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Coefficient Estimates for Certain Subclasses of m-Fold Symmetric Bi-univalent Functions Associated with Pseudo-Starlike Functions

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Abstract

In the present investigation, we introduce the subclasses $\Lambda_{\Sigma}^{m}(\eta, \leftthreetimes, \phi)$ and $\Lambda_{\Sigma}^{m}(\eta, \leftthreetimes, \delta)$ of m-fold symmetric bi-univalent function class Σ_m, which are associated with the pseudo-starlike functions and defined in the open unit disk $\mathbb{U}$. Moreover, we obtain estimates on the initial coefficients $|b_{m+1}|$ and $|b_{2m+1}|$ for the functions belong to these subclasses and identified correlations with some of the earlier known classes.

1 Introduction

Let $\mathcal{A} = \{f : \mathbb{U} \to \mathbb{C} : f \text{ is analytic in } \mathbb{U}, f(0) = 0 = f'(0) - 1\}$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions f univalent in $\mathbb{U}$. The Koebe one quarter theorem (see [6]) guarantee that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and

$$f(f^{-1}(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4).$$

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Indeed, the analytic extension of f^{-1} to $\mathbb{U}$ is

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \qquad (1.2)$$

Let $\Sigma = \{f \in \mathcal{A} : f(z) \text{ and } f^{-1}(z) \text{ are univalent in the unit disk } \mathbb{U}\}$ denote the class of bi-univalent functions.

In 1967, Lewin [12] introduced the class Σ bi-univalent function and proved that $|a_2| < 1.51$ for the functions $f \in \Sigma$. Afterverse, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$ and at one instance Goodman [8] claimed that $|a_n| \leq 1$ may be true for every $f \in \Sigma$ and $n \in \mathcal{N}$. However, Netanyahu [13] showed that $max_{f\in\Sigma}|a_2| = \frac{4}{3}$, Styer and Wright [27] showed existence of $f \in \Sigma$ for which $|a_2| > \frac{4}{3}$ and Tan [28] proved that $|a_2| \leq 1.485$ for the functions in the class Σ.

In 2010, Srivastava et al. [25] revived the concept of coefficient estimation problem for the functions $f \in \Sigma$. Motivated by their work, many researchers (viz. [1], [5], [7], [9], [14], [17], [18], [20], [21], [22] etc.) obtained coefficient estimates for the functions in several subclasses of Σ. But still the sharp coefficient estimation problem of $|a_n|$, $(n = 3, 4, 5, \cdots)$ for the functions belong to the subclasses of Σ is open.

Functions of the form:

$$g(z) = z + \sum_{k=1}^{\infty} b_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U};\ m \in \mathcal{N}) \qquad (1.3)$$

are said to be m-fold symmetric functions (see [11], [16]). Further, any function $h(z)$ of the form:

$$h(z) = \sqrt[m]{f(z^m)} \qquad (f \in \mathcal{S};\ z \in \mathbb{U};\ m \in \mathcal{N})$$

is univalent and maps the unit disk into a m-fold symmetric region.

Let $\mathcal{S}_m$ denote the class of all m-fold symmetric univalent functions in $\mathbb{U}$, which are of the form (1.3) and for $m = 1$, these functions reduces to functions of the class $\mathcal{S}_1$ (or simply $\mathcal{S}$) and are known as *one*-fold symmetric univalent functions.

For each $m \in \mathcal{N}$, every bi-univalent function generates an m-fold symmetric bi-univalent function. Srivastava et al. [26] showed that, for the function g as given in (1.3), the extension of g^{-1} to $\mathbb{U}$ is given by:

$$h(w) = w - b_{m+1}w^{m+1} + \left[(m+1)\, b_{m+1}^2 - b_{2m+1}\right] w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)\, b_{m+1}^3 - (3m+2)\, b_{m+1}b_{2m+1} + b_{3m+1}\right] w^{3m+1} + \cdots . \tag{1.4}$$

Clearly for $m = 1$, this equation (1.4) reduces to the equation (1.2). So that the bi-univalent function class Σ then generalized to the m-fold symmetric bi-univalent function class Σ_m. See [26] for examples of m-fold symmetric bi-univalent functions. Also see [2], [19], [23], [24], [29] etc. for coefficient problems of some new subclasses of Σ_m.

