$$

□

Note that P. Barry shows in [1] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then

$$S(x) = \frac{1}{1-x} A(\frac{x}{1-x})$$

is the generating function of the sequence $\{b_n\}$ with $b_n = \sum_{i=0}^{n} \binom{n}{i} a_i$. In our case, since

$$A(x) = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2}{1 - x - x^3}, \quad \text{see (1.10)},$$

we obtain

$$\begin{aligned} S(x) &= \frac{1}{1-x} \frac{V_0 + (V_1 - V_0)\left(\frac{x}{1-x}\right) + (V_2 - V_1)\left(\frac{x}{1-x}\right)^2}{1 - \left(\frac{x}{1-x}\right) - \left(\frac{x}{1-x}\right)^3} \\ &= \frac{V_0 + (V_1 - 3V_0)x + (2V_0 - 2V_1 + V_2)x^2}{1 - 4x + 5x^2 - 3x^3}. \end{aligned}$$

The previous lemma gives the following results as particular examples.

Corollary 3.2. *Generating functions of the binomial transform of the Narayana, Narayana-Lucas, Narayana-Perrin numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{N}_n x^n &= \frac{x-x^2}{1-4x+5x^2-3x^3},\\ \sum_{n=0}^{\infty}\widehat{U}_n x^n &= \frac{3-8x+5x^2}{1-4x+5x^2-3x^3},\\ \sum_{n=0}^{\infty}\widehat{H}_n x^n &= \frac{3-9x+8x^2}{1-4x+5x^2-3x^3},\end{aligned}$$

respectively.

We next find Binet's formula of the Binomial transform of the generalized Narayana numbers $\{V_n\}$ by the use of generating function for b_n.

Theorem 3.3. *(Binet's formula of the Binomial transform of the generalized Narayana numbers)*

$$b_n = \frac{d_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{d_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{d_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \tag{3.2}$$

where

$$\begin{aligned}d_1 &= V_0\theta_1^2+(V_1-3V_0)\theta_1+(2V_0-2V_1+V_2),\\ d_2 &= V_0\theta_2^2+(V_1-3V_0)\theta_2+(2V_0-2V_1+V_2),\\ d_3 &= V_0\theta_3^2+(V_1-3V_0)\theta_3+(2V_0-2V_1+V_2).\end{aligned}$$

Proof. By using Lemma 3.1, the proof follows from Theorem 1.2. □

Note that from (2.2) and (3.2), we have

$$\begin{aligned}b_2-(\theta_2+\theta_3)b_1+\theta_2\theta_3 b_0 &= V_0\theta_1^2+(V_1-3V_0)\theta_1+(2V_0-2V_1+V_2),\\ b_2-(\theta_1+\theta_3)b_1+\theta_1\theta_3 b_0 &= V_0\theta_2^2+(V_1-3V_0)\theta_2+(2V_0-2V_1+V_2),\\ b_2-(\theta_1+\theta_2)b_1+\theta_1\theta_2 b_0 &= V_0\theta_3^2+(V_1-3V_0)\theta_3+(2V_0-2V_1+V_2),\end{aligned}$$

or

$$\begin{aligned}(V_0+2V_1+V_2)-(\theta_2+\theta_3)(V_0+V_1)+\theta_2\theta_3V_0 &= V_0\theta_1^2+(V_1-3V_0)\theta_1\\ &\quad +(2V_0-2V_1+V_2),\\ (V_0+2V_1+V_2)-(\theta_1+\theta_3)(V_0+V_1)+\theta_1\theta_3V_0 &= V_0\theta_2^2+(V_1-3V_0)\theta_2\\ &\quad +(2V_0-2V_1+V_2),\\ (V_0+2V_1+V_2)-(\theta_1+\theta_2)(V_0+V_1)+\theta_1\theta_2V_0 &= V_0\theta_3^2+(V_1-3V_0)\theta_3\\ &\quad +(2V_0-2V_1+V_2).\end{aligned}$$

Note that we can also write

$$\begin{aligned}(b_0+2b_1+b_2)-(\theta_2+\theta_3)(b_0+b_1)+\theta_2\theta_3b_0 &= b_0\theta_1^2+(b_1-3b_0)\theta_1\\ &\quad +(2b_0-2b_1+b_2),\\ (b_0+2b_1+b_2)-(\theta_1+\theta_3)(b_0+b_1)+\theta_1\theta_3b_0 &= b_0\theta_2^2+(b_1-3b_0)\theta_2\\ &\quad +(2b_0-2b_1+b_2),\\ (b_0+2b_1+b_2)-(\theta_1+\theta_2)(b_0+b_1)+\theta_1\theta_2b_0 &= b_0\theta_3^2+(b_1-3b_0)\theta_3\\ &\quad +(2b_0-2b_1+b_2).\end{aligned}$$

Next, using Theorem 3.3, we present the Binet's formulas of binomial transform of Narayana, Narayana-Lucas, Narayana-Perrin sequences.

Corollary 3.4. *Binet's formulas of binomial transform of Narayana, Narayana-Lucas, Narayana-Perrin sequences are*

$$\begin{aligned}\widehat{N}_n &= \frac{(-1+\theta_1)\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{(-1+\theta_2)\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{(-1+\theta_3)\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)},\\ \widehat{U}_n &= \theta_1^n+\theta_2^n+\theta_3^n,\\ \widehat{H}_n &= \frac{(3\theta_1^2-7\theta_1+9)\theta_1^{n-1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{(3\theta_2^2-7\theta_2+9)\theta_2^{n-1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{(3\theta_3^2-7\theta_3+9)\theta_3^{n-1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)},\end{aligned}$$

respectively.

4 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Narayana sequence $\{W_n\}$.

