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Third Hankel Determinant Problem for Certain Subclasses of Analytic Functions Associated with Nephroid Domain

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Abstract

In this research article we consider two well known subclasses of starlike and bounded turning functions associated with nephroid domain. Our aims to find third Hankel determinant for these classes.

1 Introduction and Definitions

Let $\mathcal{A}$ be the collections of all normalized analytic functions defined in the unit disc $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathfrak{D}). \tag{1.1}$$

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Let us denote the most basic, well known and important subclass of class $\mathcal{A}$ by $\mathcal{S}$ which consists of all univalent functions in $\mathfrak{D}$. Since in the early stage of 20th century, researchers have been interested in coefficients of function f in class $\mathcal{A}$. In year 1916, Bieberbach see [1] was first to discover a coefficient conjecture for the function $f \in \mathcal{S}$ and finally in year 1985 De-Branges [2] solved Bieberbach coefficient conjecture. In era 1916 to 1985 many researcher have tried to prove or disprove this conjecture and they discovered different subclasses of the class $\mathcal{S}$ associated with different image domains. The definition of class $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ which are subclasses of class $\mathcal{S}$ can be written in terms of subordination

$$\begin{aligned} \mathcal{S}^* &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathfrak{D} \right\}, \\ \mathcal{C} &= \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathfrak{D} \right\}, \\ \mathcal{R} &= \left\{ f \in \mathcal{S} : f'(z) \prec \frac{1+z}{1-z}, \quad z \in \mathfrak{D} \right\}, \end{aligned} \tag{1.2}$$

where "$\prec$" represent subordination. Two analytic functions f and g, a function f is subordinate to g symbolically $f \prec g$ if there exist an analytic function $w(z)$ with limitation $w(0) = 0$ and $|w(z)| < |z|$ such that $f(z) = g(w(z))$. If $g \in \mathcal{S}$, then equivalence conditions

$$f(0) = g(0) \text{ and } f(\mathfrak{D}) \subset g(\mathfrak{D}).$$

Let $\mathcal{P}$ denote class of all analytic functions p such that $\operatorname{Re}(p(z)) > 0$, and of the form

$$p(z) = z + \sum_{n=1}^{\infty} c_n z^n, \quad (z \in \mathfrak{D}). \tag{1.3}$$

By changing the function right hand side of subordinations in (1.2), we obtain some subclasses of the class $\mathcal{S}$ which have interesting geometric properties, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. From among these subfamilies we recall here the families that are associated with trigonometric function as follows;

$$\mathcal{S}^*_{\mathcal{N}e} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + z - \frac{1}{3}z^3 \right\}, \quad (z \in \mathfrak{D}), \tag{1.4}$$

$$\mathcal{S}^*_{\mathcal{N}e} = \left\{ f \in \mathcal{S} : f'(z) \prec 1 + z - \frac{1}{3}z^3 \right\}, \quad (z \in \mathfrak{D}). \tag{1.5}$$

Recently, authors in [18], introduced the class $\mathcal{S}^*_{\mathcal{N}e}$ which are associated with nephriod domain.

The Hankel determinant $H_{q,n}(f)$ where parameters $q, n \in \mathbb{N} = \{1, 2, 3, \cdots\}$ for function $f \in \mathcal{S}$ of the form (1.1) was introduced by Pommerenke [14, 15] as;

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.6}$$

The growth of $H_{q,n}(f)$ has been evaluated for different subcollections of univalent functions. Exceptionally, the sharp bound of the determinant $H_{2,2}(f) = \left|a_2a_4 - a_3^2\right|$ for class $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ were found by Janteng et al. [16, 17] while for the family of close-to-convex functions the sharp estimation is still unknown (see, [19]). On the other hand, for the set of Bazilevič functions, the best estimate of $|H_{2,2}(f)|$ was proved by Krishna et al. [20]. For more work on $H_{2,2}(f)$, see [21, 22, 23, 24, 25].

