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On Subclasses of Bi-Univalent Functions Related to Quasi-Subordination

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Abstract

In this paper, we introduce and investigate two subclasses $\mathcal{M}_{\Sigma}^{q}(\lambda,\gamma,h)$ and $\mathcal{H}_{\Sigma}^{q}(\eta,\delta,h)$ of bi-univalent functions defined by quasi-subordination. We find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.

1. Introduction

Let $\mathcal{A}$ denote the analytic function class in the open unit disc $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$ which contains the shape

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (z \in U). \tag{1.1}$$

“and let S be the subclass of all univalent functions from $\mathcal{A}$ in $\mathcal{U}$. The Koebe One Quarter Theorem [5] states that the image of $\mathcal{U}$ beneath every function f from S contains a radius disk of $\frac{1}{4}$. This univalent function, therefore, has an inverse one f^{-1} which satisfies”

$$"f^{-1}\left(f(z)\right) = z, \ (z \in U) \ \text{and} \ f(f^{-1}(w)) = w, \ \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)"$$

where

$$"f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 ." \tag{1.2}$$

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A function $f \in \mathcal{A}$ "is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disc $\mathcal{U}$".

The notion of quasi-subordination is studied in 1970 by Robertson [2].

For two "analytic functions f and h, the function" f is "quasi-subordination to $h$written" as "$f(z) \prec_q h(z), z \in \mathcal{U}$" in the event of an analytical function $\mathcal{V}$ and $\mathcal{W}$ with $|\mathcal{V}(z)| \le 1, \mathcal{W}(0) = 0$ and $|\mathcal{W}(z)| < 1$ such that

$$\frac{f(z)}{\mathcal{V}(z)} \prec h(z), \qquad \equiv \quad f(z) = \mathcal{V}(z)h(\mathcal{W}(z)),\ \ z \in \mathcal{U}.$$

"Note that if $\mathcal{V}(z) = 1$, then $f(z) = h(\mathcal{W}(z))$, so that $f(z) \prec h(z)$ also if $\mathcal{W}(z) = z$ then $f(z) = \mathcal{V}(z)h(z)$ and it is said that $f(z)$ is majorized by $h(z)$ and written as $f(z) \ll h(z)$ in "$\mathcal{U}$". Hence it is perceptible that the quasi-subordination is a popularization of the usual subordination as well as majorization". "The labor on quasi-subordination is very extensive and that includes some recent investigations" [4, 6, 7, 8-18, 20].

Let $h(z)$ "be analytic in $\mathcal{U}$" with $h(0) = 1$ and

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots, \qquad (B_1 \in \mathbb{R}^+) \tag{1.3}$$

$$\mathcal{V}(z) = A_0 + A_1 z + A_2 z^2 + \cdots, \qquad (|V(z)| \le 1,\ z \in U). \tag{1.4}$$

Ramadan and Darus [9] defined the following differential operator:

$$D^0_{\xi,\sigma,\theta,\tau} f(z) = f(z),$$

$$D^1_{\xi,\sigma,\theta,\tau} f(z) = [(1 - (\theta - \tau)(\sigma - \xi)]f(z) + [(\theta - \tau)(\sigma - \xi)]z\, f'(z),$$

$$D^\delta_{\xi,\sigma,\theta,\tau} f(z) = z + \sum_{n=2}^{\infty} [(n-1)(\theta - \tau)(\sigma - \xi) + 1]^\delta a_n z^n,$$

where $f(z) \in S,\ \xi, \sigma, \theta, \tau \ge 0,\ \theta > \tau, \sigma > \xi,\ \delta = 0,1,2,3\ldots$

Definition 1.1. If $f \in \Sigma$, then $f \in \mathcal{M}^q_\Sigma(\lambda, \gamma, h)(0 \le \gamma \le 1, \lambda \ge 0, \delta = 0,1,2,3\ldots)$ if the following quasi-subordination hold:

$$\left(\lambda\gamma\left(\frac{z(D^\delta_{\xi,\sigma,\theta,\tau} f(z))''}{(D^\delta_{\xi,\sigma,\theta,\tau} f(z))'} - 2\right) + (\gamma(\lambda+1) + \lambda)\frac{z(D^\delta_{\xi,\sigma,\theta,\tau} f(z))'}{D^\delta_{\xi,\sigma,\theta,\tau} f(z)} + (1-\lambda)(1-\gamma)\frac{D^\delta_{\xi,\sigma,\theta,\tau} f(z)}{z}\right) - 1 \prec_q (h(z) - 1),$$

$$\left(\lambda\gamma\left(\frac{w(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))''}{(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))'}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{w(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))'}{D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}+(1-\lambda)(1-\gamma)\frac{D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)$$
$$-1 \prec_q (h(z)-1),$$

where $g = f^{-1}$.