Theorem 4.1 (Simson Formula of Generalized Tribonacci Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. (4.1) is given in Soykan [22]. □

Taking $\{W_n\} = \{b_n\}$ in the above theorem and considering $b_{n+3} = 4b_{n+2} - 5b_{n+1} + 3b_n$, $r = 4, s = -5, t = 3$, we have the following proposition.

Proposition 4.2. *For all integers n, Simson formula of binomial transforms of generalized Narayana numbers is given as*

$$\begin{vmatrix} b_{n+2} & b_{n+1} & b_n \\ b_{n+1} & b_n & b_{n-1} \\ b_n & b_{n-1} & b_{n-2} \end{vmatrix} = 3^n \begin{vmatrix} b_2 & b_1 & b_0 \\ b_1 & b_0 & b_{-1} \\ b_0 & b_{-1} & b_{-2} \end{vmatrix}.$$

The previous proposition gives the following results as particular examples.

Corollary 4.3. *For all integers n, Simson formula of binomial transforms of the Narayana, Narayana-Lucas, Narayana-Perrin numbers are given as*

$$\begin{vmatrix} \widehat{N}_{n+2} & \widehat{N}_{n+1} & \widehat{N}_{n} \\ \widehat{N}_{n+1} & \widehat{N}_{n} & \widehat{N}_{n-1} \\ \widehat{N}_{n} & \widehat{N}_{n-1} & \widehat{N}_{n-2} \end{vmatrix} = -3^{n-2},$$

$$\begin{vmatrix} \widehat{U}_{n+2} & \widehat{U}_{n+1} & \widehat{U}_{n} \\ \widehat{U}_{n+1} & \widehat{U}_{n} & \widehat{U}_{n-1} \\ \widehat{U}_{n} & \widehat{U}_{n-1} & \widehat{U}_{n-2} \end{vmatrix} = -31 \times 3^{n-2},$$

$$\begin{vmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_{n} \\ \widehat{H}_{n+1} & \widehat{H}_{n} & \widehat{H}_{n-1} \\ \widehat{H}_{n} & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{vmatrix} = -53 \times 3^{n-2},$$

respectively.

5 Some Identities

In this section, we obtain some identities of binomial transforms of Narayana, Narayana-Lucas, Narayana-Perrin numbers. First, we can give a few basic relations between $\{\widehat{N}_n\}$ and $\{\widehat{U}_n\}$.

Lemma 5.1. *The following equalities are true:*

$$\begin{aligned} 279\widehat{N}_n &= -13\widehat{U}_{n+4} + 82\widehat{U}_{n+3} - 104\widehat{U}_{n+2} \\ 93\widehat{N}_n &= 10\widehat{U}_{n+3} - 13\widehat{U}_{n+2} - 13\widehat{U}_{n+1} \\ 31\widehat{N}_n &= 9\widehat{U}_{n+2} - 21\widehat{U}_{n+1} + 10\widehat{U}_{n} \\ 31\widehat{N}_n &= 15\widehat{U}_{n+1} - 35\widehat{U}_{n} + 27\widehat{U}_{n-1} \\ 31\widehat{N}_n &= 25\widehat{U}_{n} - 48\widehat{U}_{n-1} + 45\widehat{U}_{n-2} \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} 9\widehat{U}_n &= -16\widehat{N}_{n+4}+49\widehat{N}_{n+3}-20\widehat{N}_{n+2} \\ 3\widehat{U}_n &= -5\widehat{N}_{n+3}+20\widehat{N}_{n+2}-16\widehat{N}_{n+1} \\ \widehat{U}_n &= 3\widehat{N}_{n+1}-5\widehat{N}_n \\ \widehat{U}_n &= 7\widehat{N}_n-15\widehat{N}_{n-1}+9\widehat{N}_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$\widehat{N}_n = a\times\widehat{U}_{n+4}+b\times\widehat{U}_{n+3}+c\times\widehat{U}_{n+2}$$

and solving the system of equations

$$\begin{aligned} \widehat{N}_0 &= a\times\widehat{U}_4+b\times\widehat{U}_3+c\times\widehat{U}_2 \\ \widehat{N}_1 &= a\times\widehat{U}_5+b\times\widehat{U}_4+c\times\widehat{U}_3 \\ \widehat{N}_2 &= a\times\widehat{U}_6+b\times\widehat{U}_5+c\times\widehat{U}_4 \end{aligned}$$

we find that $a=-\frac{13}{279}, b=\frac{82}{279}, c=-\frac{104}{279}$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{\widehat{N}_n\}$ and $\{\widehat{H}_n\}$.

Lemma 5.2. *The following equalities are true:*

$$\begin{aligned} 477\widehat{N}_n &= -47\widehat{H}_{n+4}+185\widehat{H}_{n+3}-142\widehat{H}_{n+2}, \\ 159\widehat{N}_n &= -\widehat{H}_{n+3}+31\widehat{H}_{n+2}-47\widehat{H}_{n+1}, \\ 53\widehat{N}_n &= 9\widehat{H}_{n+2}-14\widehat{H}_{n+1}-\widehat{H}_n, \\ 53\widehat{N}_n &= 22\widehat{H}_{n+1}-46\widehat{H}_n+27\widehat{H}_{n-1}, \\ 53\widehat{N}_n &= 42\widehat{H}_n-83\widehat{H}_{n-1}+66\widehat{H}_{n-2}, \end{aligned}$$

and

$$\begin{aligned}
9\widehat{H}_n &= -19\widehat{N}_{n+4} + 70\widehat{N}_{n+3} - 53\widehat{N}_{n+2} \\
3\widehat{H}_n &= -2\widehat{N}_{n+3} + 14\widehat{N}_{n+2} - 19\widehat{N}_{n+1} \\
\widehat{H}_n &= 2\widehat{N}_{n+2} - 3\widehat{N}_{n+1} - 2\widehat{N}_n \\
\widehat{H}_n &= 5\widehat{N}_{n+1} - 12\widehat{N}_n + 6\widehat{N}_{n-1} \\
\widehat{H}_n &= 8\widehat{N}_n - 19\widehat{N}_{n-1} + 15\widehat{N}_{n-2}
\end{aligned}$$

Now, we give a few basic relations between $\{\widehat{U}_n\}$ and $\{\widehat{H}_n\}$.

