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Strong Insertion of a Contra-continuous Function between Two Comparable Contra-precontinuous (Contra-semi-continuous) of Real-valued Functions

Majid Mirmiran[1] and Binesh Naderi[2]

[1]Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran
e-mail: mirmir@sci.ui.ac.ir

[2]Department of General Courses, School of Management and Medical Information Sciences, Isfahan University of Medical Sciences, Isfahan, Iran
e-mail: naderi@mng.mui.ac.ir

Abstract

Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense or nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term, preopen, was used for the first time by Mashhour et al. [21], while the concept of a, locally dense, set was introduced by Corson and Michael [4].

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The concept of a semi-open set in a topological space was introduced by Levine in 1963 [18]. A subset A of a topological space (X, τ) is called *semi-open* [10] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

A generalized class of closed sets was considered by Maki in [20]. He investigated the sets that can be represented as union of closed sets and called them *V*-sets. Complements of *V*-sets, i.e., sets that are intersection of open sets are called Λ-sets [20].

Recall that a real-valued function *f* defined on a topological space *X* is called *A*-continuous [28] if the preimage of every open subset of $\mathbb{R}$ belongs to *A*, where *A* is a collection of subsets of *X*. Most of the definitions of function used throughout this paper are consequences of the definition of *A*-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev in [6] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 12, 13, 26].

Hence, a real-valued function *f* defined on a topological space *X* is called *contra-continuous* (resp. *contra-semi-continuous*, *contra-precontinuous*) if the preimage of every open subset of $\mathbb{R}$ is closed (resp. semi-closed, pre-closed) in *X* [6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that Λ-sets or kernel of sets are open [20].

If *g* and *f* are real-valued functions defined on a space *X*, we write $g \leq f$ in case $g(x) \leq f(x)$ for all *x* in *X*.

The following definitions are modifications of conditions considered in [16].

A property *P* defined relative to a real-valued function on a topological space is a *cc-property* provided that any constant function has property *P* and provided that the sum of a function with property *P* and any contra-continuous function also has property *P*. If P_1

and P_2 are *cc*-properties, the following terminology is used: (i) A space *X* has the *weak cc-insertion property* for (P_1, P_2) if and only if for any functions *g* and *f* on *X* such that $g \leq f$, *g* has property P_1 and *f* has property P_2, then there exists a contra-continuous function *h* such that $g \leq h \leq f$. (ii) A space *X* has the *strong cc-insertion property* for (P_1, P_2) if and only if for any functions *g* and *f* on *X* such that $g \leq f$, *g* has property P_1 and *f* has property P_2, then there exists a contra-continuous function *h* such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any *x* in *X*, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose Λ-sets or kernel of sets are open, is given a sufficient condition for the weak *cc*-insertion property. Also for a space with the weak *cc*-insertion property, we give necessary and sufficient conditions for the space to have the strong *cc*-insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the insertion and strong insertion of a contra-α-continuous function and insertion of a contra-continuous function between two comparable real-valued functions has also recently considered by the authors in [22, 23, 24].

2. The Main Result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

The abbreviations *cc*, *cpc* and *csc* are used for contra-continuous, contra-precontinuous and contra-semi-continuous, respectively.

Definition 2.1. Let *A* be a subset of a topological space (X, τ). We define the subsets A^{Λ} and A^{V} as follows:

$$A^{\Lambda} = \bigcap\{O : O \supseteq A, O \in (X, \tau)\} \text{ and } A^{V} = \bigcup\{F : F \subseteq A, F^{c} \in (X, \tau)\}.$$

In [7, 19, 25], A^{Λ} is called the *kernel* of *A*.

The family of all preopen, preclosed, semi-open and semi-closed will be denoted by $pO(X, \tau)$, $pC(X, \tau)$, $sO(X, \tau)$, and $sC(X, \tau)$, respectively.

