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# Decomposition of Goursat Matrices and Subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$

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**Abstract**

Given the number of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$, we deduce the Goursat matrix. The purpose of this paper is two-fold. A first and more concrete aim is to demonstrate that the triangular decomposition of the Goursat matrix may also be written out explicitly, and furthermore that the same is true of the inverse of these triangular factors. A second and more abstract aim provides a containment relation property between subgroups of a direct product . Namely, if $U_2 \leq U_1 \leq \mathbb{Z}_m \times \mathbb{Z}_n$ , we provide necessary and sufficient conditions for $U_2 \leq U_1$.

## 1 Introduction

One of the most major problems of the combinatorial abelian group theory is to investigate the number of subgroups of a finite abelian group. This topic has enjoyed a constant evolution starting with the first half of the $20^{th}$ century. In 1897 Goursat proved that every subgroup of the direct product of two groups is determined by an isomorphism between factors groups of the given groups [8]. Given Goursat's lemma for groups, we use as consequences, by purely number theoretical arguments, explicit formulas for the total number of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ and $f_p(i,j)$ the number of subgroups of $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$. This allows us to deduce a new type of symmetric matrix $G_p(n)$ with $f_p(i,j)_{i,j}$ as coefficient, also called Goursat's matrix of order $n+1$. Expressing a matrix as a product of a lower triangular matrix $L$ and an upper triangular matrix $U$ is called an $LU$ factorization. Such a factorization is typically obtained by Gaussian elimination. If $L$ is a lower triangular with unit main diagonal and $U$ is an upper triangular, the $LU$ factorization of a matrix is unique [6].

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**Structure of the paper.** In Section 2 we expand on the notation introduced in Goursat's Lemma, by presenting results that will provide a better understanding of mappings that will be used in our containment characterization. This section may seem very technical, and it is indeed. To follow the information presented here, the reader should recall the Isomorphism Theorems [15] and the projections and intersections that will be introduced after Goursat's Lemma. To unify notation, we are explicit.

In Section 3 , the author gives a simpler alternative approach to the $LU$ factorization of the Goursat matrix, and obtains explicit formulas of the triangular factors.

Section 4, concludes the paper. This is the theoretical backbone of the dissertation. There is a characterization of containment of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ and a theorem that will allow one to see when two subgroups are isomorphic and have the same subgroup lattice structure. Suppose $U_1$ and $U_2$ are subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$. The characterization provides necessary and sufficient conditions for $U_2 \leq U_1$.

## 2 Goursat's Lemma for Groups

Let $L, R$ be groups. The neutral element of each group $L$ and $R$, with slight abuse of notation, will be written $e$ or 1. Let $\pi_1 : L \times R \to L$, $\pi_2 : L \times R \to R$, be the natural projections and $\iota_1 : L \to L \times R$, $\iota_2 : R \to L \times R$, be the usual inclusions. Suppose that $U$ is a subgroup of $L \times R$. Write

$$\begin{aligned} A &= \pi_1(U) = \{a \in L | (a, b) \in U, \text{for } b \in R\}; \\ B &= i_1^{-1}(U) = \{a \in L | (a, e) \in U\}. \end{aligned}$$

Similarly for $C = \pi_2(U), D = i_2^{-1}(U)$. Goursat's lemma for groups ([17], p. 2) can be stated as follows:

**Lemma 2.1.** *Let $L$ and $R$ be arbitrary groups. Then there is a bijection between the set $\mathcal{S}$ of all subgroups of $L \times R$ and the set $\mathcal{T}$ of all quintuple $(A, B, C, D, \theta)$, where $B \trianglelefteq A \leq L$, $D \trianglelefteq C \leq R$ and $\theta : A/B \to C/D$ is a bijective homomorphism (here $\leq$ denotes subgroup and $\trianglelefteq$ denotes normal subgroup). More precisely, the subgroup corresponding to $(A, B, C, D, \theta)$ is*

$$U = \{(g, h) \in A \times C : \theta(gB) = hD\}.$$

**Definition 2.2.** For a subgroup $U$ of $L \times R$. We say that the corresponding 5-tuple $Q_5(U) = (A, B, C, D, \theta)$ of Lemma 2.1 is the Goursat decomposition of $U$.