Lemma 5.3. *The following equalities are true:*

$$\begin{aligned}
477\widehat{U}_n &= 226\widehat{H}_{n+4} - 646\widehat{H}_{n+3} + 287\widehat{H}_{n+2}, \\
159\widehat{U}_n &= 86\widehat{H}_{n+3} - 281\widehat{H}_{n+2} + 226\widehat{H}_{n+1}, \\
53\widehat{U}_n &= 21\widehat{H}_{n+2} - 68\widehat{H}_{n+1} + 86\widehat{H}_n, \\
53\widehat{U}_n &= 16\widehat{H}_{n+1} - 19\widehat{H}_n + 63\widehat{H}_{n-1}, \\
53\widehat{U}_n &= 45\widehat{H}_n - 17\widehat{H}_{n-1} + 48\widehat{H}_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
279\widehat{H}_n &= 98\widehat{U}_{n+4} - 425\widehat{U}_{n+3} + 505\widehat{U}_{n+2}, \\
93\widehat{H}_n &= -11\widehat{U}_{n+3} + 5\widehat{U}_{n+2} + 98\widehat{U}_{n+1}, \\
31\widehat{H}_n &= -13\widehat{U}_{n+2} + 51\widehat{U}_{n+1} - 11\widehat{U}_n, \\
31\widehat{H}_n &= -\widehat{U}_{n+1} + 54\widehat{U}_n - 39\widehat{U}_{n-1}, \\
31\widehat{H}_n &= 50\widehat{U}_n - 34\widehat{U}_{n-1} - 3\widehat{U}_{n-2}.
\end{aligned}$$

6 Sum Formulas

6.1 Sums of Terms with Positive Subscripts

The following proposition presents some formulas of binomial transform of generalized Narayana numbers with positive subscripts.

Proposition 6.1. *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} b_k = b_{n+3} - 3b_{n+2} + 2b_{n+1} - b_2 + 3b_1 - 2b_0.$

(b) $\sum_{k=0}^{n} b_{2k} = \frac{1}{13}(6b_{2n+2} - 17b_{2n+1} + 21b_{2n} - 6b_2 + 17b_1 - 8b_0).$

(c) $\sum_{k=0}^{n} b_{2k+1} = \frac{1}{13}(7b_{2n+2} - 9b_{2n+1} + 18b_{2n} - 7b_2 + 22b_1 - 18b_0).$

Proof. Take $r = 4, s = -5, t = 3$ in Theorem 2.1 in [24] (or take $x = 1, r = 4, s = -5, t = 3$ in Theorem 2.1 in [25]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Narayana numbers (take $b_n = \widehat{N}_n$ with $\widehat{N}_0 = 0, \widehat{N}_1 = 1, \widehat{N}_2 = 3$).

Corollary 6.2. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{N}_k = \widehat{N}_{n+3} - 3\widehat{N}_{n+2} + 2\widehat{N}_{n+1}.$

(b) $\sum_{k=0}^{n} \widehat{N}_{2k} = \frac{1}{13}(6\widehat{N}_{2n+2} - 17\widehat{N}_{2n+1} + 21\widehat{N}_{2n} - 1).$

(c) $\sum_{k=0}^{n} \widehat{N}_{2k+1} = \frac{1}{13}(7\widehat{N}_{2n+2} - 9\widehat{N}_{2n+1} + 18\widehat{N}_{2n} + 1).$

Taking $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 3, \widehat{U}_1 = 4, \widehat{U}_2 = 6$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of Narayana-Lucas numbers.

Corollary 6.3. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{U}_k = \widehat{U}_{n+3} - 3\widehat{U}_{n+2} + 2\widehat{U}_{n+1}.$

(b) $\sum_{k=0}^{n} \widehat{U}_{2k} = \frac{1}{13}(6\widehat{U}_{2n+2} - 17\widehat{U}_{2n+1} + 21\widehat{U}_{2n} + 8).$

(c) $\sum_{k=0}^{n} \widehat{U}_{2k+1} = \frac{1}{13}(7\widehat{U}_{2n+2} - 9\widehat{U}_{2n+1} + 18\widehat{U}_{2n} - 8).$

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Narayana-Perrin numbers (take $b_n = \widehat{H}_n$ with $\widehat{H}_0 = 3, \widehat{H}_1 = 3, \widehat{H}_2 = 5$).

Corollary 6.4. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{H}_k = \widehat{H}_{n+3} - 3\widehat{H}_{n+2} + 2\widehat{H}_{n+1} - 2.$

(b) $\sum_{k=0}^{n} \widehat{H}_{2k} = \frac{1}{13}(6\widehat{H}_{2n+2} - 17\widehat{H}_{2n+1} + 21\widehat{H}_{2n} - 3).$

(c) $\sum_{k=0}^{n} \widehat{H}_{2k+1} = \frac{1}{13}(7\widehat{H}_{2n+2} - 9\widehat{H}_{2n+1} + 18\widehat{H}_{2n} - 23).$

6.2 Sums of Terms with Negative Subscripts

The following proposition presents some formulas of binomial transform of generalized Narayana numbers with negative subscripts.

