Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 6, Number 2, 2021, Pages 225-237
https://doi.org/10.34198/ejms.6221.225237

Special Subfamilies of Holomorphic and Bi-univalent Functions related to Quasi-subordination

S. R. Swamy[1,*] and Y. Sailaja[2]

[1] Department of Computer Science and Engineering, RV College of Engineering,
Bengaluru - 560 059, Karnataka, India
ORCID: https://orcid.org/0000-0002-8088-4103
e-mail: mailtoswamy@rediffmail.com

[2] Department of Mathematics, RV College of Engineering, Bengaluru - 560 059, Karnataka, India
ORCID: https://orcid.org/0000-0002-9155-9146
e-mail: sailajay@rvce.edu.in

Abstract

In the current work, special subfamilies of holomorphic bi-univalent functions based on quasi-subordination are introduced. Initial coefficient estimates for functions belonging to these subfamilies are established. Several consequences of our results and connections to known families are indicated.

1 Preliminaries

Let $\mathcal{A}$ be the set of normalized holomorphic functions that have the form

$$s(z) = z + \sum_{k=2}^{\infty} d_k z^k, \tag{1.1}$$

in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the set of all elements of $\mathcal{A}$ that are univalent in $\mathfrak{D}$. Let $\varsigma(z)$ be holomorphic in $\mathfrak{D}$ with $|\varsigma(z)| \leq 1$, $z \in \mathfrak{D}$, such that

$$\varsigma(z) = R_0 + R_1 z + R_2 z^2 + \cdots \tag{1.2}$$

Received: February 25, 2021; Accepted: March 7, 2021
2010 Mathematics Subject Classification: 30C45, 30C50, 30D60.
Keywords and phrases: quasi-subordination, holomorphic function, coefficient estimates, bi-univalent function.

*Corresponding author

where $R_0, R_1, R_2, \cdots$ are real and let $\mathfrak{h}(z)$ be holomorphic in $\mathfrak{D}$, with $\mathfrak{h}(0) = 1, \mathfrak{h}'(0) > 0$, having positive real part, such that

$$\mathfrak{h}(z) = 1 + Q_1 z + Q_2 z^2 + \cdots \tag{1.3}$$

where $Q_1, Q_2, Q_3, \cdots$ are real and $Q_1 > 0$. Throughout this work we shall consider ς and $\mathfrak{h}$ follow the above mentioned conditions unless otherwise mentioned. One - quarter theorem of Koebe [6], assures that the image of $\mathfrak{D}$ under every function $s \in \mathcal{S}$ contains a disc of radius 1/4. According to this, every function $s \in \mathcal{S}$ has an inverse $g = s^{-1}$ satisfying $s^{-1}(s(z)) = z$, $z \in \mathfrak{D}$ and $s(s^{-1}(\omega)) = \omega$, $|\omega| < r_0(s)$, $r_0(s) \geq 1/4$ and is in fact given by

$$g(\omega) = s^{-1}(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \cdots \tag{1.4}$$

A member s of $\mathcal{A}$ is bi-univalent in $\mathfrak{D}$ if both s and s^{-1} are univalent in $\mathfrak{D}$. We symbolize the set of bi-univalent functions of the form (1.1), by $\sum$. In [9], Lewin examined the bound for $|d_2|$ of elements of the family $\sum$ and proved that $|d_2| < 1.51$ and in [3], Brannan et al. conjectured that $|d_2| < \sqrt{2}$. Brannan and Taha in [4], proposed bi-starlike and bi-convex functions which are similar to the subfamilies of univalent functions. They have obtained non-sharp estimates on $|d_2|$ and $|d_3|$ for members of such families. For various subfamilies of the class $\sum$, coefficient estimates and other properties of functions in these subfamilies, one can refer the works of [2], [5], [12], [17], and [22].

We recall the rule of subordination and also the rule of majorization, between holomorphic functions $s(z)$ and $\tau(z)$ in $\mathfrak{D}$. We say that $s(z)$ is subordinate to $\tau(z)$, indicated as $s(z) \prec \tau(z)$, $z \in \mathfrak{D}$, if there is a $\psi(z)$ holomorphic in $\mathfrak{D}$, with $\psi(0) = 0$ and $|\psi(z)| < 1$, $z \in \mathfrak{D}$, such that $s(z) = \tau(\psi(z))$. Moreover $s(z) \prec \tau(z)$ is equivalent to $s(0) = \tau(0)$ and $s(\mathfrak{D}) \subset \tau(\mathfrak{D})$, if τ is univalent in $\mathfrak{D}$. We know that $s(z)$ is majorized by $\tau(z)$, indicated as $s(z) \prec\prec \tau(z)$, $z \in \mathfrak{D}$, if there exists a $\varsigma(z)$ holomorphic in $\mathfrak{D}$, with $|\varsigma(z)| \leq 1$, satisfying $s(z) = \varsigma(z)\tau(z)$, $z \in \mathfrak{D}$.