Motivated by [15], we have the following alternate characterization.

**Theorem 2.3.** *Let $U_1, U_2 \leq L \times R$ where $U_1$ is given by the Goursat decomposition $Q_5(U_1) = (A_1, B_1, C_1, D_1, \theta_1)$ and $U_2$ is given by the Goursat decomposition $Q_5(U_2) = (A_2, B_2, C_2, D_2, \theta_2)$. $U_2 \leq U_1$ if and only if*

$(i)$ $A_2 \leq A_1$, $B_2 \leq B_1$, *and* $C_2 \leq C_1$, $D_2 \leq D_1$;

$(ii)$ $\left|\frac{A_2 B_1}{B_1}\right| = \left|\frac{C_2 D_1}{D_1}\right| = \left|\frac{A_2}{A_2 \cap B_1}\right| = \left|\frac{C_2}{C_2 \cap D_1}\right|$;

$(iii)$ $\theta_1(\frac{A_2 B_1}{B_1}) = \frac{C_2 D_1}{D_1}$ *and* $\theta_2(\frac{A_2 \cap B_1}{B_2}) = \frac{C_2 \cap D_1}{D_2}$.

# 3 The $LU$ Factorization of the Goursat Matrix

In the following let us denote by $f_p(i, j)$ the total number of all subgroups of the finite abelian p-group $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$, $(i \leq j)$, concerning general properties of the subgroup lattice of finite Abelian groups, see Tòth [22], Tãrnãuceanu [25]. For every $p \in \mathbb{P}$ one has

$$f_p(i, j) = \sum_{k=0}^{i} (j + i - 2k + 1)p^k$$
$$= \frac{(j - i + 1)p^{i+2} - (j - i - 1)p^{i+1} - (i + j + 3)p + (i + j + 1)}{(p - 1)^2}.$$

Put $f_p(i, j) = f_p(j, i)$, for all $i > j$, and let $n$ be a fixed positive integer. We denote the Goursat matrix of order $n + 1$

$$G_p(n) = \begin{pmatrix} f_p(0,0) & f_p(0,1) & \cdots & f_p(0,n-1) & f_p(0,n) \\ f_p(1,0) & f_p(1,1) & \cdots & f_p(1,n-1) & f_p(1,n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_p(n,0) & f_p(n,1) & \cdots & f_p(n,n-1) & f_p(n,n) \end{pmatrix}$$

by $(f_p(i, j))_{0 \leq i,j \leq n}$. Because

$$\det G_p(n) = (p - 1)p^{n-1} \det G_p(n - 1),$$

by induction on $n$ one easily obtains

$$\det G_p(n) = (p-1)^n p^{\frac{n(n-1)}{2}}, \text{ for any } \quad n \geq 1.$$

An $L_p(n)U_p(n)$ decomposition of a matrix $G_p(n)$ is the product of lower triangular matrix and an upper triangular matrix that is equal to $G_p(n)$.

**Example 3.1.**

$$\begin{aligned} G_p(3) &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3+p & 4+2p & 5+3p \\ 3 & 4+2p & 5+3p+p^2 & 6+4p+2p^2 \\ 4 & 5+3p & 6+4p+2p^2 & 7+5p+3p^2+p^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & p-1 & 2(p-1) & 3(p-1) \\ 0 & 0 & p(p-1) & 2p(p-1) \\ 0 & 0 & 0 & (p-1)p^2 \end{pmatrix}. \end{aligned}$$

**Theorem 3.2.** *Given the Goursat matrix of order $n+1$,*

$$(f_p(i,j))_{0 \leq i,j \leq n},$$

*where $G_p(n) = L_p(n)U_p(n)$. Then the triangular factors are*

$$L_p(n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & \cdots & 0 & 0 \\ 3 & 2 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n+1 & n & n-1 & \cdots & 2 & 1 \end{pmatrix}$$