Proposition 6.5. *For $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} b_{-k} = -2b_{-n-1} + 2b_{-n-2} - 3b_{-n-3} + b_2 - 3b_1 + 2b_0.$

(b) $\sum_{k=1}^{n} b_{-2k} = \frac{1}{13}(-7b_{-2n+1} + 22b_{-2n} - 18b_{-2n-1} + 6b_2 - 17b_1 + 8b_0).$

(c) $\sum_{k=1}^{n} b_{-2k+1} = \frac{1}{13}(-6b_{-2n+1} + 17b_{-2n} - 21b_{-2n-1} + 7b_2 - 22b_1 + 18b_0).$

Proof. Take $r = 4, s = -5, t = 3$ in Theorem 3.1 in [24] or (or take $x = 1, r = 4, s = -5, t = 3$ in Theorem 3.1 in [25]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Narayana numbers (take $b_n = \widehat{N}_n$ with $\widehat{N}_0 = 0, \widehat{N}_1 = 1, \widehat{N}_2 = 3$).

Corollary 6.6. *For $n \geq 1$, binomial transform of Narayana numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{N}_{-k} = -2\widehat{N}_{-n-1} + 2\widehat{N}_{-n-2} - 3\widehat{N}_{-n-3}.$

(b) $\sum_{k=1}^{n} \widehat{N}_{-2k} = \frac{1}{13}(-7\widehat{N}_{-2n+1} + 22\widehat{N}_{-2n} - 18\widehat{N}_{-2n-1} + 1).$

(c) $\sum_{k=1}^{n} \widehat{N}_{-2k+1} = \frac{1}{13}(-6\widehat{N}_{-2n+1} + 17\widehat{N}_{-2n} - 21\widehat{N}_{-2n-1} - 1).$

Taking $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 3, \widehat{U}_1 = 4, \widehat{U}_2 = 6$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of Narayana-Lucas numbers.

Corollary 6.7. *For $n \geq 1$, binomial transform of Narayana-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{U}_{-k} = -2\widehat{U}_{-n-1} + 2\widehat{U}_{-n-2} - 3\widehat{U}_{-n-3}$.

(b) $\sum_{k=1}^{n} \widehat{U}_{-2k} = \frac{1}{13}(-7\widehat{U}_{-2n+1} + 22\widehat{U}_{-2n} - 18\widehat{U}_{-2n-1} - 8)$.

(c) $\sum_{k=1}^{n} \widehat{U}_{-2k+1} = \frac{1}{13}(-6\widehat{U}_{-2n+1} + 17\widehat{U}_{-2n} - 21\widehat{U}_{-2n-1} + 8)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Narayana-Perrin numbers (take $b_n = \widehat{H}_n$ with $\widehat{H}_0 = 3, \widehat{H}_1 = 3, \widehat{H}_2 = 5$).

Corollary 6.8. *For $n \geq 1$, binomial transform of Narayana-Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{H}_{-k} = -2\widehat{H}_{-n-1} + 2\widehat{H}_{-n-2} - 3\widehat{H}_{-n-3} + 2$.

(b) $\sum_{k=1}^{n} \widehat{H}_{-2k} = \frac{1}{13}(-7\widehat{H}_{-2n+1} + 22\widehat{H}_{-2n} - 18\widehat{H}_{-2n-1} + 3)$.

(c) $\sum_{k=1}^{n} \widehat{H}_{-2k+1} = \frac{1}{13}(-6\widehat{H}_{-2n+1} + 17\widehat{H}_{-2n} - 21\widehat{H}_{-2n-1} + 23)$.

6.3 Sums of Squares of Terms with Positive Subscripts

The following proposition presents some formulas of binomial transform of generalized Narayana numbers with positive subscripts.

Proposition 6.9. *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} b_k^2 = \frac{1}{13}(15b_{n+3}^2 + 167b_{n+2}^2 + 122b_{n+1}^2 - 98b_{n+3}b_{n+2} + 66b_{n+3}b_{n+1} - 252b_{n+2}b_{n+1} - 15b_2^2 - 167b_1^2 - 122b_0^2 + 98b_2b_1 - 66b_2b_0 + 252b_1b_0)$.

(b) $\sum_{k=0}^{n} b_{k+1}b_k = \frac{1}{13}(11b_{n+3}^2 + 119b_{n+2}^2 + 99b_{n+1}^2 - 71b_{n+3}b_{n+2} - 190b_{n+2}b_{n+1} + 51b_{n+3}b_{n+1} - 11b_2^2 - 119b_1^2 - 99b_0^2 + 71b_2b_1 - 51b_2b_0 + 190b_1b_0)$.

(c) $\sum_{k=0}^{n} b_{k+2}b_k = \frac{1}{13}(2b_{n+3}^2 - 2b_{n+2}^2 + 18b_{n+1}^2 - 7b_{n+3}b_{n+2} + 14b_{n+3}b_{n+1} - 18b_{n+2}b_{n+1} - 2b_2^2 + 2b_1^2 - 18b_0^2 + 7b_2b_1 - 14b_2b_0 + 18b_1b_0)$.

Proof. Take $x = 1, r = 4, s = -5, t = 3$ in Theorem 4.1 in [27], see also [26].

From the last proposition, we have the following Corollary which gives sum formulas of binomial transform of Narayana numbers (take $b_n = \widehat{N}_n$ with $\widehat{N}_0 = 0, \widehat{N}_1 = 1, \widehat{N}_2 = 3$).