Definition 1.2. If $f \in \Sigma$, then $f \in \mathcal{H}^q_\Sigma(\eta,\delta,h)$ "if the following quasi-subordination hold"

$$(\eta \geq 1, \delta = 0,1,2,3 \dots)$$

$$\left[z\left(\frac{(1-\eta)\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)'+z\left(\eta(\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}f(z)\right)''-1\right] \prec_q (h(z)-1),$$

$$\left[w\left(\frac{(1-\eta)\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)'+w\left(\eta(\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}g(w)\right)''-1\right] \prec_q (h(w)-1),$$

where $g = f^{-1}$.

Lemma 1.3 [3]. *"If $p \in \mathcal{P}$, then $|p_i| \leq 2$ for each i, where $\mathcal{P}$ is the family of all functions", analytic in U, for which $Re(p(z)) > 0$, where $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, for $z \in U$".*

2. Main Results

Theorem 2.1. *If $f \in \Sigma$ and $f \in \mathcal{M}^q_\Sigma(\lambda,\gamma,h)$ $(0 \leq \gamma \leq 1, \lambda \geq 0)$, then*

$$|a_2| \leq \min\left\{\frac{|A_0|B_1}{|4\lambda\gamma+1|([(\theta-\tau)(\sigma-\xi)+1)]^{\delta})^2|}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{|4\lambda\gamma+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}}\right\}, \quad (2.1)$$

$$|a_3| \leq \min\left\{\begin{array}{l}\frac{[|A_1|+|A_0|]B_1}{|9\gamma\lambda+(\gamma+\lambda)+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\frac{|A_0|^2B_1^2}{|4\gamma\lambda+1|^2([(\theta-\tau)(\sigma-\xi)+1)]^{\delta})^2} \\ \frac{[|A_1|+|A_0|]B_1}{|9\gamma\lambda+(\gamma+\lambda)+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\frac{|A_0|(B_1+|B_2-B_1|)}{|4\gamma\lambda+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}\end{array}\right\}. \quad (2.2)$$

Proof. Let $f \in \mathcal{M}^q_\Sigma(\lambda,\gamma,h)$, there exist the "Schwarz functions" $\mathcal{K}(z), S(w)$ with

$$\mathcal{K}(z) = c_1 z + \sum_{j=2}^{\infty} c_j z^j, \qquad S(w) = d_1 z + \sum_{j=2}^{\infty} d_j w^j$$

$\mathcal{K}(0) = S(0) = 0$ and $|\mathcal{K}(z)| < 1, |S(w)| < 1$ and an analytic function $\mathcal{V}(z)$ such that

$$\left(\lambda\gamma\left(\frac{z(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))''}{(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))'}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{z(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))'}{D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}+(1-\lambda)(1-\gamma)\frac{D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)-1$$

$$=\mathcal{V}(z)\left(h\left(\mathcal{K}(z)\right)-1\right), \tag{2.3}$$

$$\left(\lambda\gamma\left(\frac{w(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))''}{(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))'}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{w(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))'}{D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}+(1-\lambda)(1-\gamma)\frac{D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)-1$$

$$=\mathcal{V}(w)\left(h\left(S(w)\right)-1\right). \tag{2.4}$$

Define the functions $\mathcal{P}(z)$, $q(w)$

$$\mathcal{P}(z)=\frac{1+\mathcal{K}(z)}{1-\mathcal{K}(z)}=1+c_1z+c_2z^2+\cdots \tag{2.5}$$

$$q(w)=\frac{1+S(w)}{1-S(w)}=1+d_1w+d_2w^2+\cdots \tag{2.6}$$

or equivalently

$$\mathcal{K}(z)=\frac{\mathcal{P}(z)-1}{\mathcal{P}(z)+1}=\frac{1}{2}\left[c_1z+(c_2-\frac{c_1}{2})z^2+\cdots\right], \tag{2.7}$$

$$S(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left[d_1w+\left(d_2-\frac{d_1}{2}\right)w^2+\cdots\right]. \tag{2.8}$$