We define the subsets $p(A^{\Lambda})$, $p(A^{V})$, $s(A^{\Lambda})$ and $s(A^{V})$ as follows:

$$p(A^{\Lambda}) = \bigcap\{O : O \supseteq A, O \in pO(X, \tau)\},$$

$$p(A^V) = \bigcup\{F : F \subseteq A,\ F \in pC(X, \tau)\},$$

$$s(A^\Lambda) = \bigcap\{O : O \supseteq A,\ O \in sO(X, \tau)\}$$

and

$$s(A^V) = \bigcup\{F : F \subseteq A,\ F \in sC(X, \tau)\}.$$

$p(A^\Lambda)$ (resp. $s(A^\Lambda)$) is called the *prekernel* (resp. *semi-kernel*) of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S, then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

(1) If $A_i\ \rho\ B_j$ for any $i \in \{1, ..., m\}$ and for any $j \in \{1, ..., n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1, ..., m\}$ and any $j \in \{1, ..., n\}$.

(2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.

(3) If $A\ \rho\ B$, then $A^\Lambda \subseteq B$ and $A \subseteq B^V$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. *Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the* *power set of X and if there exist lower indefinite cut sets* $A(f, t)$ *and* $A(g, t)$ *in the domain of f and g at the level t for each rational number t such that if* $t_1 < t_2$, *then* $A(f, t_1)\ \rho\ A(g, t_2)$, *then there exists a contra-continuous function h defined on X such that* $g \le h \le f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \le f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1)\ \rho\ A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2)$, $G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [15] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2)$, $H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \le h \le f$: If x is in $H(t)$, then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \le t'$, it follows that $g(x) \le t$. Hence $g \le h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \ge t$. Hence $h \le f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1, t_2)$ is closed in X, i.e., h is a contra-continuous function on X. □

The above proof used the technique of Theorem 1 in [14].

If a space has the strong cc-insertion property for (P_1, P_2), then it has the weak cc-insertion property for (P_1, P_2). The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak cc-insertion property to satisfy the strong cc-insertion property.

Theorem 2.2. *Let P_1 and P_2 be cc-property and X be a space that satisfies the weak cc-insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g \leq f$, g has property P_1 and f has property P_2. The space X has the strong cc-insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f-g, 2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that* (i) *for each n, F_n and $A(f-g, 2^{-n})$ are completely separated by contra-continuous functions, and* (ii) $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.

Proof. Suppose that there is a sequence $(A(f-g, 2^{-n}))$ of lower cut sets for $f-g$ and suppose that there is a sequence (F_n) of subsets of X such that

$$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and such that for each n, there exists a contra-continuous function k_n on X into $[0, 2^{-n}]$ with $k_n = 2^{-n}$ on F_n and $k_n = 0$ on $A(f-g, 2^{-n})$. The function k from X into $[0, 1/4]$ which is defined by

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x)$$

is a contra-continuous function by the Cauchy condition and the properties of contra-continuous functions, (1) $k^{-1}(0) = \{x \in X : (f-g)(x) = 0\}$ and (2) if $(f-g)(x) > 0$, then $k(x) < (f-g)(x)$: In order to verify (1), observe that if $(f-g)(x) = 0$, then $x \in A(f-g, 2^{-n})$ for each n and hence $k_n(x) = 0$ for each n. Thus $k(x) = 0$. Conversely, if $(f-g)(x) > 0$, then there exists an n such that $x \in F_n$ and hence $k_n(x) = 2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that

$$\{x \in X : (f-g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f-g, 2^{-n})$$

and that $(A(f-g, 2^{-n}))$ is a decreasing sequence. Thus if $(f-g)(x)>0,$ then either $x \notin A(f-g, 1/2)$ or there exists a smallest n such that $x \notin A(f-g, 2^{-n})$ and $x \in A(f-g, 2^{-j})$ for $j = 1, \ldots, n-1.$

In the former case,

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x) \le 1/4 \sum_{n=1}^{\infty} 2^{-n} < 1/2 \le (f-g)(x),$$

and in the latter,

$$k(x) = 1/4 \sum_{j=n}^{\infty} k_j(x) \le 1/4 \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \le (f-g)(x).$$