Corollary 6.10. *For $n \geq 0$, binomial transform of Narayana numbers have the following properties:*

(a) $\sum_{k=0}^{n} \widehat{N}_k^2 = \frac{1}{13}(15\widehat{N}_{n+3}^2 + 167\widehat{N}_{n+2}^2 + 122\widehat{N}_{n+1}^2 - 98\widehat{N}_{n+3}\widehat{N}_{n+2} + 66\widehat{N}_{n+3}\widehat{N}_{n+1} - 252\widehat{N}_{n+2}\widehat{N}_{n+1} - 8)$.

(b) $\sum_{k=0}^{n} \widehat{N}_{k+1}\widehat{N}_k = \frac{1}{13}(11\widehat{N}_{n+3}^2 + 119\widehat{N}_{n+2}^2 + 99\widehat{N}_{n+1}^2 - 71\widehat{N}_{n+3}\widehat{N}_{n+2} - 190\widehat{N}_{n+2}\widehat{N}_{n+1} + 51\widehat{N}_{n+3}\widehat{N}_{n+1} - 5)$.

(c) $\sum_{k=0}^{n} \widehat{N}_{k+2}\widehat{N}_k = \frac{1}{13}(2\widehat{N}_{n+3}^2 - 2\widehat{N}_{n+2}^2 + 18\widehat{N}_{n+1}^2 - 7\widehat{N}_{n+3}\widehat{N}_{n+2} + 14\widehat{N}_{n+3}\widehat{N}_{n+1} - 18\widehat{N}_{n+2}\widehat{N}_{n+1} + 5)$.

Taking $b_n = \widehat{U}_n$ with $\widehat{U}_0 = 3, \widehat{U}_1 = 4, \widehat{U}_2 = 6$ in the last Proposition, we have the following Corollary which presents sum formulas of binomial transform of Narayana-Lucas numbers.

Corollary 6.11. *For $n \geq 0$, binomial transform of Narayana-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} \widehat{U}_k^2 = \frac{1}{13}(15\widehat{U}_{n+3}^2 + 167\widehat{U}_{n+2}^2 + 122\widehat{U}_{n+1}^2 - 98\widehat{U}_{n+3}\widehat{U}_{n+2} + 66\widehat{U}_{n+3}\widehat{U}_{n+1} - 252\widehat{U}_{n+2}\widehat{U}_{n+1} - 122)$.

(b) $\sum_{k=0}^{n} \widehat{U}_{k+1}\widehat{U}_k = \frac{1}{13}(11\widehat{U}_{n+3}^2 + 119\widehat{U}_{n+2}^2 + 99\widehat{U}_{n+1}^2 - 71\widehat{U}_{n+3}\widehat{U}_{n+2} - 190\widehat{U}_{n+2}\widehat{U}_{n+1} + 51\widehat{U}_{n+3}\widehat{U}_{n+1} - 125)$.

(c) $\sum_{k=0}^{n} \widehat{U}_{k+2}\widehat{U}_k = \frac{1}{13}(2\widehat{U}_{n+3}^2 - 2\widehat{U}_{n+2}^2 + 18\widehat{U}_{n+1}^2 - 7\widehat{U}_{n+3}\widehat{U}_{n+2} + 14\widehat{U}_{n+3}\widehat{U}_{n+1} - 18\widehat{U}_{n+2}\widehat{U}_{n+1} - 70)$.

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Narayana-Perrin numbers (take $b_n = \widehat{H}_n$ with $\widehat{H}_0 = 3, \widehat{H}_1 = 3, \widehat{H}_2 = 5$).

Corollary 6.12. *For $n \geq 0$, binomial transform of Narayana-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} \widehat{H}_k^2 = \frac{1}{13}(15\widehat{H}_{n+3}^2 + 167\widehat{H}_{n+2}^2 + 122\widehat{H}_{n+1}^2 - 98\widehat{H}_{n+3}\widehat{H}_{n+2} + 66\widehat{H}_{n+3}\widehat{H}_{n+1} - 252\widehat{H}_{n+2}\widehat{H}_{n+1} - 228)$.

(b) $\sum_{k=0}^{n} \widehat{H}_{k+1}\widehat{H}_k = \frac{1}{13}(11\widehat{H}_{n+3}^2 + 119\widehat{H}_{n+2}^2 + 99\widehat{H}_{n+1}^2 - 71\widehat{H}_{n+3}\widehat{H}_{n+2} - 190\widehat{H}_{n+2}\widehat{H}_{n+1} + 51\widehat{H}_{n+3}\widehat{H}_{n+1} - 227)$.

(c) $\sum_{k=0}^{n} \widehat{H}_{k+2}\widehat{H}_k = \frac{1}{13}(2\widehat{H}_{n+3}^2 - 2\widehat{H}_{n+2}^2 + 18\widehat{H}_{n+1}^2 - 7\widehat{H}_{n+3}\widehat{H}_{n+2} + 14\widehat{H}_{n+3}\widehat{H}_{n+1} - 18\widehat{H}_{n+2}\widehat{H}_{n+1} - 137)$.