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{1.7}$$

is known as third order Hankel determinant and the estimation of this determinant $|H_{3,1}(f)|$ is so hard. In 2010, the first article on $H_{3,1}(f)$ by Babalola [26], in which he obtained the upper bound of $|H_{3,1}(f)|$ for the groups of $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$. Later on, a few creators distributed their work regarding $|H_{3,1}(f)|$ for various subcollections of holomorphic and univalent functions, see [34, 35, 36, 37, 38], which served as a base for the research in this field. Recently various authors explored some interesting classes for the said property of Hankel determinant.

Srivastava et al. [27] discussed this result for a class of Bi-valent functions defined by q-derivative and gave various interesting properties of it. Then he along with coauthors in [28] investigated the class of close to convex functions associated with lemniscate of Bernouli and evaluated its Hankel determinant. Continuing the same trend he in [29] incorporated the research on Toeplitz forms and Hankel determinant for some q-starlike functions associated with a generalized domain. Many other domains were also investigated for its Hankel determinant like a class of starlike functions associated with k-Fibonacci numbers. Whose third Hankel was evaluated by Shafiq et al. [30]. Further related work on the subject the reader is referred to [31, 32, 33]. Motivated from above discussed work on the topic we investigate $|H_{3,1}(f)|$ for classes of functions defined in the relations (1.4) and (1.5).

2 Sets of Lemma

The following lemmas are important as they help in our main results.

Lemma 1. *If* $p(z) \in \mathcal{P}$ *and it is of the form* (1.3), *then*

$$\begin{aligned} |c_n| &\leq 2 \text{ for } n \geq 1, && (2.1)\\ |c_{n+k} - \delta c_n c_k| &\leq \begin{cases} 2 & \text{for } 0 \leq \delta \leq 1, \\ 2|2\delta - 1| & \text{elsewhere.} \end{cases}, && (2.2)\\ |c_n c_m - c_l c_k| &\leq 4 \text{ for } n+m = l+k, && (2.3)\\ \left|c_{n+2k} - \delta c_n c_k^2\right| &\leq 2(1+2\delta) \text{ for } \delta \in \mathbb{R}, && (2.4)\\ \left|c_2 - \frac{c_1^2}{2}\right| &\leq 2 - \frac{|c_1|^2}{2}, && (2.5) \end{aligned}$$

and for $\xi \in \mathbb{C}$

$$\left|c_2 - \xi c_1^2\right| \leq 2 \max\{1; |2\xi - 1|\}. \tag{2.6}$$

For the results in (2.1), (2.2), (2.3), (2.4), (2.5) *see [44]. Also see [43] for* (2.6).

Lemma 2. *[46]. If $p(z) \in \mathcal{P}$ and is represented by* (1.3), *then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & (\nu \leq 0), \\ 2 & (0 \leq \nu \leq 1), \\ 4\nu - 2 & (\nu \geq 1). \end{cases}$$

Lemma 3. *[47]. If $p(z) \in \mathcal{P}$ and is represented by* (1.3), *then*

$$\left|ac_1^3 - bc_1c_2 + dc_3\right| \leq 2\left|a\right| + 2\left|b - 2a\right| + 2\left|a - b + d\right|.$$

3 Bounds of $\left|H_{3,1}(f)\right|$ for class $\mathcal{S}^*_{\mathcal{N}e}$

Theorem 1. *Let $f \in \mathcal{S}^*_{\mathcal{N}e}$ of the form* (1.1). *Then*

$$\begin{aligned} |a_2| &\leq 1, \\ |a_3| &\leq \frac{1}{2}, \\ |a_4| &\leq \frac{7}{18}, \\ |a_5| &\leq \frac{5}{12}. \end{aligned}$$

The first two bounds are sharp.