In order to prove our main results, we need the following lemma [15].

Lemma 1.1. *If $w(z) \in \mathcal{P}$, the class of functions which are analytic in $\mathbb{U}$ with $\Re(w(z)) > 0$, $(z \in \mathbb{U})$ and have the form $w(z) = 1 + w_1 z + w_2 z^2 + w_3 z^3 + \cdots$, $(z \in \mathbb{U})$; then $|w_n| \leq 2$ for each $n \in \mathcal{N}$.*

We use the m-fold symmetric function w in the class $\mathcal{P}$ (see [16]) of the form:

$$w(z) = 1 + w_m z^m + w_{2m} z^{2m} + w_{3m} z^{3m} + \cdots, \quad (z \in \mathbb{U}).$$

In the present investigation, with reference to the $\leftthreetimes$-pseudo-starlike function class defined by Babalola [3] and the work of Joshi and Yadav [10], we obtain estimates on the initial coefficients $|b_{m+1}|$ and $|b_{2m+1}|$ for functions belong to the new subclasses $\Lambda_\Sigma^m(\eta, \leftthreetimes, \phi)$ and $\Lambda_\Sigma^m(\eta, \leftthreetimes, \delta)$ of the function class Σ_m. Also, we have pointed out connections with some of the earlier known subclasses of the class Σ.

2 Coefficient Bounds for the Function Class $\Lambda_\Sigma^m(\eta, \leftthreetimes, \phi)$

Definition 2.1. A function $g(z)$ given by (1.3) is said to be in the class $\Lambda_\Sigma^m(\eta, \leftthreetimes, \phi)$ if the following conditions are fulfilled:

$$\left|\arg\left[\frac{z[g'(z)]^{\leftthreetimes}}{(1-\eta)g(z) + \eta z g'(z)}\right]\right| < \frac{\phi\pi}{2} \quad z \in \mathbb{U} \tag{2.1}$$

and

$$\left|\arg\left[\frac{w[h'(w)]^{\curlywedge}}{(1-\eta)h(w)+\eta wh'(w)}\right]\right| < \frac{\phi\pi}{2} \quad w \in \mathbb{U}, \tag{2.2}$$

where $g(z) \in \Sigma_m$, $m \in \mathcal{N}$, $\curlywedge \geq 1$, $0 < \phi \leq 1$, $0 \leq \eta < 1$ and $h = g^{-1}$.

Theorem 2.2. *Let $g(z)$ given by (1.3) be in the class $\Lambda_{\Sigma}^{m}(\eta, \curlywedge, \phi)$, $0 < \phi \leq 1$. Then*

$$|b_{m+1}| \leq \frac{2\phi}{\sqrt{\begin{array}{r}(\phi+1)^2 + \phi(\eta^2 + 2\eta(1-m^2-m) - m + m(m+1)\curlywedge) \\ + \curlywedge(m+1)(\curlywedge(m+1) - 2\eta - 2)\end{array}}} \tag{2.3}$$

and

$$|b_{2m+1}| \leq \frac{2\phi}{(2m+1)\curlywedge - 2\eta m - 1} + \frac{2\phi^2(m+1)}{((m+1)\curlywedge - \eta - 1)^2}. \tag{2.4}$$

Proof. Let $g \in \Lambda_{\Sigma}^{m}(\eta, \curlywedge, \phi)$. Then,

$$\frac{z[g'(z)]^{\curlywedge}}{(1-\eta)g(z)+\eta zg'(z)} = [r(z)]^{\phi}, \tag{2.5}$$

$$\frac{w[h'(w)]^{\curlywedge}}{(1-\eta)h(w)+\eta wh'(w)} = [u(w)]^{\phi}, \tag{2.6}$$

where $h = g^{-1}$ and r, u in $\mathcal{P}$ have the following forms:

$$r(z) = 1 + r_m z^m + r_{2m} z^{2m} + r_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$u(w) = 1 + u_m w^m + u_{2m} w^{2m} + u_{3m} w^{3m} + \cdots. \tag{2.8}$$

Clearly,

$$[r(z)]^{\phi} = 1 + \phi r_m z^m + \left(\phi r_{2m} + \frac{\phi(\phi-1)}{2} r_m^2\right) z^{2m} + \cdots$$

and

$$[u(w)]^{\phi} = 1 + \phi u_m w^m + \left(\phi u_{2m} + \frac{\phi(\phi-1)}{2} u_m^2\right) w^{2m} + \cdots.$$

