



Geometrical Methods in Goursat Categories

Brice René Amougou Mbarga

University of Yaoundé I, Laboratory of Algebra, Geometry and Application,

P.O.Box: 812, Yaoundé, Cameroon; e-mail: renebrice3@gmail.com

Abstract

The main aim of the paper is to show that the Little Desarguesian Theorem, the Escher Cube, Closure Lemma 1 and 3, hold in any regular Mal'tsev categories. We prove that Mal'tsev categories may be characterized through variations of the Little Desarguesian Theorem, the Escher Cube, Closure Lemma 1 and 3, that is classically expressed in terms of four congruences R, S_1, S_2 and T , and characterizes congruence modular varieties. The proof of this result in a varietal context may be obtained exclusively through the Little Desarguesian Theorem, the Escher Cube, Closure Lemma 1 and 3. This was shown by H.P. Gumm in Geometric Methods in Congruence Modular Algebras. We prove that for any $2n + 1$ -permutable category \mathcal{E} , the category $\text{Equiv}(\mathcal{E})$ of equivalence relations in \mathcal{E} is also a $2n + 1$ -permutable category.

1 Introduction

The study of Mal'tsev categories originates from a classical theorem of Mal'tsev in 1954 [5]. For a variety of universal algebras \mathcal{V} (i.e., a category of models of a finitary one-sorted algebraic theory), he proved that the following conditions are equivalent:

(i) for any pair of congruences R and S on the same algebra X in \mathcal{V} , the equality $RS = SR$ holds;

(ii) the algebraic theory of \mathcal{V} contains a ternary operation p satisfying the equations

$$\begin{cases} p(x, y, y) = x, \\ p(x, x, y) = y. \end{cases}$$

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Varieties satisfying these conditions are now commonly called Mal'tsev varieties [5] (or 2-permutable varieties), and such a term p a Mal'tsev operation.

One of the results that A. Carboni, J. Lambek, M. C. Pedicchio had in mind in [5] was the extension to Mal'tsev categories of the Goursat Lemma, due to E. Goursat and stated in the category of groups as follows:

Lemma 1.1. (*Goursat Lemma*) *Let L and R be arbitrary groups. Then there is a bijection between the set \mathcal{S} of all subgroups of $L \times R$ and the set \mathcal{T} of all quintuple (A, B, C, D, θ) , where $B \trianglelefteq A \leq L$, $D \trianglelefteq C \leq R$ and $\theta : A/B \rightarrow C/D$ is a bijective homomorphism (here \leq denotes subgroup and \trianglelefteq denotes normal subgroup). More precisely, the subgroup corresponding to (A, B, C, D, θ) is*

$$G = \{(g, h) \in A \times C : \theta(gB) = hD\}.$$

This result was used to obtain general forms of the Zassenhaus lemma and the Jordan-Hölder-Schreier theorem for normal series in [12] and was generalized to Mal'tsev varieties by J. Lambek in [12]. After an in-depth analysis, A. Carboni, J. Lambek and M. C. Pedicchio proved that this result of E. Goursat holds not only in the Mal'tsev categories but also in those regular categories wherein each relation P from A to B satisfies the equality

$$PP^oPP^o = PP^o. \tag{1.1}$$

That is how Goursat categories were born: these are the regular categories in which each relation P satisfies the condition (1.1). In 1993, A. Carboni, M. Kelly and M. C. Pedicchio observed in [8] that the condition (1.1) is in fact strictly weaker than the difunctionality $PP^oP = P$ equivalent to the Mal'tsev condition $RS = SR$ in regular categories. They also observed that condition (1.1) is equivalent in a regular category to the 3-permutability of equivalence relations

$$RSR = SRS, \tag{1.2}$$

for any pair of equivalence relations R and S on the same object.

Structure of the paper

In Section 2 we introduce the basic categorical notions and results we shall need in the following section. In particular we recall the relationship between some epimorphisms such as strong epimorphisms, regular epimorphisms and split epimorphisms. We then recall some basic facts about regular categories and relations in regular categories. We also recall the notion Barr's Metatheorem concerning the internal logic of regular categories.

In Sections 3 and 4 the main results of the present paper show that stronger versions of the Little Desarguesian Theorem, the Escher Cube, Closure Lemma 1 and 3, characterize regular categories that are Mal'tsev categories and that are Goursat categories.

In Section 5 we prove that for any $2n + 1$ -permutable category \mathcal{E} , the category $\text{Equiv}(\mathcal{E})$ of equivalence relations in \mathcal{E} is also a $2n + 1$ -permutable category.

2 Preliminaries

In this section, we recall some elementary categorical notions and properties needed in the subsequent sections.

2.1 Regular epimorphisms

In this subsection, we examine various types of epimorphisms in order to understand the notions of regular categories.

Definition 2.1. A morphism $f : A \rightarrow B$ in a category \mathcal{E} is an epimorphism if, for any pair of parallel arrows $u, v : B \rightarrow C$ such that $u \circ f = v \circ f$, one has $u = v$.

In the category **Set** of sets, the epimorphisms are precisely the surjective maps. In the category **Grp** and **Ab** of abelian groups, the epimorphisms are the surjective homomorphisms. The notion of monomorphism is defined dually:

Definition 2.2. A morphism $f : A \rightarrow B$ in a category \mathcal{E} is a monomorphism if, for any pair of parallel arrows $u, v : C \rightarrow A$ such that $f \circ u = f \circ v$, one has $u = v$.

Monomorphisms in **Top** (the category of topological spaces) are the continuous injective maps.

Definition 2.3. An arrow $f : A \rightarrow B$ in a category \mathcal{E} is a strong epimorphism if, given any commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & \swarrow t & \downarrow h \\ C & \xrightarrow{m} & D \end{array}$$

where $m : C \rightarrow D$ is a monomorphism, there exists a unique arrow $t : B \rightarrow C$ such that $m \circ t = h$ and $t \circ f = g$.

Strong epimorphisms have the following properties:

(i) if $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ are two strong epimorphisms, then $\beta\alpha : A \rightarrow C$ is a strong epimorphism.

(ii) if $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ are such that $\beta\alpha : A \rightarrow C$ is a strong epimorphism, then $\beta : B \rightarrow C$ is a strong epimorphism.

Definition 2.4. An arrow $f : A \rightarrow B$ is a regular epimorphism if it is the coequalizer of two arrows in \mathcal{E} . In other words, f is a regular epimorphism if one can find $u, v \in \mathcal{E}$ such that $f = \text{Coeg}(u, v)$.

Definition 2.5. A split epimorphism is an arrow $f : A \rightarrow B$ such that there is an arrow $i : B \rightarrow A$ with $f \circ i = 1_B$.

In any category \mathcal{E} , the following properties hold.

- (i) every split epimorphism is regular;
- (ii) every regular epimorphism is strong.

2.2 Regular categories and relations

Recall that a kernel pair of a morphism $f : A \rightarrow B$ is a pair of parallel morphisms

$$Eq(f) \begin{matrix} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{matrix} A \text{ obtained via the following pullback}$$

$$\begin{array}{ccc} Eq(f) & \xrightarrow{f_1} & A \\ \downarrow f_2 & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

Let $f : A \rightarrow B$ be a map in Set , and $Eq(f) = \{(x, y) \in A^2 / f(x) = f(y)\}$ its kernel pair.