Thus $0 \le k \le f-g$ and if $(f-g)(x) > 0,$ then $(f-g)(x) > k(x) > 0.$ Let $g_1 = g + (1/4)k$ and $f_1 = f - (1/4)k.$ Then $g \le g_1 \le f_1 \le f$ and if $g(x) < f(x),$ then

$$g(x) < g_1(x) < f_1(x) < f(x).$$

Since P_1 and P_2 are *cc*-properties, then g_1 has property P_1 and f_1 has property P_2. Since by hypothesis *X* has the weak *cc*-insertion property for $(P_1, P_2),$ then there exists a contra-continuous function h such that $g_1 \le h \le f_1.$ Thus $g \le h \le f$ and if $g(x) < f(x),$ then $g(x) < h(x) < f(x).$ Therefore *X* has the strong *cc*-insertion property for $(P_1, P_2).$ (The technique of this proof is by Lane [16].)

Conversely, assume that *X* satisfies the strong *cc*-insertion for $(P_1, P_2).$ Let g and f be functions on *X* satisfying P_1 and P_2 respectively such that $g \le f.$ Thus there exists a contra-continuous function h such that $g \le h \le f$ and such that if $g(x) < f(x)$ for any x in *X*, then $g(x) < h(x) < f(x).$ We follow an idea contained in Powderly [27]. Now consider the functions 0 and $f-h.$ 0 satisfies property P_1 and $f-h$ satisfies property $P_2.$ Thus there exists a contra-continuous function h_1 such that $0 \le h_1 \le f-h$ and if $0 < (f-h)(x)$ for any x in *X*, then $0 < h_1(x) < (f-h)(x).$ We next show that

$$\{x \in X : (f-g)(x) > 0\} = \{x \in X : h_1(x) > 0\}.$$

If x is such that $(f - g)(x) > 0,$ then $g(x) < f(x).$ Therefore $g(x) < h(x) < f(x).$ Thus $f(x) - h(x) > 0$ or $(f - h)(x) > 0.$ Hence $h_1(x) > 0.$ On the other hand, if $h_1(x) > 0,$ then since $(f - h) \geq h_1$ and $f - g \geq f - h,$ therefore $(f - g)(x) > 0.$ For each n, let

$$A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\},$$

$$F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\}$$

and

$$k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.$$

Since $\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\},$ it follows that

$$\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.$$

We next show that k_n is a contra-continuous function which completely separates F_n and $A(f - g, 2^{-n})$. From its definition and by the properties of contra-continuous functions, it is clear that k_n is a contra-continuous function. Let $x \in F_n.$ Then, from the definition of $k_n,$ $k_n(x) = 2^{-n}.$ If $x \in A(f - g, 2^{-n}),$ then since $h_1 \leq f - h \leq f - g,$ $h_1(x) \leq 2^{-n}.$ Thus $k_n(x) = 0,$ according to the definition of $k_n.$ Hence k_n completely separates F_n and $A(f - g, 2^{-n})$. □

Theorem 2.3. *Let P_1 and P_2 be cc-properties and assume that the space X satisfied the weak cc-insertion property for (P_1, P_2). The space X satisfies the strong cc-insertion property for (P_1, P_2) if and only if X satisfies the strong cc-insertion property for (P_1, cc) and for (cc, P_2).*

Proof. Assume that X satisfies the strong cc-insertion property for (P_1, cc) and for (cc, P_2). If g and f are functions on X such that $g \leq f,$ g satisfies property $P_1,$ and f satisfies property $P_2,$ then since X satisfies the weak cc-insertion property for (P_1, P_2) there is a contra-continuous function k such that $g \leq k \leq f.$ Also, by hypothesis there exist contra-continuous functions h_1 and h_2 such that $g \leq h_1 \leq k$ and if $g(x) < k(x),$

then $g(x) < h_1(x) < k(x)$ and such that $k \leq h_2 \leq f$ and if $k(x) < f(x)$, then $k(x) < h_2(x) < f(x)$. If a function h is defined by $h(x) = (h_2(x) + h_1(x))/2$, then h is a contra-continuous function, $g \leq h \leq f$, and if $g(x) < f(x)$, then $g(x) < h(x) < f(x)$. Hence X satisfies the strong cc-insertion property for (P_1, P_2).