Also

$$\frac{z[g'(z)]^{\leftthreetimes}}{(1-\eta)g(z)+\eta z g'(z)} = 1 + ((m+1)\leftthreetimes -\eta - 1)b_{m+1}z^m + \Bigg((2m+1)\leftthreetimes - 2\eta m - 1\Bigg)b_{2m+1}z^{2m} + \Bigg(1 + 2\eta + \eta^2 - \eta\leftthreetimes(m+1) - \frac{\leftthreetimes(3+m)(m+1)}{2} + \frac{\leftthreetimes^2(m+1)^2}{2}\Bigg)b_{m+1}^2 z^{2m} + \cdots$$

$$\frac{w[h'(w)]^{\leftthreetimes}}{(1-\eta)h(w)+\eta w h'(w)} = 1 - ((m+1)\leftthreetimes -\eta - 1)b_{m+1}w^m - \Bigg((2m+1)\leftthreetimes - 2\eta m - 1\Bigg)b_{2m+1}w^{2m} + \Bigg(1 + 2\eta + \eta^2 - (m+1)\leftthreetimes\eta + (\leftthreetimes - \eta)(m+1)(2m+1) - (m+1)(1-\eta) - \frac{\leftthreetimes(m+1)(m+3)}{2} + \frac{\leftthreetimes^2(m+1)^2}{2}\Bigg)b_{m+1}^2 w^{2m} + \cdots .$$

Comparing the coefficients in (2.5) and (2.6), we have:

$$((m+1)\leftthreetimes -\eta - 1)b_{m+1} = \phi r_m, \tag{2.9}$$

$$\left(1 + 2\eta + \eta^2 - \eta\leftthreetimes(m+1) - \frac{\leftthreetimes(3+m)(m+1)}{2} + \frac{\leftthreetimes^2(m+1)^2}{2}\right)b_{m+1}^2 + ((2m+1)\leftthreetimes -2\eta m - 1)b_{2m+1} = \phi r_{2m} + \frac{\phi(\phi-1)}{2}r_m^2, \tag{2.10}$$

$$-((m+1)\leftthreetimes -\eta - 1)b_{m+1} = \phi u_m, \tag{2.11}$$

$$\Big(1+2\eta+\eta^2-(m+1)\leftthreetimes\eta+(\leftthreetimes-\eta)(m+1)(2m+1)-(m+1)(1-\eta)$$
$$-\frac{\leftthreetimes(m+1)(m+3)}{2}+\frac{\leftthreetimes^2(m+1)^2}{2}\Big)b_{m+1}^2+(1+2\eta m-(2m+1)\leftthreetimes)b_{2m+1}$$
$$=\phi u_{2m}+\frac{\phi(\phi-1)}{2}u_m^2. \quad (2.12)$$

From equations (2.9) and (2.11) we have:

$$r_m=-u_m \quad (2.13)$$

and

$$2((m+1)\leftthreetimes-\eta-1)^2b_{m+1}^2=\phi^2(r_m^2+u_m^2). \quad (2.14)$$

By adding equations (2.10) and (2.12) we get

$$\Big(\leftthreetimes^2(m+1)^2+2\eta^2-2(m+1)\eta\leftthreetimes+\leftthreetimes(m+1)(m-2)-(m+1)(2m\eta+1)$$
$$+4\eta+2\Big)b_{m+1}^2=\phi(r_{2m}+u_{2m})+\frac{\phi(\phi-1)}{2}(r_m^2+u_m^2).$$

By using (2.14) and simplifying we get

$$\Big[(\phi+1)^2+\phi(\eta^2+2\eta(1-m^2-m)-m+m(m+1)\leftthreetimes)+\leftthreetimes(m+1)(\leftthreetimes(m+1)$$
$$-2\eta-2)\Big]b_{m+1}^2=\phi^2(r_{2m}+u_{2m}).$$

By applying Lemma 1.1 for the coefficients r_{2m} and u_{2m}, then we have

$$|b_{m+1}|\leq\frac{2\phi}{\sqrt{\begin{array}{r}(\phi+1)^2+\phi(\eta^2+2\eta(1-m^2-m)-m+m(m+1)\leftthreetimes)\\+\leftthreetimes(m+1)(\leftthreetimes(m+1)-2\eta-2)\end{array}}}.$$