Definition 2.6. [13] A finitely complete category \mathcal{E} is regular when

- (a) \mathcal{E} has coequalisers of kernel pairs;
- (b) regular epimorphisms are stable under pullbacks in \mathcal{E} : That is, given a pullback diagram

$$\begin{array}{ccc} D & \xrightarrow{g} & C \\ \downarrow u & & \downarrow v \\ A & \xrightarrow{f} & B \end{array}$$

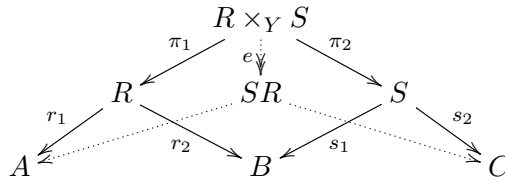
where f is regular epimorphism, then g is a regular epimorphism.

Example 2.7. Regular categories, even complete ones are abundant. The category of models of any equational theory is complete and regular. This includes the categories of sets, lattices, groups, rings, etc. The category of compact Hausdorff spaces is complete regular as is every abelian category. Also the category opposite to the category of sets is regular.

This definition of regular category is equivalent with the following one: any morphism f admits a (unique up to isomorphism) factorization $f = mr$, where r is a regular epimorphism and $m(= Im.f)$ is a monomorphism, and these factorizations are pullback stable.

$$\begin{array}{ccc} A & \xrightarrow{\forall f} & B \\ & \searrow r & \nearrow m \\ & & I \end{array}$$

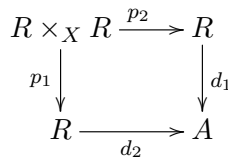
If R is a relation from A to B , namely a subobject $(r_1, r_2) : R \rightarrow A \times B$, its opposite relation, denoted R° , is a relation from B to A , the subobject $(r_2, r_1) : R \rightarrow B \times A$. Given another relation $(s_1, s_2) : S \rightarrow B \times C$ from B to C , On first builds the pullback



We obtain a composite relation $Im(r_1\pi_1, s_2\pi_2) : SR \rightarrow A \times C$.

Definition 2.8. [26] Let \mathcal{E} be a category with finite limits.

- (i) A relation $R \rightarrow A \times A$ is reflexive when there is an arrow $\delta : X \rightarrow R$ such that $d_1 \circ \delta = 1_X = d_2 \circ \delta$, or $1_A \leq R$;
- (ii) R is symmetric if there is an arrow $\sigma : R \rightarrow R$ such that $d_1 \circ \sigma = d_2$ and $d_2 \circ \sigma = d_1$, or $R^\circ \leq R$;
- (iii) Consider the pullback



the relation R is transitive if there is an arrow $\tau : R \times_X R \rightarrow R$ such that $d_1 \circ \tau = d_1 \circ p_1$ and $d_2 \circ \tau = d_2 \circ p_2$ or $RR \leq R$.

An (internal) equivalence relation is a reflexive, symmetric and transitive relation.

In particular, the kernel pair $\langle f_1, f_2 \rangle : Eq(f) \rightrightarrows A \times A$ of a morphism $f : A \rightarrow B$ (obtained by building the pullback of f along f) is an equivalence relation. The equivalence relations that occur as kernel pairs of some morphism in a category \mathcal{E} are called effective.

We denoted by $Equiv(\mathcal{E})$ the category whose objects are equivalence relations in \mathcal{E} and arrows from $\langle r_1, r_2 \rangle : R \rightarrow A \times A$ to $\langle s_1, s_2 \rangle : S \rightarrow B \times B$ are pairs (f, g) of arrows in \mathcal{E} making the following diagram commute

$$\begin{array}{ccc}
 R & \xrightarrow{g} & S \\
 r_1 \downarrow & & \downarrow s_1 \\
 & r_2 & \downarrow s_2 \\
 A & \xrightarrow{f} & B
 \end{array}$$

$s_1g = fr_1$ and $s_2g = fr_2$.

Definition 2.9. Let \mathcal{E} be a regular category, (R, r_1, r_2) a relation on A and $f : A \rightarrow B$ a regular epimorphism. We define the regular image of R along $f : A \rightarrow B$ to be the relation $f(R)$ on B induced by the (regular epimorphism, monomorphism) factorization $\langle s_1, s_2 \rangle \psi$ of the composite $(f \times f)\langle r_1, r_2 \rangle$:

$$\begin{array}{ccc}
 R & \overset{\psi}{\dashrightarrow} & f(R) \\
 \langle r_1, r_2 \rangle \downarrow & & \downarrow \langle s_1, s_2 \rangle \\
 A \times A & \xrightarrow{f \times f} & B \times B
 \end{array}$$

Note that the regular image $f(R)$ can be obtained as the relational composite $f(R) = fRf^o = fr_2r_1^o f^o$. When R is an equivalence relation, $f(R)$ is also reflexive and symmetric.

2.3 Barr’s embedding Metatheorems

The following celebrated lemma is crucial for the whole of category theory.

Lemma 2.10. (Yoneda lemma) *Let $G : \mathcal{E} \rightarrow \mathbf{Set}$ be a functor defined on an arbitrary category \mathcal{E} . For every object $C \in \mathcal{E}$, there exists a bijection $F(C) \cong \text{Nat}(\mathcal{E}(C, -), F)$ between the set $F(C)$ and the set of natural transformations from the representable functor $\mathcal{E}(C, -)$ to the functor F . These bijections are natural both in C and in F .*

Then given two objects $C, D \in \mathcal{E}$, there are bijections

$$\mathcal{E}(D, C) \cong \text{Nat}(\mathcal{E}(C, -), \mathcal{E}(D, -))$$

and

$$\mathcal{E}(C, D) \cong \text{Nat}(\mathcal{E}(-, C), \mathcal{E}(-, D)).$$

All this allows defining the "Yoneda embedding of \mathcal{E} " [6] which is the functor

$$Y_{\mathcal{E}} : \mathcal{E} \longrightarrow [\mathcal{E}^{op}, \mathbf{Set}], \quad C \longmapsto \mathcal{E}(-, C).$$

This functor $Y_{\mathcal{E}}$ is a full and faithful embedding of \mathcal{E} in the category $[\mathcal{E}^{op}, \mathbf{Set}]$ of contravariant functors from \mathcal{E} to \mathbf{Set} and natural transformations between them.

These well-known facts allow at once a powerful set-theoretical-like technique for proving various results in a category by just proving them in the category of sets.

Metatheorem 2.11. *For every small regular category \mathcal{E} there is a small category \mathcal{D} and a full and faithful embedding*

$$Z : \mathcal{E} \rightarrow [\mathcal{D}^{op}, \mathbf{Set}],$$

which preserves and reflects finite limits and regular epimorphisms.

Metatheorem 2.12. *Let P be a statement of the form $a \implies b$, where a and b can be expressed as conjunctions of properties in the following list:*

- (i) *some finite diagram is commutative;*
- (ii) *some morphism is a monomorphism;*
- (iii) *some morphism is an isomorphism;*
- (iv) *some finite diagram is a limit diagram;*
- (v) *some arrow $f : A \longrightarrow B$ factors (of course, uniquely) through some specified monomorphism $s : S \longrightarrow B$.*

If this statement P is valid in the category of sets, it is valid in every category.