Robertson [15] innovated a concept called quasi-subordination, which generalizes both the concepts of majorization and subordination. For holomorphic functions $s(z)$ and $\tau(z)$, $s(z)$ is quasi-subordinate to $\tau(z)$, indicated as $s(z) \prec_q$

$\tau(z)$, $z \in \mathfrak{D}$, if there exists two holomorphic functions ς and ψ with $|\varsigma(z)| \leq 1, \psi(0) = 0$ and $|\psi(z)| < 1$ such that $s(z) = \varsigma(z)\tau(\psi(z))$, $z \in \mathfrak{D}$. Observe that if $\varsigma(z) = 1$, then $s(z) = \tau(\psi(z))$, $z \in \mathfrak{D}$, so that $s(z) \prec \tau(z)$ in $\mathfrak{D}$. Also note that if $\psi(z) = z$, then $s(z) = \varsigma(z)\tau(z)$, $z \in \mathfrak{D}$ and hence $s(z) \prec\prec \tau(z)$ in $\mathfrak{D}$. There are more studies related to quasi-subordination such as [1], [7], [8], [11], [14], [16], [19] and [21].

Motivated by the papers [18], [20] and earlier works on quasi-subordination, we now define new special families $\mathfrak{S}^q_{\sum}(\mu, \nu, \eta, \mathfrak{h})$ and $\mathfrak{M}^q_{\sum}(\mu, \nu, \eta, \mathfrak{h})$.

Definition 1.1. For $0 \leq \nu \leq 1$, $\mu \geq 0$, $\mu \geq \nu$, $\eta \in \mathbb{C} - \{0\}$, we say that s in $\sum$, belongs to $\mathfrak{S}^q_{\sum}(\mu, \nu, \eta, \mathfrak{h})$, if

$$\frac{1}{\eta}\left(\frac{zs'(z) + \mu z^2 s''(z)}{(1-\nu)s(z) + \nu z s'(z)} - 1\right) \prec_q (\mathfrak{h}(z) - 1),\ z \in \mathfrak{D}$$

and

$$\frac{1}{\eta}\left(\frac{\omega g'(\omega) + \mu\omega^2 g''(\omega)}{(1-\nu)g(\omega) + \nu\omega g'(\omega)} - 1\right) \prec_q (\mathfrak{h}(\omega) - 1),\ \omega \in \mathfrak{D},$$

where $\mathfrak{h}$ and $g(\omega) = s^{-1}(\omega)$ are as stated in (1.3) and (1.4) respectively.

We observe that certain values of ν and μ lead the class $\mathfrak{S}^q_{\sum}(\mu, \nu, \eta, \mathfrak{h})$ to the following few subfamilies:

1. $\mathscr{K}^q_{\sum}(\eta, \mathfrak{h}) = \mathfrak{S}^q_{\sum}(\frac{1}{2}, \frac{1}{2}, \eta, \mathfrak{h})$ is the family of $s \in \sum$ of the form (1.1) satisfying

$$\frac{1}{\eta}\left(\frac{(z^2 s'(z))'}{(zs(z))'} - 1\right) \prec_q (\mathfrak{h}(z) - 1) \quad \text{and} \quad \frac{1}{\eta}\left(\frac{(w^2 g'(\omega))'}{(\omega g(\omega))'} - 1\right) \prec_q (\mathfrak{h}(\omega) - 1),$$

where z, $\omega \in \mathfrak{D}$, $\mathfrak{h}$ and $g(\omega) = s^{-1}(\omega)$ are as stated in (1.3) and (1.4) respectively.

2. $\mathscr{I}^q_{\sum}(\eta, \mathfrak{h}) = \mathfrak{S}^q_{\sum}(\frac{1}{2}, 0, \eta, \mathfrak{h})$ is the family of $s \in \sum$ of the form (1.1) satisfying

$$\frac{1}{\eta}\left(\frac{(z^2 s'(z))'}{2s(z)} - 1\right) \prec_q (\mathfrak{h}(z) - 1) \quad \text{and} \quad \frac{1}{\eta}\left(\frac{(\omega^2 g'(\omega))'}{2g(\omega)} - 1\right) \prec_q (\mathfrak{h}(\omega) - 1),$$

where z, $\omega \in \mathfrak{D}$, $\mathfrak{h}$ and $g(\omega) = s^{-1}(\omega)$ are as stated in (1.3) and (1.4) respectively.