The converse is obvious since any contra-continuous function must satisfy both properties P_1 and P_2. (The technique of this proof is by Lane [17].) □

3. Applications

Before stating the consequences of Theorems 2.1, 2.2, and 2.3 we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. *If for each pair of disjoint preopen* (*resp. semi-open*) *sets* G_1, G_2 *of* X, *there exist closed sets* F_1 *and* F_2 *of* X *such that* $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then* X *has the weak cc-insertion property for* (cpc, cpc) (*resp.* (csc, csc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are cpc (resp. csc), and $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $p(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $s(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a preopen (resp. semi-open) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. semi-closed) set, it follows that $p(A(f, t_1)^{\Lambda}) \subseteq p(A(g, t_2)^{V})$ (resp. $s(A(f, t_1)^{\Lambda}) \subseteq s(A(g, t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f, t_1)\, \rho\, A(g, t_2)$. The proof follows from Theorem 2.1. □

Corollary 3.2. *If for each pair of disjoint preopen* (*resp. semi-open*) *sets* G_1, G_2, *there exist closed sets* F_1 *and* F_2 *such that* $G_1 \subseteq F_1, G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then every contra-precontinuous* (*resp. contra-semi-continuous*) *function is contra-continuous.*

Proof. Let f be a real-valued contra-precontinuous (resp. contra-semi-continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function h such that $g = h = f$. □

Corollary 3.3. *If for each pair of disjoint preopen (resp. semi-open) sets* G_1, G_2 *of* X, *there exist closed sets* F_1 *and* F_2 *of* X *such that* $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \emptyset$, *then X has the cc-insertion property for* (cpc, cpc) *(resp.* (csc, csc)*).*

Proof. Let g and f be real-valued functions defined on the X, such that f and g are *cpc* (resp. *csc*), and $g < f$. Set $h = (f + g)/2$, thus $g \le h \le f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are contra-continuous functions hence h is a contra-continuous function. □

Corollary 3.4. *If for each pair of disjoint subsets* G_1, G_2 *of X, such that* G_1 *is preopen and* G_2 *is semi-open, there exist closed subsets* F_1 *and* F_2 *of X such that* $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \emptyset$, *then X have the weak cc-insertion property for* (cpc, csc) *and* (csc, cpc).

Proof. Let g and f be real-valued functions defined on X, such that g is *cpc* (resp. *csc*) and f is *csc* (resp. *cpc*), with $g \le f$. If a binary relation ρ is defined by $A \, \rho \, B$ in case $s(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $p(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \le t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \le t_1\}$ is a semi-open (resp. preopen) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. semi-closed) set, it follows that $s(A(f, t_1)^{\Lambda}) \subseteq p(A(g, t_2)^{V})$ (resp. $p(A(f, t_1)^{\Lambda}) \subseteq s(A(g, t_2)^{V})$). Hence $t_1 < t_2$, implies that $A(f, t_1) \, \rho \, A(g, t_2)$. The proof follows from Theorem 2.1. □

Before stating consequences of Theorems 2.2, 2.3 we state and prove the necessary lemmas.

Lemma 3.1. *The following conditions on the space X are equivalent*:

(i) *For each pair of disjoint subsets* $G_1,\ G_2$ *of X, such that* G_1 *is preopen and* G_2 *is semi-open, there exist closed subsets* $F_1,\ F_2$ *of X such that* $G_1 \subseteq F_1,\ G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$.

(ii) *If G is a semi-open* (*resp. preopen*) *subset of X which is contained in a preclosed* (*resp. semi-closed*) *subset F of X, then there exists a closed subset H of X such that* $G \subseteq H \subseteq H^{\Lambda} \subseteq F$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where *G* and *F* are semi-open (resp. preopen) and preclosed (resp. semi-closed) subsets of *X*, respectively. Hence, F^c is a preopen (resp. semi-open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint closed subsets $F_1,\ F_2$ such that $G \subseteq F_1$ and $F^C \subseteq F_2$. But

$$F^C \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an open subset containing F_1, we conclude that $F_1^{\Lambda} \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^{\Lambda} \subseteq F.$$

By setting $H = F_1$, condition(ii) holds.