2.4 Shifting Lemma, Little Desarguesian Theorem, Little Pappian Theorem, Escher Cube, Closure Lemma 1 and 3

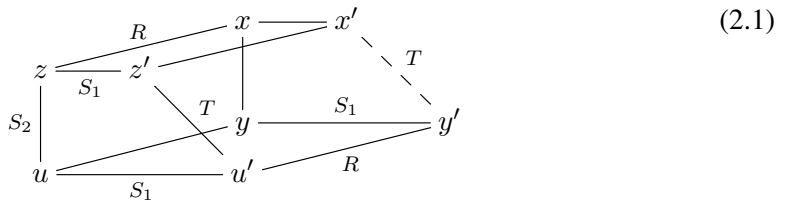
For a variety \mathcal{V} of universal algebras, Gumm's Shifting Lemma [16] is stated as follows. Given congruences R, S_1 and T on the same algebra X in \mathcal{V} such that $R \wedge S_1 \leq T$, whenever x, y, t, z are element in X with $(x, y) \in R \wedge T, (x, t) \in S_1, (y, z) \in S_1$ and

$(t, z) \in R$, it then follows that $(t, z) \in T$:

$$T \left(\begin{array}{ccc} x & \xrightarrow{S_1} & t \\ \downarrow R & & \downarrow R \\ y & \xrightarrow{S_1} & z \end{array} \right) T$$

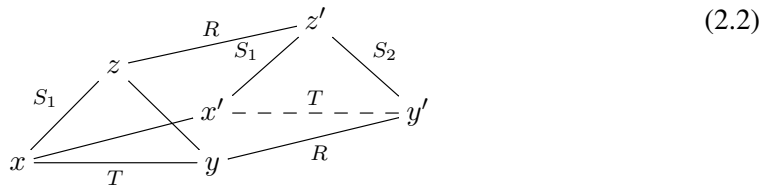
Closure Lemma 1:

A variety \mathcal{V} of universal algebras satisfies the Closure Lemma 1, [27] if given congruences R, S_i and T on the same algebra X in \mathcal{V} such that $R \wedge S_i \leq T$, whenever $x, y, z, x', y', z', u, u'$ are element in X with $(x, z), (x', z'), (u, y), (u', y') \in R, (z, z'), (x', x), (u, u'), (y, y') \in S_1, (z, u), (x, y) \in S_2$ and $(z', u') \in T$, it then follows that $(x', y') \in T$ we have



Mainly we are interested in the following two special case. Firstly, letting $x = x'$ and $z = z'$ we obtain:

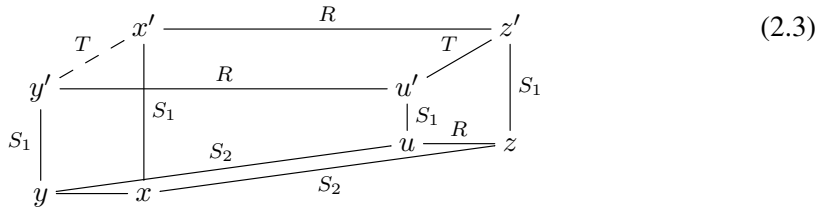
A variety \mathcal{V} of universal algebras satisfies the Little Desarguesian Theorem [27] if given congruences R, S_i and T on the same algebra X in \mathcal{V} such that $R \wedge S_i \leq T$, whenever x, y, z, x', y', z' are element in X with $(x, z), (x', z') \in S_1, (z, y), (z', y') \in S_2, (z, z'), (x, x'), (y, y') \in R$ and $(x, y) \in T$, it then follows that $(x', y') \in T$:



Escher Cube or Closure Lemma 2 :

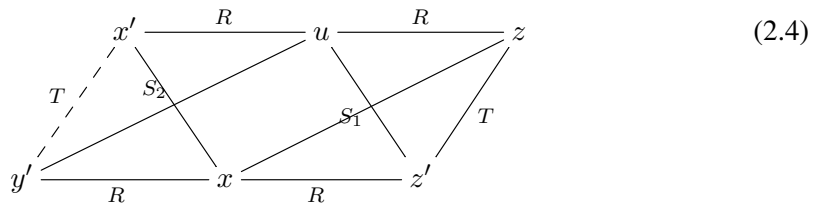
A variety \mathcal{V} of universal algebras satisfies the Closure Lemma 2, [27] if given congruences R, S_i and T on the same algebra X in \mathcal{V} such that $R \wedge S_i \leq T$,

whenever $x, y, z, x', y', z', u, u'$ are element in X with $(y, y'), (u', u), (z, z'), (x, x') \in S_1, (u, y), (x, z) \in S_2, (y', u'), (x', z') \in R$ and $(u', z') \in T$, one has $(x', y') \in T$:



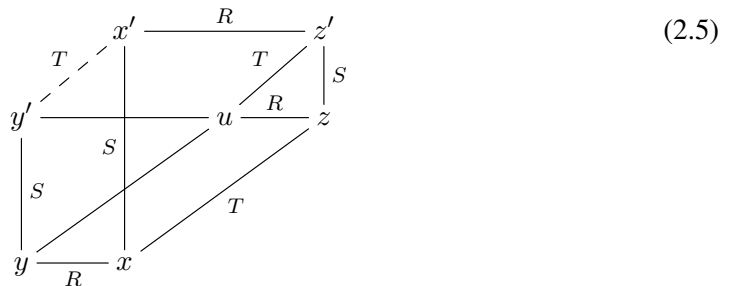
The little Pappian Theorem

A variety \mathcal{V} of universal algebras satisfies the little Pappian Theorem, [27] if given congruences R, S_i and T on the same algebra X in \mathcal{V} such that $R \wedge S_i \leq T$, whenever x, y, u, z, x', y', z' are element in X with $(u, y'), (x, z) \in S_1, (x', x), (u, z') \in S_2, (x', u), (u, z), (y', x), (x, z') \in R$ and $(z, z') \in T$, then $(x', y') \in T$:



Similarly, on identifying S_2 with T and u with z' we obtain:

A variety \mathcal{V} of universal algebras satisfies the Closure Lemma 3 if given congruences R, S and T on the same algebra X in \mathcal{V} such that $R \wedge S \leq T$, one has



3 Regular Mal'tsev Categories

A finitely complete category \mathcal{E} is called a Mal'tsev category if every reflexive relation in \mathcal{E} is an equivalence relation.

- Example 3.1.** (1) Groups, Rings and Modules are Mal'tsev categories;
 (2) abelian categories are Mal'tsev categories;
 (4) any slice of a Malcev category.

These categories are also characterized by other properties on relations, as follows:

Theorem 3.2. [15] *Let \mathcal{E} be a regular category. The following statements are equivalent:*

- (i) \mathcal{E} is a Mal'tsev category;
- (ii) $\forall R, S \in \text{Equiv}(X), R \vee S = RS = SR \in \text{Equiv}(X)$, for any X ;
- (iii) every reflexive relation E in \mathcal{E} is symmetric: $E^\circ = E$;
- (iv) any relation D is difunctional: $DD^\circ D = D$;
- (v) every reflexive relation T in \mathcal{E} is transitive: $TT = T$.

Theorem 3.3. *Let \mathcal{E} be a regular Mal'tsev category. Then*

- (1) the Little Desarguesian Theorem holds true in \mathcal{E} .
- (2) the Escher Cube holds true in \mathcal{E} ;
- (3) the Closure Lemma 1 holds true in \mathcal{E} ;
- (4) the Closure Lemma 3 holds true in \mathcal{E} ;
- (5) the Little Pappian Theorem holds true in \mathcal{E} .