3. $\mathscr{L}^q_{\sum}(\eta, \mathfrak{h}) = \mathfrak{S}^q_{\sum}(1, \frac{1}{2}, \eta, \mathfrak{h})$ is the family of $s \in \sum$ of the form (1.1) satisfying

$$\frac{1}{\eta}\left(\frac{2z(zs'(z))'}{(zs(z))'} - 1\right) \prec_q (\mathfrak{h}(z) - 1) \quad \text{and} \quad \frac{1}{\eta}\left(\frac{2\omega(\omega g'(\omega))'}{(\omega g(\omega))'} - 1\right) \prec_q (\mathfrak{h}(\omega) - 1),$$

where $z,\,\omega \in \mathfrak{D}$, $\mathfrak{h}$ and $g(\omega) = s^{-1}(\omega)$ are as stated in (1.3) and (1.4) respectively.

4. $\mathscr{R}^q_{\sum}(\eta,\mathfrak{h}) = \mathfrak{S}^q_{\sum}(\mu,0,\eta,\mathfrak{h})$ is the family of $s \in \sum$ of the form (1.1) satisfying

$$\frac{1}{\eta}\left(\left(\frac{zs'(z)}{s(z)}\right)\left(1+\mu\frac{zs''(z)}{s'(z)}\right)-1\right) \prec_q (\mathfrak{h}(z)-1),\; z \in \mathfrak{D}$$

and

$$\frac{1}{\eta}\left(\left(\frac{\omega g'(\omega}{g(\omega)}\right)\left(1+\mu\frac{\omega g''(\omega)}{g'(\omega)}\right)-1\right) \prec_q (\mathfrak{h}(\omega)-1),\; \omega \in \mathfrak{D},$$

where $\mathfrak{h}$ and $g(\omega) = s^{-1}(\omega)$ are as stated in (1.3) and (1.4) respectively.

Definition 1.2. For $0 \le \nu \le 1, \mu \ge 0, \mu \ge \nu, \eta \in \mathbb{C} - \{0\}$, we say that s in $\sum$ belongs to $\mathfrak{M}^q_{\sum}(\mu,\nu,\eta,\mathfrak{h})$, if

$$\frac{1}{\eta}\left(\frac{zs'(z)+\mu z^2 s''(z)}{(1-\nu)z+\nu zs'(z)}-1\right) \prec_q \mathfrak{h}(z)-1),\; z \in \mathfrak{D}$$

and

$$\frac{1}{\eta}\left(\frac{\omega g'(\omega)\,+\,\mu\omega^2 g''(\omega)}{(1-\nu)\omega\,+\,\nu\omega g'(\omega)}-1\right) \prec_q (\mathfrak{h}(\omega)-1),\; \omega \in \mathfrak{D},$$

where $\mathfrak{h}$ and $g(\omega) = s^{-1}(\omega)$ are given by (1.3) and (1.4) respectively.

It is easy to see that certain choices of ν lead the family $\mathfrak{M}^q_{\sum}(\mu,\nu,\eta,\mathfrak{h})$ to the following few subfamilies.

1. $\mathfrak{N}^q_{\sum}(\mu,\eta,\mathfrak{h}) = \mathfrak{M}^q_{\sum}(\mu,0,\eta,\mathfrak{h})$ is the family of $s \in \sum$ of the form (1.1) satisfying

$$\frac{1}{\eta}(s'(z)+\mu zs''(z)-1) \prec_q (\mathfrak{h}(z)-1) \quad \text{and} \quad \frac{1}{\eta}(g'(\omega)+\mu\omega g''(\omega)-1) \prec_q (\mathfrak{h}(\omega)-1),$$

where $z,\,\omega \in \mathfrak{D}$, $\mathfrak{h}$ and $g(\omega) = s^{-1}(\omega)$ are as stated in (1.3) and (1.4) respectively.

2. $\mathfrak{N}^q_{\sum}(\mu,\eta,\mathfrak{h}) = \mathfrak{M}^q_{\sum}(\mu,1,\eta,\mathfrak{h})$ is the family of $s \in \sum$ of the form (1.1) satisfying

$$1+\frac{\mu}{\eta}\left(\frac{zs''(z)}{s'(z)}\right) \prec_q \mathfrak{h}(z) \quad \text{and} \quad 1+\frac{\mu}{\eta}\left(\frac{\omega g''(\omega)}{s'(\omega)}\right) \prec_q \mathfrak{h}(\omega),$$

where $z,\,\omega \in \mathfrak{D}$, $\mathfrak{h}$ and $g(\omega) = s^{-1}(\omega)$ are as stated in (1.3) and (1.4) respectively.

In the second section, we find bounds on $|d_2|$ and $|d_3|$ in the Taylor-Maclaurin's expansion belonging to the family $\mathfrak{S}^q_\sum(\mu,\nu,\eta,\mathfrak{h})$. We also present results related to four families defined above. In the third section, we obtain bounds on $|d_2|$ and $|d_3|$ in the Taylor-Maclaurin's expansion belonging to the family $\mathfrak{M}^q_\sum(\mu,\nu,\eta,\mathfrak{h})$. We also point out results related to two families defined above.