Further, to obtain $|b_{2m+1}|$, we subtract (2.12) from (2.10), we get

$$(m+1)\left[(1-\eta)-(\leftthreetimes-\eta)(2m+1)\right]b_{m+1}^2+2[(2m+1)\leftthreetimes-2\eta m-1]b_{2m+1}$$
$$=\phi(r_{2m}-u_{2m})+\frac{\phi(\phi-1)}{2}(r_m^2-u_m^2).$$

Then, in view of (2.13) and (2.14) and applying Lemma 1.1 for coefficients r_m, r_{2m}, u_m, u_{2m}, we have

$$|b_{2m+1}|\leq\frac{2\phi}{(2m+1)\leftthreetimes-2\eta m-1}+\frac{2\phi^2(m+1)}{((m+1)\leftthreetimes-\eta-1)^2}$$

which completes the proof of Theorem 2.2. □

3 Coefficient Bounds for the Function Class $\mathit{\Lambda}_{\Sigma}^{m}(\eta,\leftthreetimes,\delta)$

Definition 3.1. A function $g(z)$ given by (1.3) is said to be in the class $\mathit{\Lambda}_{\Sigma}^{m}(\eta,\leftthreetimes,\delta)$ if the following condition are fulfilled:

$$\Re\left[\frac{z[g'(z)]^{\leftthreetimes}}{(1-\eta)g(z)+\eta zg'(z)}\right]>\delta\quad z\in\mathbb{U}\tag{3.1}$$

and

$$\Re\left[\frac{w[h'(w)]^{\leftthreetimes}}{(1-\eta)h(w)+\eta wh'(w)}\right]>\delta\quad w\in\mathbb{U},\tag{3.2}$$

where $g(z)\in\Sigma_m$, $m\in\mathcal{N}$, $\leftthreetimes\geq 1$, $0\leq\delta<1$, $0\leq\eta<1$ and $h=g^{-1}$.

Theorem 3.2. *Let $g(z)$ given by (1.3) be in the class $\mathit{\Lambda}_{\Sigma}^{m}(\eta,\leftthreetimes,\delta)$, $0\leq\delta<1$. Then,*

$$|b_{m+1}|\leq\frac{2\sqrt{1-\delta}}{\sqrt{\begin{array}{r}\leftthreetimes^2(m+1)^2+2\eta^2-2(m+1)\eta\leftthreetimes+\leftthreetimes(m+1)(m-2)\\-(m+1)(2m\eta+1)+4\eta+2\end{array}}}\tag{3.3}$$

and

$$|b_{2m+1}|\leq\frac{2(m+1)(1-\delta)^2}{((m+1)\leftthreetimes-\eta-1)^2}+\frac{2(1-\delta)}{(2m+1)\leftthreetimes-2\eta m-1}.\tag{3.4}$$

Proof. Let $g \in \Lambda_{\Sigma}^{m}(\eta, \curlywedge, \delta)$. Then,

$$\frac{z[g'(z)]^{\curlywedge}}{(1-\eta)g(z)+\eta z g'(z)} = \delta + (1-\delta)r(z), \tag{3.5}$$

$$\frac{w[h'(w)]^{\curlywedge}}{(1-\eta)h(w)+\eta w h'(w)} = \delta + (1-\delta)u(z), \tag{3.6}$$

where $h = g^{-1}$ and r, u in $\mathcal{P}$ have the following forms:

$$r(z) = 1 + r_m z^m + r_{2m} z^{2m} + r_{3m} z^{3m} + \cdots$$

and

$$u(w) = 1 + u_m w^m + u_{2m} w^{2m} + u_{3m} w^{3m} + \cdots .$$

Clearly,

$$\delta + (1-\delta)r(z) = 1 + (1-\delta)r_m z^m + (1-\delta)r_{2m} z^{2m} + \cdots$$

and

$$\delta + (1-\delta)u(w) = 1 + (1-\delta)u_m w^m + (1-\delta)u_{2m} w^{2m} + \cdots .$$