Proof. In a regular context, it suffices to give a proof in set-theoretical terms (see Metatheorem 2.12 and [27], for instance). We show that the Little Desarguesian Theorem holds true in \mathcal{E} .

Let R, S_1, S_2 and T be equivalence relations on the same object X in \mathcal{E} such that $R \wedge S_i \leq T$ and assume that x, y, z, x', y', z' , are related as in (2.2). The pairs

$(y, y), (y', y'), (x, z), (x', z')$ are elements of S_1 . Defining equivalence relations P, Q and F on S_1 by:

$$\begin{aligned} (u, v)P(r, s) &\Leftrightarrow vS_2s \\ (u, v)Q(r, s) &\Leftrightarrow uTr \text{ and } vS_2s \\ (u, v)F(r, s) &\Leftrightarrow uRr \text{ and } vRs \end{aligned}$$

We obtain

$$Q \left(\begin{array}{ccc} (x, z) & \xrightarrow{F} & (x', z') \\ \left| P \right. & & \left. P \right| \\ (y, y) & \xrightarrow{F} & (y', y') \end{array} \right)$$

If we want to apply the Shifting Lemma we have to check that $F \wedge P \leq Q$. Thus, let $(a, b)F \wedge P(c, d)$. First of all $(a, b) \in S_1$ and $(c, d) \in S_1$. Moreover $(b, d) \in R \wedge S_2$. Hence $(b, d) \in T$ by assumption. Hence

$$T \left(\begin{array}{ccc} b & \xrightarrow{S_1} & a \\ \left| R \right. & & \left. R \right| \\ d & \xrightarrow{S_1} & c \end{array} \right) \begin{array}{l} \backslash \\ | \\ / \end{array} T$$

We may apply the Shifting Lemma to

$$Q \left(\begin{array}{ccc} (x, z) & \xrightarrow{F} & (x', z') \\ \left| P \right. & & \left. P \right| \\ (y, y) & \xrightarrow{F} & (y', y') \end{array} \right) \begin{array}{l} \backslash \\ | \\ / \end{array} Q$$

We conclude that $(x', z')Q(y', y')$ and, consequently, that $(x', y') \in T$.

Similarly we show that the Closure Lemma 1 holds true in \mathcal{E} ;

Let R, S_1, S_2 and T be equivalence relations on the same object X in \mathcal{E} such that $R \wedge S_i \leq T$ and assume that x, y, z, x', y', z' , are related as in (2.1). The pairs

$(x, x'), (y, y'), (z', z), (u, u')$ are elements of S_1 . We obtain

$$\begin{array}{ccc}
 (x, x') & \xrightarrow{F} & (z, z') \\
 Q \left(\begin{array}{c} | \\ P \\ | \end{array} \right. & & \left. \begin{array}{c} | \\ P \\ | \end{array} \right) Q \\
 (y, y') & \xrightarrow{F} & (u, u')
 \end{array}$$

Similarly we can show (3), \dots , (5). □

Using the fact that in a Mal'tsev category reflexive relations coincide with equivalence relations, or with symmetric relations, we are now going to show that regular Mal'tsev categories may be characterized through a stronger version of the diagram (2.2) where, in the assumption, the equivalence relations are replaced by reflexive relations. Note that, for diagram such as (2.2) where R, S or T are not symmetric, the relations are always to be considered from left to right and from top to bottom. To avoid ambiguity with the interpretation of such diagrams, from now on we will write $a \xrightarrow{E} b$ to mean that $(a, b) \in E$ whenever E is a non-symmetric relation.

Theorem 3.4. *Let \mathcal{E} be a regular category. The following conditions are equivalent:*

- (1) \mathcal{E} is a Mal'tsev category;
- (2) the Little Desarguesian Theorem holds in \mathcal{E} when R, T, S_1, S_2 are reflexive relations on X ;
- (3) the Escher Cube holds in \mathcal{E} when R, T, S_1, S_2 are reflexive relations on X ;
- (4) the Closure Lemma 1 holds in \mathcal{E} when R, T, S_1, S_2 are reflexive relations on X .
- (5) the Little Pappian Theorem holds true in \mathcal{E} when R, T, S_1, S_2 are reflexive relations on X ;
- (6) the Closure Lemma 3 holds true in \mathcal{E} when R, T, S_1, S_2 are reflexive relations on X .

Proof. (1) \Rightarrow (2) : Since \mathcal{E} is a Mal'tsev category, reflexive relations are necessarily equivalence relations. Since by Theorem 3.2, the diagram (2.2) holds for any reflexive

relations in \mathcal{E} .

(2) \Rightarrow (1) : We shall prove that every reflexive relation $\langle e_1, e_2 \rangle : E \rightarrow X \times X$ is symmetric (Theorem 3.2(iii)). Suppose that $(x, y) \in E$, and consider the reflexive relations T and R on E defined by the following pullbacks:

$$\begin{array}{ccc}
 R \longrightarrow E & & T \longrightarrow E \\
 \langle r_1, r_2 \rangle \downarrow & & \langle t_1, t_2 \rangle \downarrow \\
 E \times E \xrightarrow{e_1 \times e_2} X \times X & \text{and} & E \times E \xrightarrow{e_2 \times e_1} X \times X \xrightarrow{\langle \pi_2, \pi_1 \rangle} X \times X \\
 & & \downarrow \langle e_1, e_2 \rangle
 \end{array}$$

We have

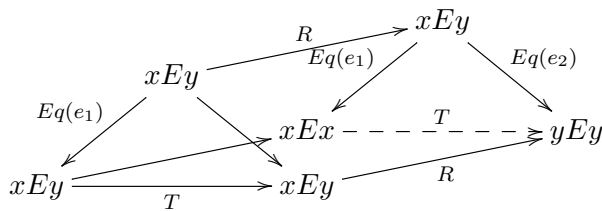
$$(aEb, cEd) \in R \text{ if and only if } (a, d) \in E,$$

and

$$(aEb, cEd) \in T \text{ if and only if } (c, b) \in E.$$

The reflexive relation on E we consider are the kernel pair $Eq(e_1), Eq(e_2)$ of e_1 and e_2 respectively.

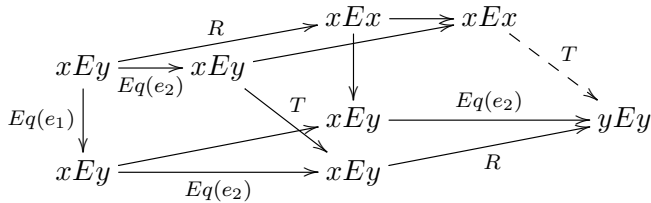
$Eq(e_1), Eq(e_2)$, are equivalence relation, with the property that $Eq(e_i) \leq R$ and $Eq(e_i) \leq T$, so that $R \wedge Eq(e_i) = Eq(e_i) \leq T, i = 1, 2$. We may apply the assumption to the following relations given in solid lines



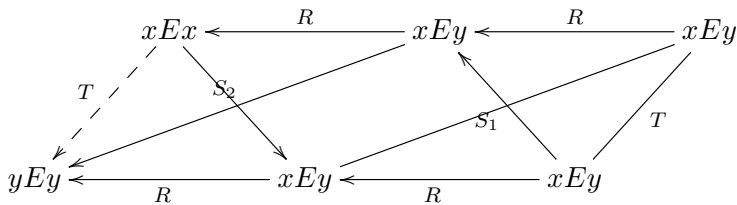
$(xEx$ and yEy by the reflexivity of the relation E). We conclude that $(xEx, yEy) \in T$ and, consequently, that $(y, x) \in E$.