(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint subsets of *X*, such that G_1 is preopen and G_2 is semi-open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is a preclosed subset of *X*. Hence by (ii) there exists a closed set *H* such that $G_2 \subseteq H \subseteq H^{\Lambda} \subseteq G_1^c$.

But

$$H \subseteq H^{\Lambda} \Rightarrow H \cap (H^{\Lambda})^c = \varnothing$$

and

$$H^{\Lambda} \subseteq G_1^c \Rightarrow G_1 \subseteq (H^{\Lambda})^c.$$

Furthermore, $(H^{\Lambda})^c$ is a closed subset of X. Hence $G_2 \subseteq H$, $G_1 \subseteq (H^{\Lambda})^c$ and $H \cap (H^{\Lambda})^c = \varnothing$. This means that condition (i) holds. □

Lemma 3.2. *Suppose that X is a topological space. If each pair of disjoint subsets* G_1, G_2 *of X, where* G_1 *is preopen and* G_2 *is semi-open, can be separated by closed subsets of X, then there exists a contra-continuous function* $h : X \to [0, 1]$ *such that* $h(G_2) = \{0\}$ *and* $h(G_1) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is semi-open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is a preclosed subset of X containing the semi-open subset G_2 of X, by Lemma 3.1, there exists a closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq G_1^c.$$

Note that $H_{1/2}$ is also a preclosed subset of X and contains G_2, and G_1^c is a preclosed subset of X and contains the semi-open subset $H_{1/2}^{\Lambda}$ of X. Hence, by Lemma 3.1, there exists closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq H_{1/4}^{\Lambda} \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq H_{3/4} \subseteq H_{3/4}^{\Lambda} \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X$, $0 \leq h(x) \leq 1$, i.e., h maps X into $[0, 1]$. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contra-continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$, then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$, then $\{x \in X : h(x) < \alpha\} = \bigcup\{H_t : t < \alpha\}$, hence, they are closed subsets of X. Similarly, if $\alpha < 0$, then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$, then $\{x \in X : h(x) > \alpha\} = \bigcup\{(H_t^{\Lambda})^c : t > \alpha\}$ hence, every of them is a closed subset. Consequently h is a contra-continuous function. □

Lemma 3.3. *Suppose that X is a topological space. If each pair of disjoint subsets* G_1, G_2 *of X, where* G_1 *is preopen and* G_2 *is semi-open, can separate by closed subsets of X, and* G_1 *(resp.* G_2*) is a closed subsets of X, then there exists a contra-continuous function* $h : X \to [0, 1]$ *such that,* $h^{-1}(0) = G_1$ *(resp.* $h^{-1}(0) = G_2$*) and* $h(G_2) = \{1\}$ *(resp.* $h(G_1) = \{1\}$*).*

Proof. Suppose that G_1 (resp. G_2) is a closed subset of X. By Lemma 3.2, there exists a contra-continuous function $h : X \to [0, 1]$ such that, $h(G_1) = \{0\}$ (resp. $h(G_2) = \{0\}$) and $h(X \setminus G_1) = \{1\}$ (resp. $h(X \setminus G_2) = \{1\}$). Hence, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and since $G_2 \subseteq X \setminus G_1$ (resp. $G_1 \subseteq X \setminus G_2$), therefore $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$). □

Lemma 3.4. *Suppose that X is a topological space such that every two disjoint semi-open and preopen subsets of X can be separated by closed subsets of X. The following conditions are equivalent:*

(i) *For every two disjoint subsets* G_1 *and* G_2 *of X, where* G_1 *is preopen and* G_2 *is semi-open, there exists a contra-continuous function* $h : X \to [0, 1]$ *such that,* $h^{-1}(0) = G_1$ *(resp.* $h^{-1}(0) = G_2$*) and* $h^{-1}(1) = G_2$ *(resp.* $h^{-1}(1) = G_1$*).*

(ii) *Every preopen (resp. semi-open) subset of X is a closed subsets of X.*

(iii) *Every preclosed (resp. semi-closed) subset of X is an open subsets of X.*

Proof. (i) $\Rightarrow$ (ii) Suppose that G is a preopen (resp. semi-open) subset of X. Since $\varnothing$ is a semi-open (resp. preopen) subset of X, by (i) there exists a contra-continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G$. Set $F_n = \left\{x \in X : h(x) < \frac{1}{n}\right\}$. Then for every $n \in \mathbb{N}$, F_n is a closed subset of X and $\bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G$.