2 Initial Coefficients for the Family $\mathfrak{S}^q_\sum(\mu,\nu,\eta,\mathfrak{h})$

Theorem 2.1. *Let $\mu \geq 0$, $\mu \geq \nu, 0 \leq \nu \leq 1$ and $\eta \in \mathbb{C} - \{0\}$. If the function $s \in \mathcal{A}$ belongs to the class $\mathfrak{S}^q_\sum(\mu,\nu,\eta,\mathfrak{h})$, then*

$$|d_2| \leq \frac{|\eta||R_0|Q_1\sqrt{Q_1}}{\sqrt{\left[|(\nu^2 - 2(1+\mu)\nu + 1 + 4\mu)\eta R_0 Q_1^2 - (Q_2 - Q_1)(1-\nu+2\mu)^2|\right]}} \tag{2.1}$$

and

$$|d_3| \leq \frac{|\eta|Q_1}{2(1-\nu+3\mu)}\left\{|R_0|\left[1 + \frac{2|\eta|(1-\nu+3\mu)|R_0|Q_1}{(1-\nu+2\mu)^2}\right] + |R_1|\right\}. \tag{2.2}$$

Proof. Let $s \in \mathfrak{S}^q_\sum(\mu,\nu,\eta,\mathfrak{h})$. Then there exists a function $\varsigma(z)$ holomorphic in $\mathfrak{D}$ and holomorphic functions $u, v : \mathfrak{D} \to \mathfrak{D}$ with $u(0) = 0, |u(z)| < 1, v(0) = 0$, $|v(z)| < 1$ satisfying

$$\frac{1}{\eta}\left(\frac{zs'(z) + \mu z^2 s''(z)}{(1-\nu)s(z) + \nu z s'(z)} - 1\right) = \varsigma(z)(\mathfrak{h}(u(z)) - 1), \tag{2.3}$$

$$\frac{1}{\eta}\left(\frac{\omega g'\omega) + \mu\omega^2 g''(\omega)}{(1-\nu)g(\omega) + \nu\omega g'\omega)} - 1\right) = \varsigma(\omega)(\mathfrak{h}(v(\omega)) - 1). \tag{2.4}$$

Define $M(z)$ and $N(z)$ by

$$M(z) = \frac{1+u(z)}{1-u(z)} = 1 + m_1 z + m_2 z^2 + \cdots \tag{2.5}$$

and

$$N(z) = \frac{1+v(z)}{1-v(z)} = 1 + n_1 z + n_2 z^2 + \cdots \tag{2.6}$$

or equivalently

$$u(z)=\frac{M(z)-1}{M(z)+1}=\frac{1}{2}\left[m_1z+\left(m_2-\frac{m_1^2}{2}\right)z^2+\cdots\right] \tag{2.7}$$

and

$$v(z)=\frac{N(z)-1}{N(z)+1}=\frac{1}{2}\left[n_1z+\left(n_2-\frac{n_1^2}{2}\right)z^2+\cdots\right]. \tag{2.8}$$

It is apparent that two functions $M(z)$ and $N(z)$ are holomorphic having positive real parts in $\mathfrak{D}$ with $M(0)=1=N(0)$. In view of (2.3) - (2.8), one gets

$$\frac{1}{\eta}\left(\frac{zs'(z)+\mu z^2s''(z)}{(1-\nu)s(z)+\nu zs'(z)}-1\right)=\varsigma(z)\left[\mathfrak{h}\left(\frac{M(z)-1}{M(z)+1}\right)-1\right] \tag{2.9}$$

and

$$\frac{1}{\eta}\left(\frac{\omega g'(\omega)\ +\ \mu\omega^2g''(\omega)}{(1-\nu)g(\omega)\ +\ \nu\omega g'(\omega)}-1\right)=\varsigma(\omega)\left[\mathfrak{h}\left(\frac{N(\omega)-1}{N(\omega)+1}\right)-1\right]. \tag{2.10}$$

The Maclaurin series expansions of functions in (2.9) and (2.10) provide us

$$\begin{aligned}&\frac{1}{\eta}\left\{(1-\nu+2\mu)d_2z+[2(1-\nu+3\mu)d_3-(1+\nu)(1-\nu+2\mu)d_2^2]z^2+\cdots\right\}\\&\quad=\frac{R_0Q_1m_1}{2}z+\left[\frac{R_1Q_1m_1}{2}+\frac{R_0Q_1}{2}(m_2-\frac{m_1^2}{2})+\frac{R_0Q_2m_1^2}{4}\right]z^2+\cdots\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\frac{1}{\eta}\left\{-(1-\nu+2\mu)d_2\omega+\left[2(1-\nu+3\mu)(2d_2^2-d_3)\right.\right.\\&\qquad\left.\left.-\ (1+\nu)(1-\nu+2\mu)d_2^2\right]\omega^2+\cdots\right\}\\&\quad=\frac{R_0Q_1n_1}{2}\omega+\left[\frac{R_1Q_1n_1}{2}+\frac{R_0Q_1}{2}\left(n_2-\frac{n_1^2}{2}\right)+\frac{R_0Q_2n_1^2}{4}\right]\omega^2+\cdots.\end{aligned} \tag{2.12}$$