The proof of (1) \Leftrightarrow (3) is similar. Since the Little Desarguesian Theorem and the Closure Lemma 2 are immediate consequences of Closure Lemma 1 with $x = x'$ and $z = z'$.

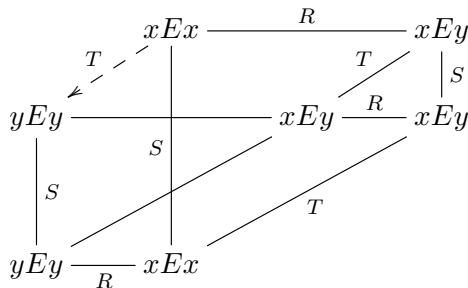
(3) \Rightarrow (1) : evident, (3) \Rightarrow (1) With the assumption of previous proof, we obtain



We conclude that $(xEx, yEy) \in T$, then $(y, x) \in E$. (5) \Rightarrow (1) obvious, (5) \Rightarrow (1) we observe with $S_1 = Eq(e_2)$ and $S_2 = Eq(e_1)$



(6) \Rightarrow (1) : we observe with $S = Eq(e_2)$



□

Corollary 3.5. Let \mathcal{E} be a regular category. The following conditions are equivalent:

- (1) \mathcal{E} is a Mal'tsev category;
- (2) the Little Desarguesian Theorem holds in \mathcal{E} when R, T , are reflexive relations and S_1, S_2 are equivalence relations on X ;

- (3) the Escher Cube holds in \mathcal{E} when R, T , are reflexive relations and S_1, S_2 are equivalence relations on X ;
- (4) the Closure Lemma 1 holds in \mathcal{E} when R, T , are reflexive relations and S_1, S_2 are equivalence relations on X ;
- (5) the Little Pappian Theorem holds true in \mathcal{E} when R, T , are reflexive relations and S_1, S_2 are equivalence relations on X ;
- (6) the Closure Lemma 3 holds true in \mathcal{E} when R, T , are reflexive relations and S_1, S_2 are equivalence relations on X .

A variety of algebras \mathcal{V} is a majority category if and only if it admits a majority term, i.e., a ternary term $m(x, y, z)$ satisfying the equations:

$$\begin{aligned} m(x, x, y) &= x, \\ m(x, y, x) &= x, \\ m(y, x, x) &= x. \end{aligned}$$

For a proof of this statement, we refer the reader to [23].

Example 3.6. (i) A subvariety \mathcal{V} of the variety of rings is a majority category if and only if \mathcal{V} satisfies the equation $x^n = x$ for some $n > 2$. In particular, the category **BoRg** of Boolean rings is a majority category. Then the polynomial

$$m(x, y, z) = x - (x - y)(x - z)^{n-1},$$

is a majority term for \mathcal{V} .

(ii) The category **NReg** of von Neumann regular rings (see [21]) is the class of all rings R such that for any $a \in R$ there exists $x \in R$ such that $a = axa$. The category **NReg** is a majority category.

Suppose that A, B, C are rings and that R is a subring of $A \times B \times C$ which is a von Neumann regular ring. Let $\bar{a} = (a, b, c'), \bar{b} = (a, b', c), \bar{c} = (a', b, c)$ be any elements of R . Then since R is von Neumann regular, there exists $x = (x_1, x_2, x_3) \in R$ such that

$$(\bar{a} - \bar{b})x(\bar{a} - \bar{b}) = (\bar{a} - \bar{b}).$$

Then it is easy to see that

$$\bar{a} - (\bar{a} - \bar{b})x(\bar{a} - \bar{c}) = (a, b, c),$$

so that $(a, b, c) \in R$.

For a regular category \mathcal{E} , the property of being a majority category can be equivalently defined as follows (see [21]): for any reflexive relations R, S and T on the same object X in \mathcal{E} , the inequality

$$R \wedge (ST) \leq (R \wedge S)(R \wedge T)$$

holds.

Lemma 3.7. *Let \mathcal{E} be a majority category \mathcal{E} . Then*

- (i) *The Little Desarguesian Theorem holds true in \mathcal{E} ;*
- (ii) *Escher Cube holds true in \mathcal{E} ;*
- (iii) *Closure Lemma 1 holds true in \mathcal{E} ;*
- (iv) *Closure Lemma 3 holds true in \mathcal{E} .*

Proof. The proof of this result is based on the fact that a majority category satisfies the Shifting Lemma. □

4 Regular Goursat Categories

A regular category \mathcal{E} is called a Goursat category [14] when it is 3-permutable, i.e. for any pair of equivalence relations R and S on the same object in \mathcal{E} one has $RSR = SRS$.

Example 4.1. Mal'tsev categories.

As examples of Goursat categories that are not regular Mal'tsev categories we have the category of implication algebras

Theorem 4.2. [14] *Let \mathcal{E} be a regular category. The following statements are equivalent:*

- (i) *\mathcal{E} is a Goursat category;*
- (ii) *$\forall R, S \in \text{Equiv}(X), R \vee S = RSR (= SRS) \in \text{Equiv}(X)$;*

- (iii) every relation $P \rightarrow X \times Y$ in \mathcal{E} , $PP^\circ PP^\circ = PP^\circ$;
- (iv) every reflexive relation F in \mathcal{E} , $F^\circ F = FF^\circ \in \text{Equiv}(X)$;
- (v) every reflexive and positive relation $R = U^\circ U$ in \mathcal{E} is an equivalence relation;
- (vi) $E^\circ \leq EE$, for any reflexive relation E ;
- (vii) $(1_X \wedge T)T^\circ(1_X \wedge T) \leq TT$, for any relation T on an object X .

Proposition 4.3. *Let \mathcal{E} be a regular Goursat categories.*

- (1) *The little desarguesian theorem (2.2) holds in \mathcal{E} when S_1 is a reflexive relation and S_2, R and T are equivalence relations.*
- (2) *The Closure Lemma 1 (2.1) holds in \mathcal{E} when S_1 is a reflexive relation and S_2, R and T are equivalence relations.*
- (3) *The Closure Lemma 2 (2.3) holds in \mathcal{E} when S_1 is a reflexive relation and S_2, R and T are equivalence relations.*
- (4) *The Little Pappian Theorem holds true in \mathcal{E} when S_1 is a reflexive relation and S_2, R and T are equivalence relations.*
- (5) *The Closure Lemma 3 holds true in \mathcal{E} when S is a reflexive relation and R and T are equivalence relations.*

Proof. The proof is based on Proposition 3.4 in [15] which stipule that a Goursat category satisfies the Shifting Lemma when S_1 is a reflexive relation and R and T are equivalence relations.

In a regular context, it suffices to give a proof in set-theoretical terms (see Metatheorem 2.12 and [27], for instance). We show that the Little Desarguesian Theorem holds true in \mathcal{E} .