*and*

$$U_p(n) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n & n+1 \\ 0 & p-1 & 2(p-1) & \cdots & (n-1)(p-1) & n(p-1) \\ 0 & 0 & (p-1)p & \cdots & (n-2)(p-1)p & (n-1)(p-1)p \\ 0 & 0 & 0 & \cdots & (n-3)(p-1)p^2 & (n-2)(p-1)p^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (p-1)p^{n-1} & 2(p-1)p^{n-1} \\ 0 & 0 & 0 & \cdots & 0 & (p-1)p^n \end{pmatrix}$$

*Proof.* It is convenient to use the following notation for the general square matrix $A$ of order $k$ (see [6])

$$\begin{aligned} A &= (a_1 b_2 c_3 \cdots v_{k-1} w_k) \\ &= \begin{pmatrix} a_1 & b_1 & c_1 & \cdots & v_1 & w_1 \\ a_2 & b_2 & c_2 & \cdots & v_2 & w_2 \\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ a_k & b_k & c_k & \cdots & v_k & w_k \end{pmatrix} \end{aligned}$$

We denote the determinant of this matrix by $|a_1 b_2 c_3 \cdots w_k|$ . Thus

$$|b_1 d_3 h_4| = \det \begin{pmatrix} b_1 & d_1 & h_1 \\ b_3 & d_3 & h_3 \\ b_4 & d_4 & h_4 \end{pmatrix}.$$

It is known (Turnbull [27], p.369) that for the usual triangular decomposition $A = LU$, the elements of $L$ and $U$ can be expressed in terms of determinants involving the elements of $A$ as follows (it is assumed that the decomposition is possible and $L$ has been chosen to have units in the principal diagonal) :

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{|a_2|}{|a_1|} & 1 & 0 & \cdots & & \\ \frac{|a_3|}{|a_1|} & \frac{|a_1 b_3|}{|a_1 b_2|} & 1 & \cdots & & \\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \frac{|a_m|}{|a_1|} & \frac{|a_1 b_m|}{|a_1 b_2|} & \frac{|a_2 b_2 c_m|}{|a_1 b_2 c_3|} & \cdots & 1 & 0 \\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \frac{|a_k|}{|a_1|} & \frac{|a_1 b_k|}{|a_1 b_2|} & \frac{|a_1 b_2 c_k|}{|a_1 b_2 c_3|} & \cdots & \frac{|a_1 b_2 c_3 \cdots v_k|}{|a_1 b_2 c_3 \cdots v_{k-1}|} & 1 \end{pmatrix}$$

and

$$U = \begin{pmatrix} |a_1| & |b_1| & |c_1| & \cdots & |v_1| & |w_1| \\ 0 & \frac{|a_1 b_2|}{|a_1|} & \frac{|a_1 c_2|}{|a_1|} & \cdots & \frac{|a_1 v_2|}{|a_1|} & \frac{|a_1 w_2|}{|a_1|} \\ 0 & 0 & \frac{|a_1 b_2 c_3|}{|a_1 b_2|} & \cdots & \frac{|a_1 b_3 v_3|}{|a_1 b_2|} & \frac{|a_1 b_2 w_3|}{|a_1 b_2|} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{|a_1 b_2 \cdots u_{k-2} v_{k-1}|}{|a_1 b_2 \cdots u_{k-1}|} & \frac{|a_1 b_2 \cdots u_{k-2} w_{k-1}|}{|a_1 b_2 \cdots v_{k-1}|} \\ 0 & 0 & 0 & \cdots & 0 & \frac{|a_1 b_2 \cdots v_{k-1} w_k|}{|a_1 b_2 \cdots v_{k-1}|} \end{pmatrix}.$$

□

**Remark 3.3.** *$L_p(n) = L'^2$ with $L' = (l_{i,j})_{0 \le i,j \le n}$ the lower triangular matrices such that $l_{i,j} = 1$. And*

$$G_p(n) = L_p(n) D_p(n) L_p(n)^T,$$

*where $D_p(n) = diag\{1, p-1, p^2(p-1), p^3(p-1), \cdots, p^n(p-1)\}$. $L_p(2)$ is factorized into 1-lower banded matrices, $L_p(2) = L^{(0)} L^{(1)} L^{(2)}$ where*