(ii) $\Rightarrow$ (i) Suppose that G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is semi-open. By Lemma 3.3, there exists a contra-continuous function $f : X \to [0, 1]$ such that, $f^{-1}(0) = G_1$ and $f(G_2) = \{1\}$. Set $G = \left\{x \in X : f(x) < \frac{1}{2}\right\}$, $F = \left\{x \in X : f(x) = \frac{1}{2}\right\}$, and $H = \left\{x \in X : f(x) > \frac{1}{2}\right\}$. Then $G \bigcup F$ and $H \bigcup F$ are two open subsets of X and $(G \bigcup F) \bigcap G_2 = \varnothing$. By Lemma 3.3, there exists a contra-continuous function $g : X \to \left[\frac{1}{2}, 1\right]$ such that, $g^{-1}(1) = G_2$ and $g(G \bigcup F) = \left\{\frac{1}{2}\right\}$. Define h by $h(x) = f(x)$ for $x \in G \bigcup F$, and $h(x) = g(x)$ for $x \in H \bigcup F$. Then h is well-defined and a contra-continuous function, since $(G \bigcup F) \bigcap (H \bigcup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \bigcup F) \bigcup (H \bigcup F) = X$, hence h defined on X and maps to $[0, 1]$. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every open subset of X is a closed subset of X and complement of every closed subset of X is an open subset of X, the equivalence is hold. □

Corollary 3.5. *If for every two disjoint subsets* G_1 *and* G_2 *of* X, *where* G_1 *is preopen* (*resp. semi-open*) *and* G_2 *is semi-open* (*resp. preopen*), *there exists a contra-continuous function* $h : X \to [0, 1]$ *such that,* $h^{-1}(0) = G_1$ *and* $h^{-1}(1) = G_2$, *then* X *has the strong cc-insertion property for* (cpc, csc) (*resp.* (csc, cpc)).

Proof. Since for every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen (resp. semi-open) and G_2 is semi-open (resp. preopen), there exists a contra-continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define

$F_1 = \left\{x \in X : h(x) < \frac{1}{2}\right\}$ and $F_2 = \left\{x \in X : h(x) > \frac{1}{2}\right\}$. Then F_1 and F_2 are two disjoint closed subsets of *X* that contain G_1 and G_2, respectively. Hence by Corollary 3.4, *X* has the weak *cc*-insertion property for (cpc, csc) and (csc, cpc). Now, assume that *g* and *f* are functions on *X* such that $g \leq f$, *g* is *cpc* (resp. *csc*) and *f* is *cc*. Since $f - g$ is *cpc* (resp. *csc*), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a preopen (resp. semi-open) subset of *X*. Now setting $H_n = \{x \in X : (f - g)(x) > 2^{-n}\}$ for every $n \in \mathbb{N}$, then by Lemma 3.4, H_n is an open subset of *X* and we have $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ and for every $n \in \mathbb{N}$, H_n and $A(f - g, 2^{-n})$ are disjoint subsets of *X*. By Lemma 3.2, H_n and $A(f - g, 2^{-n})$ can be completely separated by contra-continuous functions. Hence by Theorem 2.2, *X* has the strong *cc*-insertion property for (cpc, cc) (resp. (csc, cc)).

By an analogous argument, we can prove that *X* has the strong *cc*-insertion property for (cc, csc) (resp. (cc, cpc)). Hence, by Theorem 2.3, *X* has the strong cc-insertion property for (cpc, csc) (resp. (csc, cpc)). □

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