Comparing the coefficients in (2.11) and (2.12), we get

$$\frac{(1-\nu+2\mu)d_2}{\eta}=\frac{1}{2}R_0Q_1m_1, \tag{2.13}$$

$$\begin{aligned}\frac{\left[2(1-\nu+3\mu)d_3-(1+\nu)(1-\nu+2\mu)d_2^2\right]}{\eta}=&\ \tfrac{R_1Q_1m_1}{2}+\tfrac{R_0Q_1}{2}\left(m_2-\tfrac{m_1^2}{2}\right)\\&+\tfrac{R_0Q_2m_1^2}{4}\end{aligned} \tag{2.14}$$

and

$$-\frac{(1-\nu+2\mu)d_2}{\eta}=\frac{1}{2}R_0Q_1n_1, \tag{2.15}$$

$$\frac{\left[2(1-\nu+3\mu)(2d_2^2-d_3)-(1+\nu)(1-\nu+2\mu)d_2^2\right]}{\eta}=\frac{R_1Q_1n_1}{2}+\frac{R_0Q_1}{2}\left(n_2-\frac{n_1^2}{2}\right)+\frac{R_0Q_2n_1^2}{4}. \tag{2.16}$$

Solving the equations (2.13), (2.14),(2.15), and (2.16), we obtain

$$d_2^2=\frac{\eta^2R_0^2Q_1^3(m_2+n_2)}{4\left\{[\nu^2-2(1+\mu)\nu+(1+4\mu)]\eta R_0Q_1^2-(Q_2-Q_1)(1-\nu+2\mu)^2\right\}} \tag{2.17}$$

and

$$d_3=\frac{\eta}{4(1-\nu+3\mu)}\left[R_1Q_1m_1+\frac{1}{2}R_0Q_1(m_2-n_2)+\frac{\eta(1-\nu+3\mu)}{(1-\nu+2\mu)^2}R_0^2Q_1^2m_1^2\right]. \tag{2.18}$$

Using well-known inequalities $|m_i|\leq 2$ and $|n_i|\leq 2\,(i=1,2)$ [13] we get (2.1) and (2.2) from (2.17) and (2.18) respectively. This ends the proof. □

Remark 2.1. For $\mu=\nu=0$, in Theorem 2.1 we get the estimates [10, Corollary 9] and for $\mu=\nu$, $0\leq\nu\leq 1$, in Theorem 2.1, we obtain [23, Theorem 2.1]. Also, for $\gamma=\mu=1$, Theorem 2.1 coincide with estimates [10, Corollary 11].

We now present below the four consequences of Theorem 2.1.

Corollary 2.1. *If the function $s\in\mathscr{K}_{\sum}^{q}(\eta,\mathfrak{h})$, $\eta\in\mathbb{C}-\{0\}$, then*

$$|d_2|\leq\frac{2|\eta||R_0|Q_1\sqrt{Q_1}}{\sqrt{[|7\eta R_0Q_1^2-(Q_2-Q_1)|]}}$$

and

$$|d_3|\leq\frac{|\eta|Q_1}{4}\left[|R_0|\left(1+\frac{8}{9}|\eta||R_0|Q_1\right)+|R_1|\right].$$

Corollary 2.2. *If the function $s\in\mathscr{J}_{\sum}^{q}(\eta,\mathfrak{h})$, $\eta\in\mathbb{C}-\{0\}$, then*

$$|d_2|\leq\frac{|\eta||R_0|Q_1\sqrt{Q_1}}{\sqrt{[|3\eta R_0Q_1^2-4(Q_2-Q_1)|]}}$$

and

$$|d_3| \leq \frac{|\eta|Q_1}{5}\left[|R_0|\left(1+\frac{5}{4}|\eta||R_0|Q_1\right)+|R_1|\right].$$

Corollary 2.3. *If the function* $s \in \mathscr{L}^q_{\sum}(\eta, \mathfrak{h})$, $\eta \in \mathbb{C} - \{0\}$, *then*

$$|d_2| \leq \frac{2|\eta||R_0|Q_1\sqrt{Q_1}}{\sqrt{[|13\eta R_0 Q_1^2 - 25(Q_2-Q_1)|]}}$$

and

$$|d_3| \leq \frac{|\eta|Q_1}{7}\left[|R_0|\left(1+\frac{28}{25}|\eta||R_0|Q_1\right)+|R_1|\right].$$