"It is clear that $\mathcal{P}(z), q(w)$ are analytic and have positive real parts in U. In view of" (2.3), (2.4), (2.7) and (2.8) clearly

$$\left(\lambda\gamma\left(\frac{z(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))''}{(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))'}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{z(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))'}{D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}+(1-\lambda)(1-\gamma)\frac{D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)-1$$

$$=\mathcal{V}(z)\left(h\left(\frac{\mathcal{P}(z)-1}{\mathcal{P}(z)+1}\right)-1\right), \tag{2.9}$$

$$\left(\lambda\gamma\left(\frac{w(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))''}{(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))'}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{w(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))'}{D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}+(1-\lambda)(1-\gamma)\frac{D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)-1$$

$$=\mathcal{V}(w)\left(h\left(\frac{q(w)-1}{q(w)+1}\right)-1\right). \tag{2.10}$$

where $f(z)$ and $g(w)$ as given in (1.1) and (1.2) respectively.

$$\left(\lambda\gamma\left(\frac{z(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))''}{(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))'}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{z(D^{\delta}_{\xi,\sigma,\theta,\tau}f(z))'}{D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}+(1-\lambda)(1-\gamma)\frac{D^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)-1$$

$$=(4\gamma\lambda+1)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_2z$$

$$+\left[(9\gamma\lambda+(\gamma+\lambda)+1)a_3-\big(5\gamma\lambda+(\gamma+\lambda)\big)a_2^2\right][2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}z^2+\cdots, \quad (2.11)$$

$$\left(\lambda\gamma\left(\frac{w(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))''}{(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))'}-2\right)+(\gamma(\lambda+1)+\lambda)\frac{w(D^{\delta}_{\xi,\sigma,\theta,\tau}g(w))'}{D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}+(1-\lambda)(1-\gamma)\frac{D^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)-1$$

$$=-(4\gamma\lambda+1)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_2w$$

$$+[(13\gamma\lambda+(\gamma+\lambda)+2)a_2^2-(9\gamma\lambda+(\gamma+\lambda)+1)a_3][2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}w^2-\cdots. \quad (2.12)$$

Using (2.5) and (2.6) together with (1.3) and (1.4)

$$\mathcal{V}(z)\left(h\left(\frac{\mathcal{P}(z)-1}{\mathcal{P}(z)+1}\right)-1\right)=\frac{1}{2}A_0B_1c_1z+\left[\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1(c_2+\frac{c_1^2}{2})+\frac{A_0B_2c_1^2}{4}\right]z^2+\cdots, \quad (2.13)$$

$$\mathcal{V}(w)\left(h\left(\frac{q(w)-1}{q(w)+1}\right)-1\right)=\frac{1}{2}A_0B_1d_1w+\left[\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1(d_2+\frac{d_1^2}{2})+\frac{A_0B_2d_1^2}{4}\right]w^2+\cdots. \quad (2.14)$$

From (2.9) we get (2.11) = (2.13)

$$(4\gamma\lambda+1)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_2=\frac{1}{2}A_0B_1c_1, \quad (2.15)$$

$$\left[(9\gamma\lambda+(\gamma+\lambda)+1)a_3-\big(5\gamma\lambda+(\gamma+\lambda)\big)a_2^2\right][2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}$$

$$=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1(c_2+\frac{c_1^2}{2})+\frac{A_0B_2c_1^2}{4}. \quad (2.16)$$

Similarly, (2.10) we get (2.12) = (2.14)

$$-(4\gamma\lambda+1)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_2=\frac{1}{2}A_0B_1d_1, \quad (2.17)$$

$$[(13\gamma\lambda+(\gamma+\lambda)+2)a_2^2-(9\gamma\lambda+(\gamma+\lambda)+1)a_3][2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}$$

$$=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1(d_2+\frac{d_1^2}{2})+\frac{A_0B_2d_1^2}{4}. \quad (2.18)$$