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

It does not seem to have been noticed that $L_p^{-1}(n)$ and $U_p^{-1}(n)$ can similarly be expressed explicitly as follows :

$$L_p^{-1}(3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}, \quad U_p^{-1}(3) = \begin{pmatrix} 1 & \frac{-2}{p-1} & \frac{1}{(p-1)p} & 0 \\ 0 & \frac{1}{p-1} & \frac{-2}{(p-1)p} & \frac{1}{(p-1)p^2} \\ 0 & 0 & \frac{1}{p(p-1)} & \frac{-2}{(p-1)p^2} \\ 0 & 0 & 0 & \frac{1}{(p-1)p^2} \end{pmatrix}.$$

**Theorem 3.4.** *Given the Goursat matrix of order $n+1$,*

$$G_p(n) = (f_p(i,j))_{0 \le i,j \le n},$$

*where $G_p(n) = L_p(n)U_p(n)$. Then*

$$L_p^{-1}(n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{pmatrix}$$

*and*

$$U_p^{-1}(n) = \begin{pmatrix} 1 & \frac{-2}{p-1} & \frac{1}{(p-1)p} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{p-1} & \frac{-2}{(p-1)p} & \frac{1}{(p-1)p^2} & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{p(p-1)} & \frac{-2}{(p-1)p^2} & \cdots & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & 0 & 0 & \cdots & \frac{-2}{(p-1)p^{n-2}} & \frac{1}{(p-1)p^{n-1}} \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{(p-1)p^{n-2}} & \frac{-2}{(p-1)p^{n-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{(p-1)p^{n-1}} \end{pmatrix}.$$

*Proof.* Compute

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & & \\ -\frac{|a_2|}{|a_1|} & 1 & & & \\ \frac{|a_2 b_3|}{|a_1 b_2|} & -\frac{|a_1 b_3|}{|a_1 b_2|} & 1 & & \\ -\frac{|a_2 b_3 c_4|}{|a_1 b_2 c_3|} & \frac{|a_1 b_3 c_4|}{|a_1 b_2 c_3|} & -\frac{|a_1 b_2 c_4|}{|a_1 b_2 c_3|} & 1 & \\ \multicolumn{5}{c}{\dotfill} \\ (-1)^{k-1}\frac{|a_2 b_3 \cdots v_k|}{|a_1 b_2 \cdots v_{k-1}|} & (-1)^k \frac{|a_1 b_3 \cdots v_k|}{|a_1 b_2 \cdots v_{k-1}|} & (-1)^{k-1}\frac{|a_1 b_2 \cdots v_k|}{|a_1 b_2 \cdots v_{k-1}|} & \cdots & 1 \end{pmatrix}$$

and

$$U^{-1} = \begin{pmatrix} \frac{|1|}{|a_1|} & -\frac{|b_1|}{|a_1b_2|} & \frac{|b_1c_2|}{|a_1b_2c_3|} & -\frac{|b_1c_2d_3|}{|a_1b_2c_3d_4|} & \cdots & (-1)^{k-1}\frac{|b_1c_2d_3\cdots w_{k-1}|}{|a_1b_2c_3\cdots w_k|} \\ 0 & \frac{|a_1|}{|a_1b_1|} & -\frac{|a_1c_2|}{|a_1b_2c_3|} & \frac{|a_1c_2d_3|}{|a_1b_2c_3d_4|} & \cdots & (-1)^{k}\frac{|a_1c_2d_3\cdots w_{k-1}|}{|a_1b_2c_3\cdots w_k|} \\ 0 & 0 & \frac{|a_1b_2|}{|a_1b_2c_3|} & -\frac{|a_1b_2d_3|}{|a_1b_2c_3d_4|} & \cdots & (-1)^{k-1}\frac{|a_1b_2d_3\cdots w_{k-1}|}{|a_1b_2c_3\cdots w_k|} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{|a_1b_2\cdots v_{k-1}|}{|a_1b_2\cdots w_k|} \end{pmatrix}.$$

□

In some problems one wishes to decompose a matrix into a product of the form $UL$ rather than in the more usual form $LU$ .