Corollary 2.4. *If the function* $s \in \mathscr{R}^q_{\sum}(\mu, \eta, \mathfrak{h})$, $\eta \in \mathbb{C} - \{0\}$, *then*

$$|d_2| \leq \frac{|\eta||R_0|Q_1\sqrt{Q_1}}{\sqrt{\left[|(1+4\mu)\eta R_0 Q_1^2 - (Q_2-Q_1)(1+2\mu)^2|\right]}}$$

and

$$|d_3| \leq \frac{|\eta|Q_1}{2(3\mu+1)}\left\{|R_0|\left[1+\frac{|\eta|(3\mu+1)|R_0|Q_1}{(1+2\mu)^2}\right]+|R_1|\right\}.$$

3 Coefficient Estimates for the Family $\mathfrak{M}^q_{\sum}(\mu, \nu, \eta, \mathfrak{h})$

Theorem 3.1. *Let* $0 \leq \nu \leq 1, \mu \geq 0, \mu \geq \nu$ *and* $\eta \in \mathbb{C} - \{0\}$. *If the function* $s \in \mathcal{A}$ *belongs to* $\mathfrak{M}^q_{\sum}(\mu, \nu, \eta, \mathfrak{h})$, *then*

$$|d_2| \leq \frac{R_0 Q_1\sqrt{Q_1}|\eta|}{\sqrt{|\left(4\nu^2-(7+4\mu)\nu+3(1+2\mu)\right)\eta R_0 Q_1^2 - 4(Q_2-Q_1)(1-\nu+\mu)^2|}} \tag{3.1}$$

and

$$|d_3| \leq \frac{|\eta|Q_1}{3(1-\nu+2\mu)}\left[|R_1|+|R_0|\left(1+\frac{3|\eta|(1-\nu+2\mu)|R_0|Q_1}{4(1-\nu+\mu)^2}\right)\right]. \tag{3.2}$$

Proof. Let $s \in \mathfrak{M}^q_{\sum}(\mu, \nu, \eta, \mathfrak{h})$. Then there exists a function $\varsigma(z)$ holomorphic in $\mathfrak{D}$ and holomorphic functions $u, v : \mathfrak{D} \to \mathfrak{D}$ with $u(0) = 0, |u(z)| < 1, v(0) = 0$, $|v(z)| < 1$ satisfying

$$\frac{1}{\eta}\left(\frac{zs'(z)+\mu z^2 s''(z)}{(1-\nu)z+\nu zs'(z)}-1\right) \prec \varsigma(z)(\mathfrak{h}(u(z))-1) \tag{3.3}$$

$$\frac{1}{\eta}\left(\frac{\omega g'(\omega)+\mu\omega^2 g''(\omega)}{(1-\nu)\omega+\nu\omega g'(\omega)}-1\right) \prec \varsigma(\omega)(\mathfrak{h}(v(\omega))-1). \tag{3.4}$$

Following the steps of Theorem 2.1, with *M(z)* and *N(z)* as defined in (2.5) and (2.6), respectively, one gets in view of (3.3) and (3.4)

$$\frac{1}{\eta}\left(\frac{zs'(z)+\mu z^2 s''(z)}{(1-\nu)z+\nu zs'(z)}-1\right) = \varsigma(z)\left[\mathfrak{h}\left(\frac{M(z)-1}{M(z)+1}\right)-1\right] \tag{3.5}$$

and

$$\frac{1}{\eta}\left(\frac{\omega g'(\omega)\ +\ \mu\omega^2 g''(\omega)}{(1-\nu)\omega\ +\ \nu\omega g'(\omega)}-1\right) = \varsigma(\omega)\left[\mathfrak{h}\left(\frac{N(\omega)-1}{N(\omega)+1}\right)-1\right]. \tag{3.6}$$

The Maclaurin series expansions of functions in (3.5) and (3.6) provide us

$$\begin{aligned}&\frac{1}{\eta}\left\{(1-\nu+\mu)2d_2z+[3(1-\nu+2\mu)d_3-(1-\nu+\mu)4\nu d_2^2]z^2+\cdots\right\}\\ &=\frac{R_0Q_1m_1}{2}z+\left[\frac{R_1Q_1m_1}{2}+\frac{R_0Q_1}{2}\left(m_2-\frac{m_1^2}{2}\right)+\frac{R_0Q_2m_1^2}{4}\right]z^2+\cdots\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\frac{1}{\eta}\left\{-(1-\nu+\mu)2d_2\omega+\left[3(1-\nu+2\mu)(2d_2^2-d_3)-(1-\nu+\mu)4\nu d_2^2\right]\omega^2+\cdots\right\}\\ &\quad=\frac{R_0Q_1n_1}{2}\omega+\left[\frac{R_1Q_1n_1}{2}+\frac{R_0Q_1}{2}\left(n_2-\frac{n_1^2}{2}\right)+\frac{R_0Q_2n_1^2}{4}\right]\omega^2+\cdots.\end{aligned} \tag{3.8}$$