Also

$$\frac{z[g'(z)]^{\curlywedge}}{(1-\eta)g(z)+\eta z g'(z)} = 1 + ((m+1)\curlywedge - \eta - 1)b_{m+1} z^m + \Bigg((2m+1)\curlywedge - 2\eta m - 1\Bigg) b_{2m+1} z^{2m} + \Bigg(1 + 2\eta + \eta^2 - \eta\curlywedge(m+1) - \frac{\curlywedge(3+m)(m+1)}{2} + \frac{\curlywedge^2(m+1)^2}{2}\Bigg) b_{m+1}^2 z^{2m} + \cdots$$

$$\frac{w[h'(w)]^{\curlywedge}}{(1-\eta)h(w)+\eta w h'(w)} = 1 - ((m+1)\curlywedge - \eta - 1)b_{m+1} w^m - \Bigg((2m+1)\curlywedge - 2\eta m - 1\Bigg) b_{2m+1} w^{2m} + \Bigg(1 + 2\eta + \eta^2 - (m+1)\curlywedge\eta + (\curlywedge - \eta)(m+1)(2m+1) - (m+1)(1-\eta) - \frac{\curlywedge(m+1)(m+3)}{2} + \frac{\curlywedge^2(m+1)^2}{2}\Bigg) b_{m+1}^2 w^{2m} + \cdots .$$

From (3.5) and (3.6), we obtain

$$((m+1)\curlywedge -\eta - 1)b_{m+1} = (1-\delta)r_m, \tag{3.7}$$

$$\left(1 + 2\eta + \eta^2 - \eta \curlywedge (m+1) - \frac{\curlywedge(3+m)(m+1)}{2} + \frac{\curlywedge^2(m+1)^2}{2}\right) b_{m+1}^2 + ((2m+1)\curlywedge -2\eta m - 1)b_{2m+1} = (1-\delta)r_{2m}, \tag{3.8}$$

$$-((m+1)\curlywedge -\eta - 1)b_{m+1} = (1-\delta)u_m, \tag{3.9}$$

$$\left(1 + 2\eta + \eta^2 - (m+1)\curlywedge \eta + (\curlywedge - \eta)(m+1)(2m+1) - (m+1)(1-\eta) - \frac{\curlywedge(m+1)(m+3)}{2} + \frac{\curlywedge^2(m+1)^2}{2}\right) b_{m+1}^2 + (1 + 2\eta m - (2m+1)\curlywedge)b_{2m+1} = (1-\delta)u_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we have

$$r_m = -u_m \tag{3.11}$$

and

$$2((m+1)\curlywedge -\eta - 1)^2 b_{m+1}^2 = (1-\delta)^2(r_m^2 + u_m^2). \tag{3.12}$$

Adding (3.8) and (3.10), we have

$$(\curlywedge^2(m+1)^2 + 2\eta^2 - 2(m+1)\eta\curlywedge + \curlywedge(m+1)(m-2) - (m+1)(2m\eta+1) + 4\eta + 2) b_{m+1}^2 = (1-\delta)(r_{2m} + u_{2m}).$$

Therefore, we get

$$b_{m+1}^2 = \frac{(1-\delta)(r_{2m} + u_{2m})}{\curlywedge^2(m+1)^2 + 2\eta^2 - 2(m+1)\eta\curlywedge + \curlywedge(m+1)(m-2) - (m+1)(2m\eta+1) + 4\eta + 2}. \tag{3.13}$$

By applying Lemma 1.1 for the coefficients r_{2m} and u_{2m}, then we have

$$|b_{m+1}| \leq \frac{2\sqrt{1-\delta}}{\sqrt{\begin{array}{r}\leftthreetimes^2(m+1)^2+2\eta^2-2(m+1)\eta\leftthreetimes+\leftthreetimes(m+1)(m-2)\\ -(m+1)(2m\eta+1)+4\eta+2\end{array}}}.$$

Also, to find the bound on $|b_{2m+1}|$, using the relations (3.8) and (3.10), we obtain

$$\begin{aligned}-(m+1)\left[(2m+1)\leftthreetimes-2\eta m-1\right]b_{m+1}^2+2[(2m+1)\leftthreetimes-2\eta m-1]b_{2m+1}\\ =(1-\delta)(r_{2m}-u_{2m}).\end{aligned}$$