**Example 3.5.** Write $G_p(n) = U_p(n)L_p(n), n = 1, 2$, then

$$G_p(1) = \begin{pmatrix} \frac{p-1}{p+3} & 2 \\ 0 & 3+p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2}{3+p} & 1 \end{pmatrix}.$$

$$U_p(2) = \begin{pmatrix} \frac{(p-1)^2 p}{-1-2p+2p^2+p^3} & \frac{2(p^2-1)}{5+3p+p^2} & 3 \\ 0 & \frac{-1-2p+2p^2+p^3}{5+3p+p^2} & 4+2p \\ 0 & 0 & 5+3p+p^2 \end{pmatrix}$$

and

$$L_p(2) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2(p^2-1)}{-1-2p-2p^2+p^3} & 1 & 0 \\ \frac{3}{5+3p+p^2} & \frac{4+2p}{5+3p+p^2} & 1 \end{pmatrix}.$$

In the following let us denote by $h_p(i, j)$ the number $f_p(i, j) - i - j$. Let $n$ be a fixed positive integer and $H_p(n)$ be the matrix $(h_p(i, j))_{0 \le i,j \le n}$ . Then $H_p(n)$ induces a quadratic form

$$\forall x = (x_0, \cdots, x_n) \in \mathbb{R}^{n+1}, \ \ q(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} h_p(i, j) x_i x_j.$$

Because

$$\det H_p(n) = p^n \det G_p(n-1)$$

by induction on $n$ one easily obtains

$$\det H_p(n) = (p-1)^{n-1} p^{\frac{n^2-n+2}{2}}, \quad \text{for any } n \geq 1.$$

Hence, we have proved the next corollary.

**Corollary 3.6.** *The quadratic form $q(x)$ induced by the matrix $H_p(n)$ is positive definite, and all eigenvalues of the matrix $H_p(n)$ are positive, for all $n \geq 1$ .*

**Remark 3.7.** $h_p(i,j)$ is the number of all chains of p-group $\mathbb{Z}_{p^i} \times \mathbb{Z}_{p^j}$ $(i \leq j)$, determined in Theorem 4.5 .

**Corollary 3.8.** *Given the matrix of order $n+1$, $(h_p(i,j))_{0 \leq i,j \leq n}$ where $H_p(n) = L_p(n)U_p(n)$. Then the triangular factors are*

$$L_p(n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & n & n-1 & \cdots & 2 & 1 \end{pmatrix}$$

*and*

$$U_p(n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & p & 2p & \cdots & (n-1)p & np \\ 0 & 0 & (p-1)p & \cdots & (n-2)(p-1)p & (n-1)(p-1)p \\ 0 & 0 & 0 & \cdots & (n-3)(p-1)p^2 & (n-2)(p-1)p^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (p-1)p^{n-2} & 2(p-1)p^{n-2} \\ 0 & 0 & 0 & \cdots & 0 & (p-1)p^{n-1} \end{pmatrix}.$$

**Corollary 3.9.** *Given the matrix of order $n+1$, $(h_p(i,j))_{0 \leq i,j \leq n}$ where $H_p(n) =$*

$L_p(n)U_p(n)$. *Then*

$$L_p^{-1}(n) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{pmatrix}.$$

*and*

$$U_p^{-1}(n) = \begin{pmatrix} 1 & \frac{-1}{p} & \frac{1}{(p-1)p} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{p} & \frac{-2}{(p-1)p} & \frac{1}{(p-1)p^2} & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{p(p-1)} & \frac{-2}{(p-1)p^2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-2}{(p-1)p^{n-2}} & \frac{1}{(p-1)p^{n-1}} \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{(p-1)p^{n-2}} & \frac{-2}{(p-1)p^{n-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{(p-1)p^{n-1}} \end{pmatrix}.$$