Comparing the coefficients in (3.7) and (3.8), we get

$$\frac{(1-\nu+\mu)2d_2}{\eta}=\frac{1}{2}R_0Q_1m_1 \tag{3.9}$$

$$\frac{\left[3(1-\nu+2\mu)d_3-(1-\nu+\mu)4\nu d_2^2\right]}{\eta} = \frac{R_1Q_1m_1}{2}+\frac{R_0Q_1}{2}\left(m_2-\frac{m_1^2}{2}\right)+\frac{R_0Q_2m_1^2}{4} \tag{3.10}$$

and

$$-\frac{(1-\nu+\mu)2d_2}{\eta}=\frac{1}{2}R_0Q_1n_1 \tag{3.11}$$

$$\frac{\left[3(1-\nu+2\mu)(2d_2^2-d_3)-(1-\nu+\mu)4\nu d_2^2\right]}{\eta} = \frac{R_1Q_1n_1}{2}+\frac{R_0Q_1}{2}\left(n_2-\frac{n_1^2}{2}\right) +\frac{R_0Q_2n_1^2}{4}. \tag{3.12}$$

From (3.9) and (3.11), we have

$$m_1=-n_1. \tag{3.13}$$

By adding (3.10) and (3.12), we obtain

$$\frac{\left[4\nu^2-(7+4\mu)\nu+3(1+2\mu)\right]}{\eta}2d_2^2=\frac{R_0Q_1}{2}(m_2+n_2)+\frac{R_0(Q_2-Q_1)}{4}(m_1^2+n_1^2). \tag{3.14}$$

Using (3.9), (3.11) and (3.13) in (3.14), we get

$$d_2^2=\frac{R_0^2Q_1^3\eta^2(m_2+n_2)}{4\{[4\nu^2-4(1+\mu)\nu+3(1-\nu+2\mu)]\eta R_0Q_1^2-4(Q_2-Q_1)(1-\nu+\mu)^2\}}. \tag{3.15}$$

By subtracting (3.12) from (3.10), we get

$$d_3=\frac{\eta}{6(1-\nu+2\mu)}\left[\frac{R_1Q_1}{2}(m_1-n_1)+\frac{R_0Q_1}{2}(m_2-n_2) +\frac{R_0}{4}(Q_2-Q_1)(m_1^2-n_1^2)\right]+d_2^2. \tag{3.16}$$

By well-known inequalities $|m_i|\le 2$ and $|n_i|\le 2\,(i=1,2)$ [13], we get the results (3.1) and (3.2) using (3.9), (3.13), (3.15) and (3.16) respectively. This ends the proof. □

Remark 3.1. For $\mu=\nu=0$, in Theorem 3.1 we get the estimates [10, Corollary 3] and for $\mu=\nu$, $0\le\nu\le 1$, in Theorem 3.1, we obtain [16, Theorem 2.1]. Also, for $\gamma=\mu=1$, Theorem 3.1 coincide with [10, Corollary 11].

We now present below the two consequences of Theorem 3.1.

Corollary 3.1. *If the function* $s \in \mathfrak{R}^q_{\sum}(\mu, \eta, \mathfrak{h})$, $\eta \in \mathbb{C} - \{0\}$, $\mu \geq 0$, *then*

$$|d_2| \leq \frac{|\eta||R_0|Q_1\sqrt{Q_1}}{\sqrt{\left[|3(1+2\mu)\eta R_0 Q_1^2 - 4(Q_2 - Q_1)(1+\mu)^2|\right]}}$$

and

$$|d_3| \leq \frac{|\eta|Q_1}{3(1+2\mu)}\left\{|R_0|\left[1 + \frac{3|\eta|(1+2\mu)|R_0|Q_1}{4(1+\mu)^2}\right] + |R_1|\right\}.$$

Corollary 3.2. *If the function* $s \in \mathfrak{L}^q_{\sum}(\mu, \eta, \mathfrak{h})$, $\eta \in \mathbb{C} - \{0\}$, $\mu \geq 1$, *then*

$$|d_2| \leq \frac{|\eta||R_0|Q_1\sqrt{Q_1}}{\sqrt{\left[|2\,\mu\,\eta R_0 Q_1^2 - 4(Q_2 - Q_1)\mu^2|\right]}}$$

and

$$|d_3| \leq \frac{|\eta|Q_1}{6\mu}\left\{|R_0|\left[1 + \frac{3|\eta||R_0|Q_1}{2\mu}\right] + |R_1|\right\}.$$