Proof. Since $f \in \mathcal{S}^*_{\mathcal{N}e}$, there exists an analytic function $s(z)$, $|s(z)| \leq 1$ and $s(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} = 1 + w(z) - \frac{1}{3}(w(z))^3. \tag{3.1}$$

Denote

$$\Psi(s(z)) = 1 + w(z) - \frac{1}{3}(w(z))^3,$$

and

$$k(z) = 1 + c_1z + c_2z^2 + \cdots = \frac{1 + w(z)}{1 - w(z)}. \tag{3.2}$$

Obviously, the function $k(z) \in \mathcal{P}$, and $w(z) = \frac{k(z)-1}{k(z)+1}$. This gives

$$w(z) = \frac{k(z) - 1}{k(z) + 1} = \frac{c_1z + c_2z^2 + c_3z^3 + \cdots}{2 + c_1z + c_2z^2 + c_3z^3 + \cdots}.$$

And

$$\begin{aligned} & 1+(w(z))-\frac{1}{3}(w(z))^3 \\ = \ & 1+\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+\left(\frac{1}{12}c_1^3-\frac{1}{2}c_2c_1+\frac{1}{2}c_3\right)z^3 \\ & +\left(\frac{1}{4}c_1^2c_2-\frac{1}{2}c_3c_1-\frac{1}{4}c_2^2+\frac{1}{2}c_4\right)z^4+\cdots . \end{aligned} \tag{3.3}$$

And other side,

$$\begin{aligned} \frac{zf'(z)}{f(z)} = \ & 1+a_2z+\left(2a_3-a_2^2\right)z^2+\left(a_2^3-3a_2a_3+3a_4\right)z^3+ \\ & \left(-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2+4a_5\right)z^4+\cdots . \end{aligned} \tag{3.4}$$

On equating coefficients of (3.3), and (3.4), we get

$$a_2 = \frac{1}{2}c_1, \tag{3.5}$$

$$a_3 = \frac{1}{4}c_2, \tag{3.6}$$

$$a_4 = -\frac{1}{72}c_1^3-\frac{1}{24}c_2c_1+\frac{1}{6}c_3, \tag{3.7}$$

$$a_5 = \frac{5}{576}c_1^4-\frac{1}{48}c_1^2c_2-\frac{1}{24}c_3c_1-\frac{1}{32}c_2^2+\frac{1}{8}c_4. \tag{3.8}$$

Now using (2.1) in (3.5) and (3.6), we get

$$|a_2| \leq 1 \text{ and } |a_3| \leq \frac{1}{2}.$$

Rearrange the equation (3.7), we may write

$$|a_4| = \left|\frac{1}{12}\left(c_3-\frac{1}{2}c_1c_2\right)+\frac{1}{12}\left(c_3-\frac{1}{6}c_1^3\right)\right|.$$

Using triangle inequality along with (2.2) and (2.4), we get

$$|a_4| \leq \frac{7}{18}.$$

Now rearrange the (3.8), we may write

$$|a_5| = \left|\frac{1}{16}\left(c_4-\frac{2}{3}c_1c_3\right)+\frac{1}{16}\left(c_4-\frac{1}{2}c_2^2\right)-\frac{c_1^2}{48}\left(c_2-\frac{5}{12}c_1^2\right)\right|$$

Application of triangle inequality along with (2.1) and (2.2), we get

$$|a_5| \leq \frac{5}{12}.$$

The first two bounds are sharp for function define as

$$f(z) = z \exp\left(z - \frac{z^3}{9}\right) = z + z^2 + \frac{z^3}{2} + \cdots.$$

□

Theorem 2. *Let $f(z) \in \mathcal{S}^*_{\mathcal{N}e}$ be of the form* (1.1). *Then*

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} \frac{1-2\lambda}{2}, & \lambda \leq 0 \\ \frac{1}{2}, & 0 \leq \lambda \leq 1 \\ \frac{2\lambda-1}{2}, & \lambda \geq 1. \end{cases}.$$