## 4 Containment of Subgroups of a Direct Product $\mathbb{Z}_m \times \mathbb{Z}_n$

Throughout the paper we use the following notation: $\mathbb{N} := \{1, 2, \cdots\}$, $\mathbb{P}$ is the set of primes; the prime power factorization of $n \in \mathbb{N}$ is $n = \prod_{p\in\mathbb{P}} p^{\nu_p(n)}$, where all but a finite number of the exponents $\nu_p(n)$ are zero; $\mathbb{Z}_n$ denotes the additive group of residue classes modulo $n$ ($n \in \mathbb{N}$); $a \wedge b$ and $a \vee b$ denote the greatest common divisor and least common multiple, respectively of $a, b$. For every $m, n \in \mathbb{N}$, let

$$J_{m,n} := \Big\{(a, b, c, d, t) \in \mathbb{N}^5 : a|m, b|a, c|n, d|c, \frac{a}{b} = \frac{c}{d}, t \le \frac{a}{b}, t \wedge \frac{a}{b} = 1\Big\}.$$

Using the condition

$$\frac{a}{b} = \frac{c}{d}.$$

Note that

$$(b \wedge d) \cdot (a \vee c) = ad, \text{ and } b \wedge d | a \vee c.$$

For $k = 1, 2$ and $(a_k, b_k, c_k, d_k, t_k) \in J_{m,n}$, define

$$U_k = U_{a_k,b_k,c_k,d_k,t_k} = \Big\{ \Big( \frac{i_k m}{a_k} \; , \; i_k t_k \frac{n}{c_k} + j_k \frac{n}{d_k} \Big) : 0 \le i_k \le a_k, 0 \le j_k \le d_k \Big\}.$$

Then $|U_k| = a_k d_k$ and $\mathbb{Z}_{b_k \wedge d_k} \times \mathbb{Z}_{a_k \vee c_k} \cong U_k \le \mathbb{Z}_m \times \mathbb{Z}_n$ .

**Theorem 4.1.** *Let* $m, n \in \mathbb{N}$ *and* $U_2, U_1 \le \mathbb{Z}_m \times \mathbb{Z}_n$. *Then* $U_2 \le U_1$ *if and only if*

(1) $a_2|a_1, \; b_2|b_1, \; c_2|c_1, \; d_2|d_1, \; a_2|b_1 \wedge d_1$ *and* $c_2|a_1 \vee c_1$*;*

(2) $\dfrac{b_1 \vee a_2}{b_1} = \dfrac{a_2}{a_2 \wedge b_1} = \dfrac{d_1 \vee c_2}{d_1} = \dfrac{c_2}{c_2 \wedge d_1}$ .

*Proof.* Let $k = 1, 2$ and $U_k = (\theta_k : A_k/B_k \to C_k/D_k)$ be a subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$. With the notations of Theorem 2.1, let $|A_k| = a_k, |B_k| = b_k, |C_k| = c_k, |D_k| = d_k$, where $a_k|m, b_k|a_k$ , $c_k|n, d_k|c_k$. Thus $U_2 \le U_1$ if and only if we verify the following 2 steps of Theorem 2.3.

**Step** 1. $A_2 \le A_1$, $B_2 \le B_1$, and $C_2 \le C_1$, $D_2 \le D_1$. Hence $a_2|a_1, \; b_2|b_1, \; c_2|c_1$ $d_2|d_1$. And $U_2 \le U_1 \cong \mathbb{Z}_{b_1 \wedge d_1} \times \mathbb{Z}_{a_1 \vee c_1}$. Hence $a_2|b_1 \wedge d_1$ and $c_2|a_1 \vee c_1$.