References

[1] Abdul Rahman S. Juma and Mohammed H. Saloomi, Generalized differential operator on bistarlike and biconvex functions associated by quasi-subordination, *J. Phys.: Conf. Ser.* 1003 (2018), 012046. https://doi.org/10.1088/1742-6596/1003/1/012046

[2] R. M. Ali, S. K. Lee, V. Ravichandran and S. Subramaniam, Coefficient estimates for bi-univalent Ma- Minda starlike and convex functions, *Appl. Math. Lett.* 25(3) (2012), 344-351. https://doi.org/10.1016/j.aml.2011.09.012

[3] D. A. Brannan, J. Clunie and W. E. Kirwan, Coefficient estimates for a class of starlike functions, *Canad. J. Math.* 22 (1970), 476-485. https://doi.org/10.4153/CJM-1970-055-8

[4] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, *Mathematical Analysis and Its Applications* 3 (1985), 18-21.

[5] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, *J. Class. Anal.* 2(1) (2013), 49-60. https://doi.org/10.7153/jca-02-05

[6] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.

[7] S. P. Goyal and Rakesh Kumar, Coefficient estimates and quasi-subordination properties associated with certain subclasses of analytic and bi-univalent functions, *Math. Slovaca* 65(3) (2015), 533-544. https://doi.org/10.1515/ms-2015-0038

[8] S. P. Goyal, O. Singh and R. Mukherjee, Certain results on a subclass of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, *Palestine Journal of Mathematics* 5(1) (2016), 79-85.

[9] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* 18 (1967), 63-68. https://doi.org/10.1090/S0002-9939-1967-0206255-1

[10] N. Magesh, V. K. Balaji and J. Yamini, Certain subclasses of bistarlike and biconvex functions based on quasi-subordination, *Abstract and Applied Analysis* 2016 (2016), Article ID 3102960, 6 pp. https://doi.org/10.1155/2016/3102960

[11] M. H. Mohd and M. Darus, Fekete-Szego problems for quasi-subordination classes, *Abstract and Applied Analysis* 2012 (2012), Article ID 192956, 14 pp. https://doi.org/10.1155/2012/192956

[12] Z. Peng, G. Murugusundaramoorthy and T. Janani, Coefficient estimate of bi-univalent functions of complex order associated with the Hohlov operator, *Journal of Complex Analysis* 2014 (2014), Article ID 693908, 6 pp. https://doi.org/10.1155/2014/693908

[13] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.

[14] F. Y. Ren, S. Owa and S. Fukui, Some inequalities on quasi-subordinate functions, *Bull. Aust. Math. Soc.* 43(2) (1991), 317-324. https://doi.org/10.1017/S0004972700029117

[15] M. S. Robertson, Quasi-subordination and coefficient conjecture, *Bull. Amer. Math. Soc.* 76 (1970), 1-9. https://doi.org/10.1090/S0002-9904-1970-12356-4

[16] Shashi Kant, Coefficients estimate for certain subclass of bi-univalent functions associated with quasi-subordination, *J. Fract. Calc. Appl.* 9(1) (2018), 195-203.

[17] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses analytic and bi-univalent functions, *Appl. Math. Lett.* 23(10) (2010), 1188-1192. https://doi.org/10.1016/j.aml.2010.05.009

[18] S. R. Swamy, Ruscheweyh derivative and a new generalized Multiplier differential operator, *Annals of Pure and Applied Mathematics* 10(2) (2015), 229-238.

[19] S. R. Swamy and Y. Sailaja, Fekete-Szegö inequality and quasi-subordination, *Palestine Journal of Mathematics* 10(2) (2021), in press.

[20] S. R. Swamy and Y Sailaja, Horadam polynomial coefficient estimates for two families of holomorphic and bi-univalent functions, *International Journal of Mathematics Trends and Technology* 66(8) (2020), 131-138. https://doi.org/10.14445/22315373/IJMTT-V66I8P514

[21] S. R. Swamy and Y. Sailaja, Sharp bounds of Fekete-Szego functional for quasi-subordination class, *International Journal of Mathematics Trends and Technology* 66(11) (2020), 87-94. https://doi.org/10.2478/ausm-2019-0008

[22] H. Tang, G. Deng and S. Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, *J. Inequal. Appl.* 2013 (2013), Article No. 317, 10 pp. https://doi.org/10.1186/1029-242X-2013-317

[23] P. P. Vyas and S. Kant, Coefficients estimate for new subclass of bi-univalent functions associated with quasi-subordination, *AIP Conference Proceedings* 1953 (2018), 140068. https://doi.org/10.1063/1.5033243

[24] P. P. Vyas and S. Kant, Certain subclasses of bi-univalent functions associated with quasi-subordination, *Journal of Rajasthan Academy of Physical Sciences* 15(4) (2016), 315-325.