From (2.15) and (2.17), we find

$$c_1=-d_1, \quad (2.19)$$

$$a^2 = \frac{A_0^2 B_1^2 (c_1^2 + d_1^2)}{8(4\gamma\lambda + 1)^2([(\theta - \tau)(\sigma - \xi) + 1)]^\delta)^2}. \tag{2.20}$$

Adding (2.16), (2.18) we get

$$a_2^2 = \frac{2A_0B_1(c_2 + d_2) + A_0(B_2 - B_1)(\, c_1^2 + d_1^2)}{8(4\gamma\lambda + 1)[2(\theta - \tau)(\sigma - \xi) + 1)]^\delta}. \tag{2.21}$$

Lemma 1.3 is applied for $c_1, c_2\, d_1$ and d_2 follows from (2.20),(2.21) we get

$$|a_2| \leq \frac{|A_0|B_1}{|4\lambda\gamma + 1||([(\theta - \tau)(\sigma - \xi) + 1)]^\delta)^2|},$$

$$|a_2| \leq \sqrt{\frac{|A_0|(B_1 + |B_2 - B_1|)}{|\, 4\lambda\gamma + 1||[2(\theta - \tau)(\sigma - \xi) + 1)]^\delta|}},$$

$$|a_2| \leq \min\left\{\frac{|A_0|B_1}{|4\lambda\gamma + 1||([(\theta - \tau)(\sigma - \xi) + 1)]^\delta)^2|}, \sqrt{\frac{|A_0|(B_1 + |B_2 - B_1|)}{|\, 4\lambda\gamma + 1||[2(\theta - \tau)(\sigma - \xi) + 1)]^\delta|}}\right\}. \tag{2.22}$$

That provided $|a_2|$ as showed (2.1).

New further computations (2.16) to (2.18) lead to

$$a_3 = \frac{4A_1B_1c_1 + 2A_0B_1(c_2 - d_2)}{8(9\gamma\lambda + (\gamma + \lambda) + 1)[2(\theta - \tau)(\sigma - \xi) + 1)]^\delta} + a_2^2.$$

Upon substituting the value of a_2^2 from (2.20), (2.21) and Lemma 1.3 is applied for $c_1, c_2\, d_1$ and d_2, we get

$$|a_3| \leq \frac{[|A_1| + |A_0|]B_1}{|9\gamma\lambda + (\gamma + \lambda) + 1||[2(\theta - \tau)(\sigma - \xi) + 1)]^\delta|} + \frac{|A_0|^2 B_1^2}{|4\gamma\lambda + 1|^2([(\theta - \tau)(\sigma - \xi) + 1)]^\delta)^2},$$

$$|a_3| \leq \frac{[|A_1| + |A_0|]B_1}{|9\gamma\lambda + (\gamma + \lambda) + 1||[2(\theta - \tau)(\sigma - \xi) + 1)]^\delta|} + \frac{|A_0|(B_1 + |B_2 - B_1|)}{|4\gamma\lambda + 1||[2(\theta - \tau)(\sigma - \xi) + 1)]^\delta|},$$

$$|a_3| \le \min\left\{\begin{array}{l}\dfrac{[|A_1|+|A_0|]B_1}{|9\gamma\lambda+(\gamma+\lambda)+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\dfrac{|A_0|^2B_1^2}{|4\gamma\lambda+1|^2([(\theta-\tau)(\sigma-\xi)+1)]^{\delta})^2} \\ \dfrac{[|A_1|+|A_0|]B_1}{|9\gamma\lambda+(\gamma+\lambda)+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\dfrac{|A_0|(B_1+|B_2-B_1|)}{|4\gamma\lambda+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}\end{array}\right\}. \quad (2.23)$$

That provided $|a_3|$ as showed (2.2).