**Step** 2. Writing explicitly the corresponding subgroups and quotient groups we deduce : for $0 \le i_k \le \frac{a_k}{b_k} - 1, \; 1 \le t_k \le \frac{a_k}{b_k}, t_k \wedge \frac{a_k}{b_k} = 1$

$$\theta_k \Big( \frac{i_k m}{a_k} + B_k \Big) \quad = \quad i_k t_k \frac{n}{c_k} + D_k,$$

and

$$\begin{aligned} A_2 = \langle m/a_2 \rangle \quad &= \quad \Big\{ 0 \, , \, \frac{m}{a_2} \, , \, 2\frac{m}{a_2}, \cdots, (a_2 - 1)\frac{m}{a_2} \Big\} \le \mathbb{Z}_m, \\ B_1 = \langle m/b_1 \rangle \quad &= \quad \Big\{ 0 \, , \, \frac{m}{b_1} \, , \, 2\frac{m}{b_1}, \cdots, (b_1 - 1)\frac{m}{b_1} \Big\} \le \mathbb{Z}_m, \\ A_2 \cap B_1 \quad &= \quad \Big\{ \frac{im}{\delta}, 0 \le i \le \delta - 1, \delta = a_2 \wedge b_1 \Big\}, \\ \frac{A_2 + B_1}{B_1} \quad &= \quad \Big\{ \frac{im}{\lambda} + B_1, 0 \le i \le \lambda - 1, \lambda = a_2 \vee b_1 \Big\}. \end{aligned}$$

And similarly

$$
\begin{aligned}
C_2 = \langle n/c_2\rangle &= \left\{0\,,\ \frac{n}{c_2}\,,\ 2\frac{n}{c_2}, \cdots, (c_2-1)\frac{n}{c_2}\right\} \leq \mathbb{Z}_n\,,\\
D_1 = \langle n/d_1\rangle &= \left\{0\,,\ \frac{n}{d_1}\,,\ 2\frac{n}{d_1}, \cdots, (d_1-1)\frac{n}{d_1}\right\} \leq \mathbb{Z}_n\,,\\
C_2 \cap D_1 &= \left\{\frac{in}{\mu}, 0 \leq i \leq \mu-1, \mu = c_2 \wedge d_1\right\},\\
\frac{C_2+D_1}{D_1} &= \left\{\frac{in}{\nu} + D_1, 0 \leq i \leq \nu-1, \nu = c_2 \vee d_1\right\}.
\end{aligned}
$$

Since $\left|\frac{A_2+B_1}{B_1}\right| = \left|\frac{C_2+D_1}{D_1}\right| = \left|\frac{A_2}{A_2\cap B_1}\right| = \left|\frac{C_2}{C_2\cap D_1}\right|$, we deduce:

$$\frac{b_1 \vee a_2}{b_1} = \frac{a_2}{a_2 \wedge b_1} = \frac{d_1 \vee c_2}{d_1} = \frac{c_2}{c_2 \wedge d_1}\ .$$

□

**Corollary 4.2.** *Let $m, n \in \mathbb{N}$ and $U_2, U_1 \leq \mathbb{Z}_m \times \mathbb{Z}_n$. Then*

$$U_2 \cong U_1 \Leftrightarrow b_1 \wedge d_1 = b_2 \wedge d_2 \ \ and\ \ a_1 \vee c_1 = a_2 \vee c_2\ .$$

*Proof.*

$$U_1 \cong \mathbb{Z}_{b_1\wedge d_1} \times \mathbb{Z}_{a_1 \vee c_1} \cong U_2 \cong \mathbb{Z}_{b_2\wedge d_2} \times \mathbb{Z}_{a_2\vee c_2}.$$

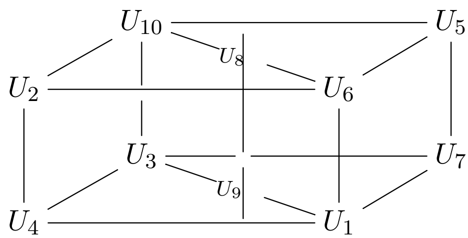
□

**Definition 4.3.** Let $G$ be a direct product of $\mathbb{Z}_m \times \mathbb{Z}_n$. Given a chain of subgroups of $G$ of the form