7 Matrices related with Binomial Transform of Generalized Narayana numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

For matrix formulation (7.1), see [14]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 4 & -5 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 3$. From (2.1) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n+1} \\ b_n \\ b_{n-1} \end{pmatrix} \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} b_2 \\ b_1 \\ b_0 \end{pmatrix}.$$

If we take $b_n = \widehat{N}_n$ in (7.2) we have

$$\begin{pmatrix} \widehat{N}_{n+2} \\ \widehat{N}_{n+1} \\ \widehat{N}_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{N}_{n+1} \\ \widehat{N}_n \\ \widehat{N}_{n-1} \end{pmatrix}. \tag{7.3}$$

For $n \geq 0$, we define

$$B_n = \begin{pmatrix} \sum_{k=0}^{n+1} \widehat{N}_k & -5\sum_{k=0}^{n} \widehat{N}_k + 3\sum_{k=0}^{n-1} \widehat{N}_k & 3\sum_{k=0}^{n} \widehat{N}_k \\ \sum_{k=0}^{n} \widehat{N}_k & -5\sum_{k=0}^{n-1} \widehat{N}_k + 3\sum_{k=0}^{n-2} \widehat{N}_k & 3\sum_{k=0}^{n-1} \widehat{N}_k \\ \sum_{k=0}^{n-1} \widehat{N}_k & -5\sum_{k=0}^{n-2} \widehat{N}_k + 3\sum_{k=0}^{n-3} \widehat{N}_k & 3\sum_{k=0}^{n-2} \widehat{N}_k \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} b_{n+1} & -5b_n + 3b_{n-1} & 3b_n \\ b_n & -5b_{n-1} + 3b_{n-2} & 3b_{n-1} \\ b_{n-1} & -5b_{n-2} + 3b_{n-3} & 3b_{n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1} \widehat{N}_k = 0, \quad \sum_{k=0}^{-2} \widehat{N}_k = \frac{1}{3}, \quad \sum_{k=0}^{-3} \widehat{N}_k = \frac{5}{9}.$$

Theorem 7.1. *For all integers $m, n \geq 0$, we have*

(a) $B_n = A^n$.

(b) $C_1 A^n = A^n C_1$.

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. **(a)** Proof can be done by mathematical induction on n.

(b) After matrix multiplication, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 4 & -5 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_n & -5b_{n-1}+3b_{n-2} & 3b_{n-1} \\ b_{n-1} & -5b_{n-2}+3b_{n-3} & 3b_{n-2} \\ b_{n-2} & -5b_{n-3}+3b_{n-4} & 3b_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{n+1} & -5b_n+3b_{n-1} & 3b_n \\ b_n & -5b_{n-1}+3b_{n-2} & 3b_{n-1} \\ b_{n-1} & -5b_{n-2}+3b_{n-3} & 3b_{n-2} \end{pmatrix} = C_n, \end{aligned}$$

i.e., $C_n = AC_{n-1}$. From the last equation, using induction, we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

□

Some properties of matrix A^n can be given as

$$A^n = 4A^{n-1} - 5A^{n-2} + 3A^{n-3} = \frac{5}{3}A^{n+1} - \frac{4}{3}A^{n+2} + \frac{1}{3}A^{n+3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 3^n$$

for all integers $m, n \geq 0$.

Theorem 7.2. *For $m, n \geq 0$, we have*

$$\begin{aligned} b_{n+m} &= b_n \sum_{k=0}^{m+1} \widehat{N}_k + b_{n-1}\left(-5\sum_{k=0}^{m} \widehat{N}_k + 3\sum_{k=0}^{m-1} \widehat{N}_k\right) + 3b_{n-2}\sum_{k=0}^{m} \widehat{N}_k \\ &= b_n \sum_{k=0}^{m+1} \widehat{N}_k + (-5b_{n-1} + 3b_{n-2})\sum_{k=0}^{m} \widehat{N}_k + 3b_{n-1}\sum_{k=0}^{m-1} \widehat{N}_k. \end{aligned}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$, we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation, we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

Corollary 7.3. *For $m, n \geq 0$, we have*

$$\begin{aligned} \widehat{N}_{n+m} &= \widehat{N}_n \sum_{k=0}^{m+1} \widehat{N}_k + \widehat{N}_{n-1}\left(-5\sum_{k=0}^{m} \widehat{N}_k + 3\sum_{k=0}^{m-1} \widehat{N}_k\right) + 3\widehat{N}_{n-2}\sum_{k=0}^{m} \widehat{N}_k, \\ \widehat{U}_{n+m} &= \widehat{U}_n \sum_{k=0}^{m+1} \widehat{N}_k + \widehat{U}_{n-1}\left(-5\sum_{k=0}^{m} \widehat{N}_k + 3\sum_{k=0}^{m-1} \widehat{N}_k\right) + 3\widehat{U}_{n-2}\sum_{k=0}^{m} \widehat{N}_k, \\ \widehat{H}_{n+m} &= \widehat{H}_n \sum_{k=0}^{m+1} \widehat{N}_k + \widehat{H}_{n-1}\left(-5\sum_{k=0}^{m} \widehat{N}_k + 3\sum_{k=0}^{m-1} \widehat{N}_k\right) + 3\widehat{H}_{n-2}\sum_{k=0}^{m} \widehat{N}_k, \end{aligned}$$

From Corollary 6.2, we know that for $n \geq 0$,

$$\sum_{k=0}^{n} \widehat{N}_k = \widehat{N}_{n+3} - 3\widehat{N}_{n+2} + 2\widehat{N}_{n+1}$$

So, Theorem 7.2 and Corollary 7.3 can be written in the following forms:

Theorem 7.4. *For $m, n \geq 0$, we have*

$$\begin{aligned} b_{n+m} &= (\widehat{N}_{m+4} - 3\widehat{N}_{m+3} + 2\widehat{N}_{m+2})b_n \\ &\quad +(-5\widehat{N}_{m+3} + 18\widehat{N}_{m+2} - 19\widehat{N}_{m+1} + 6\widehat{N}_m)b_{n-1} \\ &\quad +3(\widehat{N}_{m+3} - 3\widehat{N}_{m+2} + 2\widehat{N}_{m+1})b_{n-2}. \end{aligned} \tag{7.4}$$

Remark 7.5. By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So, for all integers m, n, (7.4) is true.