If putting $\lambda = 0$ in Theorem 2.1, we get

Corollary 2.2. *Let* $f \in \mathcal{M}_{\Sigma}^{q}(0,\gamma,h)$. *Then*

$$|a_2| \le \min\left\{\frac{|A_0|B_1}{([(\theta-\tau)(\sigma-\xi)+1)]^{\delta})^2}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{|[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}}\right\},$$

$$|a_3| \le \min\left\{\begin{array}{l}\dfrac{[|A_1|+|A_0|]B_1}{|\gamma+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\dfrac{|A_0|^2B_1^2}{|([(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|^2} \\ \dfrac{[|A_1|+|A_0|]B_1}{|\gamma+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\dfrac{|A_0|(B_1+|B_2-B_1|)}{|[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}\end{array}\right\}.$$

If putting $\mathcal{V}(z) = 1$ in Theorem 2.1, we get

Corollary 2.3. *Let* $f \in \mathcal{M}_{\Sigma}^{q}(\lambda,\gamma,h)$. *Then*

$$|a_2| \le \min\left\{\frac{B_1}{|4\lambda\gamma+1||([(\theta-\tau)(\sigma-\xi)+1)]^{\delta})^2|}, \sqrt{\frac{(B_1+|B_2-B_1|)}{|4\lambda\gamma+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}}\right\},$$

$$|a_3| \le \min\left\{\begin{array}{l}\dfrac{B_1}{|9\lambda\gamma+(\gamma+\lambda)+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\dfrac{B_1^2}{|4\lambda\gamma+1|^2|([(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|^2}, \\ \dfrac{B_1}{|9\lambda\gamma+(\gamma+\lambda)+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\dfrac{(B_1+|B_2-B_1|)}{|4\lambda\gamma+1||[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}\end{array}\right\}.$$

Theorem 2.4. *If* $f \in \Sigma$ *and* $f \in \mathcal{H}_{\Sigma}^{q}(\eta,\delta,h)$, $(\eta \ge 1, \delta = 1,2,3 \dots)$, *then*

$$|a_2| \le \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{\left|2A_0B_1^2(1+2\eta)(2[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}-(B_2-B_1)\left((1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}\right)^2\right|}}, \quad (2.24)$$

$$|a_3| \le \frac{B_1[|A_1|+|A_0|]}{|2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\frac{|A_0|^2B_1^2}{\left((1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}\right)^2}. \quad (2.25)$$

Proof. Since $f \in \mathcal{H}_{\Sigma}^{q}(\eta,\delta,h)$ and $g = f^{-1}$, "there exist Schwarz functions" $\mathcal{K}(z), S(w)$ "and an analytic function $\mathcal{V}(z)$ such that"

$$\left[z\left(\frac{(1-\eta)\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)' + z\left(\eta(\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}f(z)\right)'' - 1\right] = \mathcal{V}(z)\left(h\left(\mathcal{K}(z)\right) - 1\right), \quad (2.26)$$

$$\left[w\left(\frac{(1-\eta)\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)' + w\left(\eta(\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}g(w)\right)'' - 1\right] = \mathcal{V}(w)\left(h\left(S(w)\right) - 1\right). \quad (2.27)$$

For $\mathcal{P}(z), q(w)$ as given in (2.5), (2.6) in view of (2.25), (2.26) clearly

$$\left[z\left(\frac{(1-\eta)\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)' + z\left(\eta(\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}f(z)\right)'' - 1\right] = \mathcal{V}(z)\left(h\left(\frac{\mathcal{P}(z)-1}{\mathcal{P}(z)+1}\right) - 1\right), \quad (2.28)$$

$$\left[w\left(\frac{(1-\eta)\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)' + w\left(\eta(\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}g(w)\right)'' - 1\right] = \mathcal{V}(w)\left(h\left(\frac{q(w)-1}{q(w)+1}\right) - 1\right). \quad (2.29)$$

Since

$$\left[z\left(\frac{(1-\eta)\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}f(z)}{z}\right)' + z\left(\eta(\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}f(z)\right)'' - 1\right]$$

$$= (1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_2 z$$

$$+2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_3 z^2 + \cdots, \quad (2.30)$$

$$\left[w\left(\frac{(1-\eta)\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}g(w)}{w}\right)' + w\left(\eta(\mathcal{D}^{\delta}_{\xi,\sigma,\theta,\tau}g(w)\right)'' - 1\right]$$

$$= -(1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_2 w$$

$$+2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}(2a_2^2 - a_{3)}w^2 - \cdots. \quad (2.31)$$