Proof. Since using (3.5) and (3.6), we get

$$\left|a_3 - \lambda a_2^2\right| = \frac{1}{4}\left|c_2 - \lambda c_1^2\right|,$$

applying Lemma 2, we get the required results. □

Theorem 3. *Let $f(z) \in \mathcal{S}^*_{\mathcal{N}e}$ be of the form* (1.1). *Then for $\xi \in \mathbb{C}$, we have*

$$\left|a_3 - \xi a_2^2\right| \leq \frac{1}{2}\max\{1, |2\xi - 1|\}.$$

Proof. Since using (3.5) and (3.6), we get

$$\left|a_3 - \xi a_2^2\right| = \frac{1}{4}\left|c_2 - \xi c_1^2\right|,$$

applying Lemma 2.6, we get the required results. □

If we put $\xi = 1$, the above result become as:

Theorem 4. *Let $f(z) \in \mathcal{S}^*_{\mathcal{N}e}$ be of the form* (1.1). *Then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2}.$$

This results is sharp.

Theorem 5. *Let $f(z) \in \mathcal{S}^*_{\mathcal{N}e}$ be of the form* (1.1). *Then*

$$|a_2a_3 - a_4| \leq \frac{7}{18}.$$

Proof. Since using (3.5), (3.6) and (3.7) also rearranging term, we get

$$|a_2a_3 - a_4| = \left| \frac{1}{12}(c_3 - c_2c_1) + \frac{1}{12}\left(c_3 - \frac{1}{6}c_1^3\right) \right|,$$

Using triangle inequality along with (2.2) and (2.4), we get

$$|a_2a_3 - a_4| \leq \frac{7}{18}.$$

□

Theorem 6. *Let $f(z) \in \mathcal{S}^*_{\mathcal{N}e}$ be of the form* (1.1). *Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9}.$$

Proof. Since using (3.5), (3.6) and (3.7), we get

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &= \left| -\frac{1}{144}c_1^4 - \frac{1}{48}c_1^2c_2 + \frac{1}{12}c_3c_1 - \frac{1}{16}c_2^2 \right| \\ &= \left| \frac{1}{16}\left(c_1c_3 - c_2^2\right) + \frac{1}{48}c_1\left(c_3 - c_1c_2\right) - \frac{1}{144}c_1^4 \right|, \end{aligned}$$

applying (2.1), (2.2) and (2.3), we get the required result. □

Theorem 7. *Let $f(z) \in \mathcal{S}^*_{\mathcal{N}e}$ be of the form* (1.1). *Then*

$$|H_{3,1}(f)| \leq \frac{377}{648} \simeq 0.581\,79.$$

Proof. Since

$$H_{3,1}(f) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right),$$

by applying triangle inequality, we obtain

$$|H_{3,1}(f)| \leq |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\left|a_4 - a_2a_3\right| + |a_5|\left|a_3 - a_2^2\right|.$$

Next, from Theorems 1, 4, 5 and 6, we get the required result. □

4 Bounds of $|H_{3,1}(f)|$ for class $\mathcal{R}_{\mathcal{N}e}$

Theorem 8. *Let* $f \in \mathcal{R}_{\mathcal{N}e}$ *of the form* (1.1). *Then*

$$\begin{aligned} |a_2| &\leq \frac{1}{2}, \\ |a_3| &\leq \frac{1}{3}, \\ |a_4| &\leq \frac{1}{4}, \\ |a_5| &\leq \frac{3}{5}. \end{aligned}$$

Proof. Since $f \in \mathcal{R}_{\mathcal{N}e}$, there exists an analytic function $w(z)$, $|w(z)| \leq 1$ and $w(0) = 0$, such that

$$f'(z) = 1 + w(z) - \frac{1}{3}(w(z))^3.$$

And

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 3a_4 z^3 + 4a_5 z^4 + \cdots. \tag{4.1}$$