Corollary 7.6. *For all integers m, n, we have*

$$\begin{aligned}
N_{n+m} &= (\widehat{N}_{m+4} - 3\widehat{N}_{m+3} + 2\widehat{N}_{m+2})N_n \\
&\quad +(-5\widehat{N}_{m+3} + 18\widehat{N}_{m+2} - 19\widehat{N}_{m+1} + 6\widehat{N}_m)N_{n-1} \\
&\quad +3(\widehat{N}_{m+3} - 3\widehat{N}_{m+2} + 2\widehat{N}_{m+1})N_{n-2}, \\
U_{n+m} &= (\widehat{N}_{m+4} - 3\widehat{N}_{m+3} + 2\widehat{N}_{m+2})U_n \\
&\quad +(-5\widehat{N}_{m+3} + 18\widehat{N}_{m+2} - 19\widehat{N}_{m+1} + 6\widehat{N}_m)U_{n-1} \\
&\quad +3(\widehat{N}_{m+3} - 3\widehat{N}_{m+2} + 2\widehat{N}_{m+1})U_{n-2}, \\
H_{n+m} &= (\widehat{N}_{m+4} - 3\widehat{N}_{m+3} + 2\widehat{N}_{m+2})H_n \\
&\quad +(-5\widehat{N}_{m+3} + 18\widehat{N}_{m+2} - 19\widehat{N}_{m+1} + 6\widehat{N}_m)H_{n-1} \\
&\quad +3(\widehat{N}_{m+3} - 3\widehat{N}_{m+2} + 2\widehat{N}_{m+1})H_{n-2}.
\end{aligned}$$

Now, we consider non-positive subscript cases. For $n \geq 0$, we define

$$B_{-n} = \begin{pmatrix} -\sum_{k=0}^{n-2}\widehat{N}_{-k} & 5\sum_{k=0}^{n-1}\widehat{N}_{-k} - 3\sum_{k=0}^{n}\widehat{N}_{-k}) & -3\sum_{k=0}^{n-1}\widehat{N}_{-k} \\ -\sum_{k=0}^{n-1}\widehat{N}_{-k} & 5\sum_{k=0}^{n}\widehat{N}_{-k} - 3\sum_{k=0}^{n+1}\widehat{N}_{-k}) & -3\sum_{k=0}^{n}\widehat{N}_{-k} \\ -\sum_{k=0}^{n}\widehat{N}_{-k} & 5\sum_{k=0}^{n+1}\widehat{N}_{-k} - 3\sum_{k=0}^{n+2}\widehat{N}_{-k}) & -3\sum_{k=0}^{n+1}\widehat{N}_{-k} \end{pmatrix}$$

and

$$C_{-n} = \begin{pmatrix} b_{-n+1} & -5b_{-n} + 3b_{-n-1} & 3b_{-n} \\ b_{-n} & -5b_{-n-1} + 3b_{-n-2} & 3b_{-n-1} \\ b_{-n-1} & -5b_{-n-2} + 3b_{-n-3} & 32b_{-n-2} \end{pmatrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-1}\widehat{N}_{-k} = 0, \quad \sum_{k=0}^{-2}\widehat{N}_{-k} = -1.$$

Theorem 7.7. *For all integers $m, n \geq 0$, we have*

(a) $B_{-n} = A^{-n}$.

(b) $C_{-1}A^{-n} = A^{-n}C_{-1}$.

(c) $C_{-n-m} = C_{-n}B_{-m} = B_{-m}C_{-n}$.

Proof. **(a)** Proof can be done by mathematical induction on n.

(b) After matrix multiplication, (b) follows.

(c) We have

$$\begin{aligned} A^{-1}C_{-n-1} &= \begin{pmatrix} 4 & -5 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_{-n} & -5b_{-n-1}+3b_{-n-2} & 3b_{-n-1} \\ b_{-n-1} & -5b_{-n-2}+3b_{-n-3} & 3b_{-n-2} \\ b_{-n-2} & -5b_{-n-3}+3b_{-n-4} & 3b_{-n-3} \end{pmatrix} \\ &= \begin{pmatrix} b_{-n+1} & -5b_{-n}+3b_{-n-1} & 3b_{-n} \\ b_{-n} & -5b_{-n-1}+3b_{-n-2} & 3b_{-n-1} \\ b_{-n-1} & -5b_{-n-2}+3b_{-n-3} & 3b_{-n-2} \end{pmatrix} = C_{-n}, \end{aligned}$$

i.e. $C_{-n} = A^{-1}C_{-n-1}$. From the last equation, using induction, we obtain $C_{-n} = A^{-n-1}C_{-1}$. Now,

$$C_{-n-m} = A^{-n-m-1}C_{-1} = A^{-n-1}A^{-m}C_{-1} = A^{-n-1}C_{-1}A^{-m} = C_{-n}B_{-m}$$

and similarly,

$$C_{-n-m} = B_{-m}C_{-n}.$$

□

Some properties of matrix A^{-n} can be given as

$$A^{-n} = 4A^{-n-1} - 5A^{-n-2} + 3A^{-n-3} = \frac{5}{3}A^{-n+1} - \frac{4}{3}A^{-n+2} + \frac{1}{3}A^{-n+3}$$

and

$$A^{-n-m} = A^{-n}A^{-m} = A^{-m}A^{-n}$$

and

$$\det(A^{-n}) = 3^{-n}$$

for all integers $m, n \geq 0$.