From (2.28) we get (2.30) = (2.13)

$$(1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_2 = \frac{1}{2}A_0 B_1 c_1, \quad (2.32)$$

$$2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_3 = \frac{1}{2}A_1 B_1 c_1 + \frac{1}{2}A_0 B_1(c_2 + \frac{c_1^2}{2}) + \frac{A_0 B_2 c_1^2}{4}. \quad (2.33)$$

Also from (2.29) we get $(2.31) = (2.14)$

$$-(1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}a_2 = \frac{1}{2}A_0 B_1 d_1, \quad (2.34)$$

$$2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}(2a_2^2-a_3)$$
$$=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1(d_2+\frac{d_1^2}{2})+\frac{A_0B_2d_1^2}{4}. \quad (2.35)$$

From (2.32), (2.34) it follows that

$$c_1=-d_1, \quad (2.36)$$

$$(c_1^2+d_1^2)=\frac{8\left((1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}\right)^2 a_2^2}{A_0^2B_1^2}. \quad (2.37)$$

Adding (2.33), (2.35) and using (2.36), (2.37) we obtain

$$16A_0B_1^2a_2^2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1]^{\delta}$$
$$=2A_0^2B_1^3(c_2+d_2)+(B_2-B_1)8\left((1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}\right)^2 a_2^2,$$

$$a_2^2=\frac{2A_0^2B_1^3(c_2+d_2)}{16A_0B_1^2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}-(B_2-B_1)8\left((1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}\right)^2}. \quad (2.38)$$

Lemma 1.3 is applied for c_2, d_2

$$|a_2|\le\frac{|A_0|B_1\sqrt{B_1}}{\sqrt{\left|2A_0B_1^2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}-(B_2-B_1)8\left((1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}\right)^2\right|}}. \quad (2.39)$$

That provided $|a_2|$ as showed (2.24).

New further computations (2.33) to (2.35) lead to

$$a_3=\frac{A_1B_1(c_1-d_1)+A_0B_1(c_2-d_2)}{8(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}}+\frac{A_0^2B_1^2(c_1^2+d_1^2)}{8\left((1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}\right)^2}. \quad (2.40)$$

Lemma 1.3 is applied for c_1, c_2 d_1 and d_2, we get

$$|a_3|\le\frac{B_1[|A_1|+|A_0|]}{|2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|}+\frac{|A_0|^2B_1^2}{|(1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^{\delta}|^2}. \quad (2.41)$$

That provided $|a_3|$ as showed (2.25).

If putting $\eta=1$ in Theorem 2.4, we get

Corollary 2.5. *Let* $f\in\mathcal{H}_{\Sigma}^{q}(1,\delta,h)$. *Then*

$$|a_2| \leq \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{|6A_0B_1^2[2(\theta-\tau)(\sigma-\xi)+1)]^\delta - 32(B_2-B_1)([(\theta-\tau)(\sigma-\xi)+1)]^\delta)^2|}},$$

$$|a_3| \leq \frac{B_1[|A_1|+|A_0|]}{|6[2(\theta-\tau)(\sigma-\xi)+1)]^\delta|} + \frac{|A_0|^2B_1^2}{4|[(\theta-\tau)(\sigma-\xi)+1)]^\delta|^2}.$$

If putting $\mathcal{V}(z)=1$ in Theorem 2.4, we get

Corollary 2.6. *Let* $f \in \mathcal{H}_\Sigma^q(\eta, \delta, h)$. *Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|2B_1^2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^\delta - 8(B_2-B_1)\left((1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^\delta\right)^2\right|}},$$

$$|a_3| \leq \frac{B_1}{|2(1+2\eta)[2(\theta-\tau)(\sigma-\xi)+1)]^\delta|} + \frac{B_1^2}{|(1+\eta)[(\theta-\tau)(\sigma-\xi)+1)]^\delta|^2}.$$

Conclusion

Usually determining and estimating coefficients for two new associated with quasi-subordination very difficult, especially if the functions are bi-univalent. It is useful in complex analysis and can benefit from article in physical and chemical applications.

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