On equating coefficients of (3.3), and (4.1), we get $\left(\frac{1}{6}c_2 - \frac{1}{12}c_1^2\right)^2 = \frac{1}{144}c_1^4 - \frac{1}{36}c_1^2 c_2 + \frac{1}{36}c_2^2$

$$a_2 = \frac{1}{4}c_1, \tag{4.2}$$

$$a_3 = \frac{1}{6}c_2 - \frac{1}{12}c_1^2, \tag{4.3}$$

$$a_4 = \frac{1}{48}c_1^3 - \frac{1}{8}c_2 c_1 + \frac{1}{8}c_3, \tag{4.4}$$

$$a_5 = \frac{1}{20}c_1^2 c_2 - \frac{1}{10}c_3 c_1 - \frac{1}{20}c_2^2 + \frac{1}{10}c_4. \tag{4.5}$$

Now using (2.1) to (4.2), we get

$$|a_2| \leq \frac{1}{2}.$$

Using (2.5) to (4.3), we obtain

$$|a_3| \leq \frac{1}{3}.$$

Application of Lemma 3 to (4.4), lead us to

$$|a_4| \leq \frac{1}{4}.$$

Rearranging the (4.5), we have

$$|a_5| = \left| \frac{1}{10}\left(c_4 - \frac{1}{2}c_2^2\right) - \frac{c_1}{10}\left(c_3 - \frac{1}{2}c_1c_2\right) \right|,$$

applying (2.1) and (2.2), we get

$$|a_5| \leq \frac{3}{5}.$$

□

Theorem 9. *Let $f(z) \in \mathcal{R}_{\mathcal{N}e}$ be of the form* (1.1)*. Then for $\xi \in \mathbb{C}$, we have*

$$\left|a_3 - \xi a_2^2\right| \leq \frac{1}{3}\max\left\{1, \frac{3\,|\xi|}{4}\right\}.$$

Proof. Since using (4.2) and (4.3), we get

$$\left|a_3 - \xi a_2^2\right| = \frac{1}{6}\left|c_2 - \frac{4+3\xi}{8}c_1^2\right|,$$

application of relation (2.6), we get the required results. □

If we put $\xi = 1$, the above result become as:

Theorem 10. *Let $f(z) \in \mathcal{R}_{\mathcal{N}e}$ be of the form* (1.1)*. Then*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{3}.$$

Theorem 11. *Let $f(z) \in \mathcal{R}_{\mathcal{N}e}$ be of the form* (1.1)*. Then*

$$|a_2a_3 - a_4| \leq \frac{1}{4}.$$

Proof. Since using (4.2), (4.3) and (4.4), we get

$$|a_2a_3 - a_4| = \left|\frac{1}{24}c_1^3 - \frac{1}{6}c_2c_1 + \frac{1}{8}c_3\right|,$$

using Lemma 3, we have

$$|a_2a_3 - a_4| \leq \frac{1}{4}.$$

□

Theorem 12. *Let $f(z) \in \mathcal{R}_{\mathcal{N}e}$ be of the form* (1.1)*. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{11}{72}.$$

Proof. Since using (4.2), (4.3) and (4.4), we get

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &= \left|-\frac{1}{576}c_1^4 - \frac{1}{288}c_1^2c_2 + \frac{1}{32}c_3c_1 - \frac{1}{36}c_2^2\right| \\ &= \left|\frac{1}{36}\left(c_1c_3 - c_2^2\right) + \frac{1}{288}c_1\left(c_3 - c_1c_2\right) - \frac{1}{576}c_1^4\right|, \end{aligned}$$

application of triangle inequality along with (2.1), (2.2) and (2.3), we get the required result. □

Theorem 13. *Let $f(z) \in \mathcal{R}_{\mathcal{N}e}$ be of the form* (1.1)*. Then*

$$\left|H_{3,1}(f)\right| \leq \frac{677}{2160} \simeq 0.313\,43.$$

Proof. Since

$$H_{3,1}(f) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right),$$

by applying triangle inequality, we obtain

$$\left|H_{3,1}(f)\right| \leq |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\left|a_4 - a_2a_3\right| + |a_5|\left|a_3 - a_2^2\right|.$$

Next, from Theorems 8, 10, 11 and 12, we get the required result. □

Authors contributions

All authors jointly worked on the results and they read and approved the final manuscript.

Conflict of Interest

The authors declare that there is no conflict of interests.

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