Let R, S_1, S_2 and T be equivalence relations on the same object X in \mathcal{E} such that $R \wedge S_i \leq T$ and assume that x, y, z, x', y', z' , are related as in (2.2). The pairs

$(y, y), (y', y'), (x, z), (x', z')$ are elements of S_1 . Defining equivalence relations P, Q and F on S_1 by:

$$\begin{aligned} (u, v)P(r, s) &\Leftrightarrow vS_2s \\ (u, v)Q(r, s) &\Leftrightarrow uTr \text{ and } vS_2s \\ (u, v)F(r, s) &\Leftrightarrow uRr \text{ and } vRs \end{aligned}$$

We obtain

$$Q \left(\begin{array}{ccc} (x, z) & \xrightarrow{F} & (x', z') \\ \left| P \right. & & \left. P \right| \\ (y, y) & \xrightarrow{F} & (y', y') \end{array} \right)$$

If we want to apply the Shifting Lemma we have to check that $F \wedge P \leq Q$. Thus, let $(a, b)F \wedge P(c, d)$. First of all $(a, b) \in S_1$ and $(c, d) \in S_1$. Moreover $(b, d) \in R \wedge S_2$. Hence $(b, d) \in T$ by assumption (Proposition 3.4 in [15]). Hence

$$T \left(\begin{array}{ccc} b & \xrightarrow{S_1} & a \\ \left| R \right. & & \left. R \right| \setminus \\ d & \xrightarrow{S_1} & c \end{array} \right) \begin{array}{l} / \\ / \\ / \end{array} T$$

We may apply the Shifting Lemma to

$$Q \left(\begin{array}{ccc} (x, z) & \xrightarrow{F} & (x', z') \\ \left| P \right. & & \left. P \right| \setminus \\ (y, y) & \xrightarrow{F} & (y', y') \end{array} \right) \begin{array}{l} / \\ / \\ / \end{array} Q$$

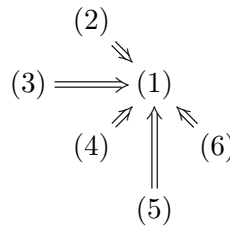
We conclude that $(x', z')Q(y', y')$ and, consequently, that $(x', y') \in T$.

□

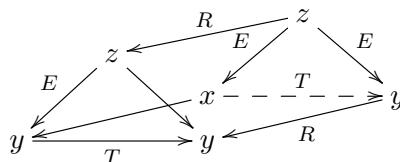
Theorem 4.4. *Let \mathcal{E} be a regular category and consider the following assertions.*

- (1) \mathcal{E} is a Goursat category;
- (2) the little desarguesian theorem (2.2) holds in \mathcal{E} when S_1, S_2 are reflexive relations and R and T are reflexive and positive relations;
- (3) The Escher Cube, (2.1) holds in \mathcal{E} when S_1, S_2 are reflexive relations and R and T are reflexive and positive relations;
- (4) The Closure Lemma 1, (2.3) holds in \mathcal{E} when S_1, S_2 are reflexive relations and R and T are reflexive and positive relations;
- (5) the Little Pappian Theorem holds true in \mathcal{E} when S_1, S_2 are reflexive relations and R and T are reflexive and positive relations;
- (6) the Closure Lemma 3 holds true in \mathcal{E} when S_1, S_2 are reflexive relations and R and T are reflexive and positive relations.

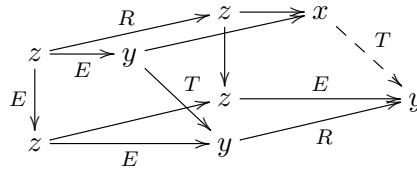
Then



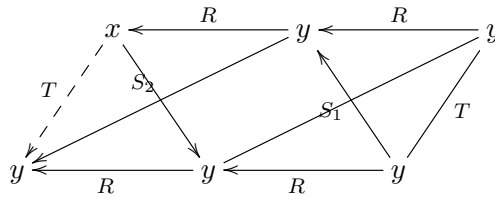
Proof. (2) \Rightarrow (1) We shall prove that for any reflexive relation E on X in \mathcal{E} , $EE^\circ = E^\circ E$ (see Theorem 4.2 (iv)). Suppose that $(x, y) \in EE^\circ$. Then, for some z in X , one has that $(z, x) \in E$ and $(z, y) \in E$. Consider the reflexive and positive relations $R = EE^\circ$ and $T = E^\circ E$, and the reflexive relation E on X . Then we have:



to conclude that $(x, y) \in E^\circ E$. The equality $E^\circ E = EE^\circ$ follows immediately. (3) \Rightarrow (1) Obvious, (4) \Rightarrow (1) With the assumption of previous proof, we obtain



Similarly (5) \Rightarrow (1) is obvious, (6) \Rightarrow (1) we observe with $S_1 = S_2 = E$



□

A variety \mathcal{V} of universal algebras is called 3-permutable when the strictly weaker equality $RSR = SRS$ holds. Such varieties are characterized by the existence of two quaternary operations p and q satisfying the identities $p(x, y, y, z) = x$, $p(u, u, v, v) = q(u, u, v, v)$, $q(x, y, y, z) = z$. We start by giving a direct proof of Little Desarguesian theorem in the varietal context. We give an alternative proof which is suitable to be extended to the categorical context of regular categories.

Theorem 4.5. *Let \mathcal{E} be a majority and 3-permutable varieties. Let x, y, z, x', y', z' , be elements of X and R, T, S_1, S_2 congruences on X with $R \wedge S_i \leq T$, for $i = 1, 2$. Then the diagram (2.2) holds in \mathcal{E} .*

Proof. Let R, S_1, S_2 and T be congruence on the same algebra X in \mathcal{E} such that $R \wedge S_i \leq T$ and assume that x, y, z, x', y', z' , are related as in (2.2). From the relations

$$\begin{array}{ll}
 xTxRx'S_1S_2x' & xTxRxS_1S_2x \\
 xTxRx'S_1S_2y' & xTxRxS_1S_2y \\
 xTyRy'S_1S_2y' & xTxRyS_1S_2y \\
 yTyRy'S_1S_2y' & yTyRyS_1S_2y
 \end{array}$$

we may deduce the following ones by applying the quaternary operations p and q , respectively:

$$p(x, x, x, y)Tp(x, x, y, y)Rp(x', x', y', y')Sp(x', y', y', y')$$

and

$$q(x, x, x, y)Tq(x, x, y, y)Rq(x', x', y', y')Sq(x', y', y', y')$$

one has $xTp(x, x, y, y)Rp(x', x', y', y')Sx'$ and $yTq(x, x, y, y)Rq(x', x', y, y)Sy'$; one has $xTxRp(x, x, y, y)Sx$ and $yTyRq(x, x, y, y)Sy$.