Theorem 7.8. *For $m, n \geq 0$, we have*

$$\begin{aligned} b_{-n-m} &= -b_{-n}\sum_{k=0}^{m-2}\widehat{N}_{-k} - b_{-n-1}\left(-5\sum_{k=0}^{m-1}\widehat{N}_{-k} + 3\sum_{k=0}^{m}\widehat{N}_{-k}\right) - 3b_{-n-2}\sum_{k=0}^{m-1}\widehat{N}_{-k} \\ &= -b_{-n}\sum_{k=0}^{m-2}\widehat{N}_{-k} - (-5b_{-n-1} + 3b_{-n-2})\sum_{k=0}^{m-1}\widehat{N}_{-k} - 3b_{-n-1}\sum_{k=0}^{m}\widehat{N}_{-k}. \end{aligned}$$

Proof. From the equation $C_{-n-m} = C_{-n}B_{-m} = B_{-m}C_{-n}$, we see that an element of C_{-n-m} is the product of row C_{-n} and a column B_{-m}. From the last equation, we say that an element of C_{-n-m} is the product of a row C_{-n} and column B_{-m}. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{-n-m} and $C_{-n}B_{-m}$. This completes the proof. □

Corollary 7.9. *For $m, n \geq 0$, we have*

$$\begin{aligned} \widehat{N}_{-n-m} &= -\widehat{N}_{-n}\sum_{k=0}^{m-2}\widehat{N}_{-k} - \widehat{N}_{-n-1}\left(-5\sum_{k=0}^{m-1}\widehat{N}_{-k} + 3\sum_{k=0}^{m}\widehat{N}_{-k}\right) \\ &\quad -3\widehat{N}_{-n-2}\sum_{k=0}^{m-1}\widehat{N}_{-k}, \\ \widehat{U}_{-n-m} &= -\widehat{U}_{-n}\sum_{k=0}^{m-2}\widehat{N}_{-k} - \widehat{U}_{-n-1}\left(-5\sum_{k=0}^{m-1}\widehat{N}_{-k} + 3\sum_{k=0}^{m}\widehat{N}_{-k}\right) \\ &\quad -3\widehat{U}_{-n-2}\sum_{k=0}^{m-1}\widehat{N}_{-k}, \\ \widehat{H}_{-n-m} &= -\widehat{H}_{-n}\sum_{k=0}^{m-2}\widehat{N}_{-k} - \widehat{H}_{-n-1}\left(-5\sum_{k=0}^{m-1}\widehat{N}_{-k} + 3\sum_{k=0}^{m}\widehat{N}_{-k}\right) \\ &\quad -3\widehat{H}_{-n-2}\sum_{k=0}^{m-1}\widehat{N}_{-k}. \end{aligned}$$

From Corollary 6.6, we know that for $n \geq 1$,

$$\sum_{k=1}^{n}\widehat{N}_{-k} = -2\widehat{N}_{-n-1} + 2\widehat{N}_{-n-2} - 3\widehat{N}_{-n-3}.$$

Since $\widehat{N}_0 = 0$, it follows that

$$\sum_{k=0}^{n} \widehat{N}_{-k} = -2\widehat{N}_{-n-1} + 2\widehat{N}_{-n-2} - 3\widehat{N}_{-n-3}.$$

So, Theorem 7.8 and Corollary 7.8 can be written in the following forms.

Theorem 7.10. *For $m, n \geq 0$, we have*

$$\begin{aligned} b_{-n-m} &= (2\widehat{N}_{-m+1} - 2\widehat{N}_{-m} + 3\widehat{N}_{-m-1})b_{-n} \\ &\quad +(-10\widehat{N}_{-m} + 16\widehat{N}_{-m-1} - 21\widehat{N}_{-m-2} + 9\widehat{N}_{-m-3})b_{-n-1} \\ &\quad +3(2\widehat{N}_{-m} - 2\widehat{N}_{-m-1} + 3\widehat{N}_{-m-2})b_{-n-2}. \end{aligned} \tag{7.5}$$

Remark 7.11. By induction, it can be proved that for all integers $m, n \leq 0$, (7.5) holds. So, for all integers m, n, (7.5) is true.

Corollary 7.12. *For all integers m, n, we have*

$$\begin{aligned} \widehat{N}_{-n-m} &= (2\widehat{N}_{-m+1} - 2\widehat{N}_{-m} + 3\widehat{N}_{-m-1})\widehat{N}_{-n} \\ &\quad +(-10\widehat{N}_{-m} + 16\widehat{N}_{-m-1} - 21\widehat{N}_{-m-2} + 9\widehat{N}_{-m-3})\widehat{N}_{-n-1} \\ &\quad +3(2\widehat{N}_{-m} - 2\widehat{N}_{-m-1} + 3\widehat{N}_{-m-2})\widehat{N}_{-n-2}, \\ \widehat{U}_{-n-m} &= (2\widehat{N}_{-m+1} - 2\widehat{N}_{-m} + 3\widehat{N}_{-m-1})\widehat{U}_{-n} \\ &\quad +(-10\widehat{N}_{-m} + 16\widehat{N}_{-m-1} - 21\widehat{N}_{-m-2} + 9\widehat{N}_{-m-3})\widehat{U}_{-n-1} \\ &\quad +3(2\widehat{N}_{-m} - 2\widehat{N}_{-m-1} + 3\widehat{N}_{-m-2})\widehat{U}_{-n-2}, \\ H_{-n-m} &= (2\widehat{N}_{-m+1} - 2\widehat{N}_{-m} + 3\widehat{N}_{-m-1})H_{-n} \\ &\quad +(-10\widehat{N}_{-m} + 16\widehat{N}_{-m-1} - 21\widehat{N}_{-m-2} + 9\widehat{N}_{-m-3})H_{-n-1} \\ &\quad +3(2\widehat{N}_{-m} - 2\widehat{N}_{-m-1} + 3\widehat{N}_{-m-2})H_{-n-2}. \end{aligned}$$

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