$$x_k \quad : 1 = U_1 \leq U_2 \leq U_3 \leq \cdots \leq U_k = G.$$

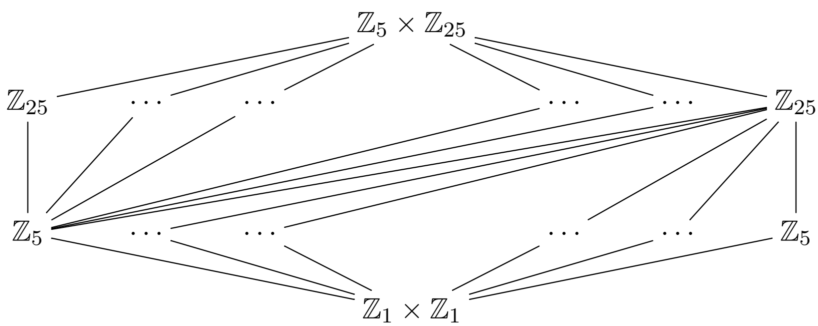
We say that $k$ is the length of the chain and $x_k$ the chain. Let $k = \varepsilon(m,n)$, and $\omega(m,n) := |\{x_{k_1}, x_{k_2}, \cdots\}|$ the number of the chain.

**Example 4.4.** Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ and let $U_1 = 1$, $U_2 = \{(0,0),(1,0),(0,2),(1,2),(0,4),(1,4)\}$, $U_3 = \{(0,0),(1,0),(0,3),(1,3)\}$, $U_4 = \{(0,0),(1,0)\}$, $U_5 = \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5)\}$, $U_6 = \{(0,0),(0,2),(0,4)\}$, $U_7 = \{(0,0),(0,3)\}$, $U_8 = \{(0,0),(0,2),(0,4),(1,1),(1,3),(1,5)\}$, $U_9 = \{(0,0),(1,3)\}$, $U_{10} = \mathbb{Z}_2 \times \mathbb{Z}_6$. See Figure for the subgroup lattice of $G$.



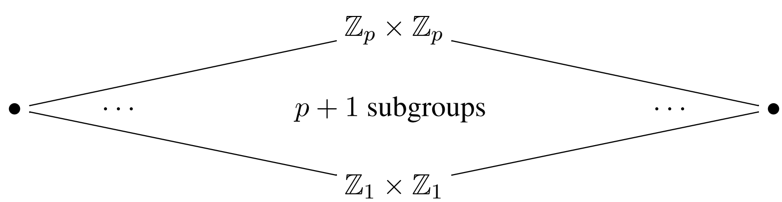

$\varepsilon(2,6) = 4, \omega(2,6) = 8.$

Consider $G = \mathbb{Z}_5 \times \mathbb{Z}_{25}$ . See Figure for the subgroup lattice of $G$.




Then $\varepsilon(5,25) = 4, \omega(5,25) = 11.$

By Goursat's Theorem, that means that there are $p+3$ subgroups of the direct product of $\mathbb{Z}_p \times \mathbb{Z}_p$. As it turns out, it is fairly straight forward to calculate $\omega(p,p) = p+1$. The general subgroup lattice $\mathbb{Z}_p \times \mathbb{Z}_p$




The following is a subgroup lattice of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$. Let $1 = U_{1,1,1,1,1}$ and $U_{p^2,p^2,p^2,p^2,1} =$

$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ one has

$$\begin{array}{lllll} 1 \subseteq & U_{p,p,1,1,1} \subseteq & U_{p^2,p^2,1,1,1} \subseteq & U_{p^2,p^2,p,p,1} \subseteq & U_{p^2,p^2,p^2,p^2,1} \\ 1 \subseteq & U_{p,p,1,1,1} \subseteq & U_{p^2,p,p,1,1} \subseteq & U_{p^2,p^2,p,p,1} \subseteq & U_{p^2,p^2,p^2,p^2,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 \subseteq & U_{p,1,p,1,1} \subseteq & U_{p^2,1,p^2,1,1} \subseteq & U_{p^2,p,p^2,p,1} \subseteq & U_{p^2,p^2,p^2,p^2,1} \end{array}$$

**Theorem 4.5.** *Let* $m, n, i, j \in \mathbb{N}$ *with* $i \leq j$. *Then*

(1) $\omega(p^i, p^j) = h_p(i, j) = f_p(i, j) - i - j$,

(2) $\varepsilon(m, n) = 1 + \sum_{p \in \mathbb{P}} \nu_p(mn)$.

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