Since \mathcal{E} is majority variety, then $R \wedge S_1S_2 \leq (R \wedge S_1)(R \wedge S_2) \leq TT = T$. We apply the Shifting Lemma to

$$T \left(\begin{array}{ccc} x & \xrightarrow{R} & x \\ \left| \begin{array}{c} S \\ \\ \\ \end{array} \right. & & \left| \begin{array}{c} S \\ \\ \\ \end{array} \right. \\ x & \xrightarrow{R} & p(x, x, y, y) \end{array} \right) \quad T \left(\begin{array}{ccc} y & \xrightarrow{R} & y \\ \left| \begin{array}{c} S \\ \\ \\ \end{array} \right. & & \left| \begin{array}{c} S \\ \\ \\ \end{array} \right. \\ y & \xrightarrow{R} & q(x, x, y, y) \end{array} \right)$$

Again, we apply the Shifting Lemma to

$$T \left(\begin{array}{ccc} x & \xrightarrow{R} & x' \\ \left| \begin{array}{c} S \\ \\ \\ \end{array} \right. & & \left| \begin{array}{c} S \\ \\ \\ \end{array} \right. \\ p(x, x, y, y) & \xrightarrow{R} & p(x', x', y', y') \end{array} \right) \quad T \left(\begin{array}{ccc} y & \xrightarrow{R} & y' \\ \left| \begin{array}{c} S \\ \\ \\ \end{array} \right. & & \left| \begin{array}{c} S \\ \\ \\ \end{array} \right. \\ q(x, x, y, y) & \xrightarrow{R} & q(x', x', y', y') \end{array} \right)$$

we obtain $x'Tp(x', x', y', y') = q(x', x', y', y')Ty'$; it follows that $(x', y') \in T$. □

We adapt this varietal proof into a categorical one using an appropriate matrix and the corresponding relations which may be deduced from it (see [16] for more details). The kind of matrix we use translates the quaternary identities into the property on relations given in Theorem 4.2(iii):

$$\left(\begin{array}{cccc|cc} x & y & y & z & x & z \\ u & u & v & v & \alpha & \alpha \end{array} \right)$$

Thus, the introduction of a new element α , to represent the identity $p(u, u, v, v) = q(u, u, v, v)(= \alpha)$.

For this matrix, the interpretation gives: for any binary relation P , if xPu, yPu, yPv and zPv , then $xP\alpha$ and $zP\alpha$, for some α ; this gives the property $PP^oPP^o \leq PP^o$. Since $PP^o \leq PP^oPP^o$ is always true, we get precisely $PP^oPP^o = PP^o$ from Theorem 4.2(iii).

Theorem 4.6. *Let \mathcal{E} be a regular majority and Goursat categories. Let x, y, z, x', y', z' , be elements of X and R, T, S_1, S_2 are equivalence relations on X with $R \wedge S_i \leq T$, for $i = 1, 2$. Then the diagram (2.2) holds in \mathcal{E} .*

Proof. We extend the proof of Theorem 4.5 to a categorical context by constructing an appropriate matrix as follows

$$\left(\begin{array}{cccc|cc} x & x & x & y & x & y \\ x & y & y & y & x & y \\ x' & y' & y' & y' & x' & y' \\ \hline x & x & y & y & \theta & \theta \\ x' & x' & y' & y' & \omega & \omega \end{array} \right)$$

We define a relation P from X^3 to X^2 by:

$$(a, b, c)P(d, e) \Leftrightarrow \exists l \text{ such that } aTdRlS_1S_2b, \quad lRe, \quad eS_1S_2c \text{ and } aS_1S_2d.$$

From the matrix we see that $(x, x, x')P(x, x')$, $(x, y, y')P(x, x')$, $(x, y, y')P(y, y')$ and $(y, y, y')P(y, y')$, we also see that $(x, x, x')PP^oPP^o(y, y, y')$ from which we conclude that $(x, x, x')PP^o(y, y, y')$. It then follows that $(x, x, x')P(\theta, \omega)$ and $(y, y, y')P(\theta, \omega)$, for some (θ, ω) . Let $S = S_1S_2$, i.e., there exist α and β such that

$$\begin{aligned} xT\theta R\alpha Sx, \quad \alpha R\omega, \quad \omega Sx' \text{ and } xS\theta \\ yT\theta R\beta Sy, \quad \beta R\omega, \quad \omega Sy' \text{ and } yS\theta \end{aligned}$$

Since \mathcal{E} is majority category, $R \wedge S_1S_2 \leq (R \wedge S_1)(R \wedge S_2) \leq TT = T$. We apply the Shifting Lemma to

$$T \left(\begin{array}{ccc} x & \xrightarrow{R} & x \\ \left| \begin{array}{cc} S & S \end{array} \right. & & \backslash T \\ \theta & \xrightarrow{R} & \alpha \end{array} \right) \quad T \left(\begin{array}{ccc} y & \xrightarrow{R} & y \\ \left| \begin{array}{cc} S & S \end{array} \right. & & \backslash T \\ \theta & \xrightarrow{R} & \beta \end{array} \right)$$

Again, we apply the Shifting Lemma to

$$T \left(\begin{array}{ccc} x & \xrightarrow{R} & x' \\ \left| \begin{array}{cc} S & S \end{array} \right. & & \backslash \\ \alpha & \xrightarrow{R} & \omega \end{array} \right) T \quad T \left(\begin{array}{ccc} y & \xrightarrow{R} & y' \\ \left| \begin{array}{cc} S & S \end{array} \right. & & \backslash \\ \beta & \xrightarrow{R} & \omega \end{array} \right) T$$

we obtain $x'T\omega Ty'$; it follows that $(x', y') \in T$.

□

5 Equivalence Relations in n -permutable Categories

In this section we investigate the category $\text{Equiv}(\mathcal{E})$ of internal equivalence relations in a regular category \mathcal{E} . We show that $\text{Equiv}(\mathcal{E})$ is a $2n + 1$ -permutable category whenever \mathcal{E} is.

For $n = 1$ (see [17]) \mathcal{E} be a Goursat category. Then the category $\text{Equiv}(\mathcal{E})$ is a also a Goursat category.

Following [1], given any pair (R, S) of reflexive relations on an object X in a regular category \mathcal{E} , let us denote by $(R; S)_n$; $n \geq 2$ the alternate composition $RSRS \dots$ of length n which is a reflexive relation as well. Clearly we have: $(R, S)_n \leq (R, S)_{n+1}$ and $(S, R)_n \leq (R, S)_{n+1}$. Then call n -permutable a regular category satisfying $(R, S)_n = (S, R)_n$ for all pairs (R, S) of equivalence relations.

Theorem 5.1. ([24], Theorem 3.5 of [9]) *Let $n \geq 2$ and let \mathcal{E} be a regular category. The following statements are equivalent:*

- (i) \mathcal{E} is n -permutable category;
- (ii) $(P, P^\circ)_{n+1} \leq (P, P^\circ)_{n-1}$ for any relation P ;
- (iii) $(R, S)_n$ is an equivalence relation and is therefore $R \vee S$;
- (iv) $(1_X \wedge T)T^\circ(1_X \wedge T) \leq T^{n-1}$, for any relation T on an objet X ;
- (v) for any reflexive endorelation $E \rightrightarrows X \times X$ in \mathcal{E} , the relation $(E, E^\circ)_{n-1}$ is an equivalence relation;

- (vi) for any such reflexive endorelation E , the relation $(E, E^\circ)_{n-1}$ is transitive;
- (vii) for any such reflexive endorelation E we have $(E, E^\circ)_{n-1} = (E^\circ, E)_{n-1}$;
- (viii) $E^\circ \leq E^{n-1}$ for any reflexive relation E .

Proposition 5.2. [4] Given any $2n$ -permutable or $(2n + 1)$ -permutable regular category \mathcal{E} , any regular epimorphism $f : X \rightarrow Y$ and any equivalence relation S on X , then the reflexive relation $f(S)^n$ is an equivalence relation in \mathcal{E} .

Another characterization of $(2n + 1)$ -permutable categories in terms of (equivalence) relations is given by the preservation of equivalence relations through the regular image by a regular epimorphism as follows:

Theorem 5.3. [4] Given any regular category \mathcal{E} , TFAE:

- (i) the regular category \mathcal{E} is $(2n + 1)$ -permutable.
- (ii) for any regular epimorphism $f : X \rightarrow Y$ in \mathcal{E} and any equivalence relation S on X , then $f(S)^n$ is an equivalence relation.

The kernel pair of a morphism (f, g) in $\text{Equiv}(\mathcal{E})$ is given by the kernel pairs $Eq(f)$ of f and $Eq(g)$ of g in \mathcal{E}

$$\begin{array}{ccccc}
 Eq(g) & \xrightarrow{\frac{g_1}{g_2}} & R & \xrightarrow{g} & S \\
 \tilde{r}_1 \downarrow \tilde{r}_2 & & r_1 \downarrow r_2 & & s_1 \downarrow s_2 \\
 Eq(f) & \xrightarrow{\frac{f_1}{f_2}} & A & \xrightarrow{f} & B
 \end{array} \tag{5.1}$$

Consequently, a morphism (f, g) is a monomorphism in $\text{Equiv}(\mathcal{E})$ if and only if both f and g are monomorphisms in \mathcal{E} . When \mathcal{E} is $(2n + 1)$ -permutable category, a similar property holds with respect to regular epimorphisms:

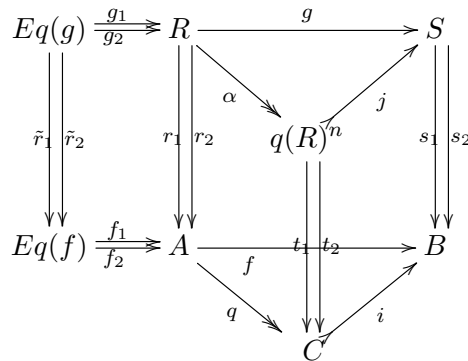
Lemma 5.4. Let R and S be two equivalence relations in a $(2n + 1)$ -permutable category \mathcal{E} and $(f, g) : R \rightarrow S$ be a morphism

$$\begin{array}{ccc}
 R & \xrightarrow{g} & S \\
 r_1 \downarrow r_2 & & s_1 \downarrow s_2 \\
 A & \xrightarrow{f} & B
 \end{array} \tag{5.2}$$

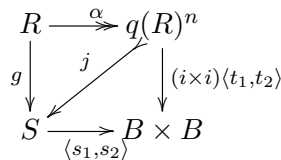
in $\text{Equiv}(\mathcal{E})$. Then (f, g) is a regular epimorphism in $\text{Equiv}(\mathcal{E})$ if and only if both f and g are regular epimorphisms in \mathcal{E} .

Proof. When f and g are regular epimorphisms in \mathcal{E} , it is not difficult to check that (f, g) is necessarily the coequalizer of its kernel pair in $\text{Equiv}(\mathcal{E})$ given in (5.1) (one uses the fact that $g = \text{coeq}(g_1, g_2)$ and $f = \text{coeq}(f_1, f_2)$ in \mathcal{E}).

Conversely, let (f, g) be a morphism in $\text{Equiv}(\mathcal{E})$ as in (5.1) that is a regular epimorphism in $\text{Equiv}(\mathcal{E})$. Consider the kernel pairs of f and g , the (regular epimorphism, monomorphism) factorization $f = iq$ of f , and the regular image $(q(R)^n, t_1, t_2)$ of (R, r_1, r_2) along q . We obtain the following commutative diagram



where $(q(R)^n, t_1, t_2) \in \text{Equiv}(\mathcal{E})$ (by Theorem 5.3) and (i, j) is the morphism in $\text{Equiv}(\mathcal{E})$ such that $(i, j)(q, \alpha) = (f, g)$. Note that j is induced by the fact that $(i \times i)\langle t_1, t_2 \rangle \alpha$ is the (regular epimorphism, monomorphism) factorization of $\langle s_1, s_2 \rangle g$, thus it is a monomorphism.



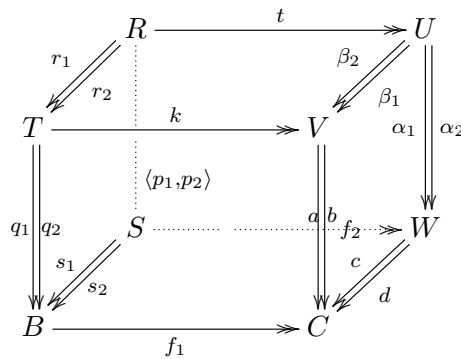
From the fact that (f, g) is the coequalizer of its kernel pair in $\text{Equiv}(\mathcal{E})$ and that (q, α) is the coequalizer of (f_1, g_1) and (f_2, g_2) in $\text{Equiv}(\mathcal{E})$ (since (i, j) is a monomorphism),

it easily follows that (i, j) is an isomorphism in $\text{Equiv}(\mathcal{E})$. This implies that f and g are regular epimorphisms in \mathcal{E} . \square

Corollary 5.5. *Let \mathcal{E} be a $(2n + 1)$ -permutable category. Then the category $\text{Equiv}(\mathcal{E})$ is a regular category.*

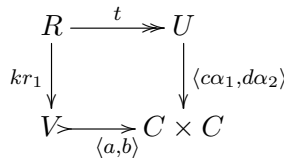
Theorem 5.6. *$\text{Equiv}(\mathcal{E})$ is a $(2n + 1)$ -permutable category whenever \mathcal{E} is.*

Proof. Let $(R, (p_1, q_1), (p_2, q_2))$ be an equivalence relation on (S, s_1, s_2) in the category $\text{Equiv}(\mathcal{E})$ and $f = (f_1, f_2)$ a regular epimorphism in $\text{Equiv}(\mathcal{E})$. We must prove that $f(R)^n$ is an equivalence relation in $\text{Equiv}(\mathcal{E})$. The relation $f(R)^n$ is obtained through the following diagram



where $U = f_2(R)^n$ and $V = f_1(T)^n$ are in $\text{Equiv}(\mathcal{E})$.

One has: $akr_1 = c\alpha_1t$ and $bkr_1 = c\alpha_2t$, thus the following diagram commutes:



Since t is a strong epimorphism and $\langle a, b \rangle$ is a monomorphism, there exists $\beta_1 : U \rightarrow V$ such that $\langle a, b \rangle \beta_1 = \langle c\alpha_1, d\alpha_2 \rangle$. The existence of β_2 is obtained similarly. Since

$$\langle \langle a, b \rangle \times \langle a, b \rangle \rangle \langle \beta_1, \beta_2 \rangle = \langle c \times c, d \times d \rangle \langle \alpha_1, \alpha_2 \rangle,$$

it follows that $\langle \beta_1, \beta_2 \rangle : U \rightarrow V \times V$ is a monomorphism and thus (U, β_1, β_2) is a relation on V . All parallel morphisms of the left face represent equivalence relations and all horizontal morphisms are regular epimorphisms, so that all parallel morphisms of the right face also represent equivalence relations (Theorem 5.3), and then $f(R)^n$ is an equivalence relation in $\text{Equiv}(\mathcal{E})$. \square

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