Then, in view of (3.11) and (3.12), also applying Lemma 1.1 for the coefficients r_m, r_{2m}, u_m, u_{2m}, we have

$$|b_{2m+1}| \leq \frac{2(m+1)(1-\delta)^2}{((m+1)\leftthreetimes-\eta-1)^2}+\frac{2(1-\delta)}{(2m+1)\leftthreetimes-2\eta m-1}, \tag{3.14}$$

which completes the proof of Theorem 3.2. □

Choosing $\eta = 0$ in Theorems 2.2 and 3.2, the classes $\Lambda_\Sigma^m(\eta,\leftthreetimes,\phi)$ and $\Lambda_\Sigma^m(\eta,\leftthreetimes,\delta)$ reduces to classes $\Lambda_\Sigma^m(\leftthreetimes,\phi)$ and $\Lambda_\Sigma^m(\leftthreetimes,\delta)$ and thus we get the following corollaries.

Corollary 3.3. *Let $g(z)$ given by* (1.3) *be in the class $\Lambda_\Sigma^m(\leftthreetimes,\phi)$, $0<\phi\leq 1$. Then*

$$|b_{m+1}| \leq \frac{2\phi}{\sqrt{1+\phi(m^2+m\leftthreetimes-m)+\leftthreetimes(m+1)(\leftthreetimes m+\leftthreetimes-2)}}$$

and

$$|b_{2m+1}| \leq \frac{2\phi}{(2m+1)\leftthreetimes-1}+\frac{2\phi^2(m+1)}{((m+1)\leftthreetimes-1)^2}.$$

Corollary 3.4. *Let $g(z)$ given by* (1.3) *be in the class $\Lambda_\Sigma^m(\leftthreetimes,\delta)$, $0\leq\delta<1$. Then,*

$$|b_{m+1}| \leq \frac{2\sqrt{1-\delta}}{\sqrt{\leftthreetimes^2(1+m)^2+\leftthreetimes(m+1)(m-2)-(m+1)+2}}$$

and

$$|b_{2m+1}| \leq \frac{2(m+1)(1-\delta)^2}{((m+1)\leftthreetimes-1)^2}+\frac{2(1-\delta)}{(2m+1)\leftthreetimes-1}.$$

The following classes $\Lambda_{\Sigma}^{m}(\leftthreetimes,\phi)$ and $\Lambda_{\Sigma}^{m}(\leftthreetimes,\delta)$ are defined as follows:

Definition 3.5. A function $g(z)$ given by (1.3) is said to be in the class $\Lambda_{\Sigma}^{m}(\leftthreetimes,\phi)$ if the following condition are fulfilled:

$$\left|\arg\left[\frac{z[g'(z)]^{\leftthreetimes}}{g(z)}\right]\right| < \frac{\phi\pi}{2} \quad z\in\mathbb{U}$$

and

$$\left|\arg\left[\frac{w[h'(w)]^{\leftthreetimes}}{h(w)}\right]\right| < \frac{\phi\pi}{2} \quad w\in\mathbb{U},$$

where $g(z)\in\Sigma_m$, $m\in\mathcal{N}$, $\leftthreetimes\geq 1$, $0<\phi\leq 1$ and $h=g^{-1}$.

Definition 3.6. A function $g(z)$ given by (1.3) is said to be in the class $\Lambda_{\Sigma}^{m}(\leftthreetimes,\delta)$ if the following condition are fulfilled:

$$\Re\left[\frac{z[g'(z)]^{\leftthreetimes}}{g(z)}\right] > \delta \quad z\in\mathbb{U}$$

and

$$\Re\left[\frac{w[h'(w)]^{\leftthreetimes}}{h(w)}\right] > \delta \quad w\in\mathbb{U},$$

where $g(z)\in\Sigma_m$, $m\in\mathcal{N}$, $\leftthreetimes\geq 1$, $0\leq\delta<1$ and $h=g^{-1}$.

For $m=1$ (*one*-fold) symmetric bi-univalent function, Theorems 2.2 and 3.2 gives Corollaries 3.7 and 3.8, respectively, which were investigated by Joshi and Yadav [10].

Corollary 3.7. *Let $f(z)$ given by (1.1) be in the class $\Lambda_{\Sigma}(\eta,\leftthreetimes,\phi)$, $0<\phi\leq 1$. Then*

$$|b_2| \leq \frac{2\phi}{\sqrt{[(\eta+1)^2+\phi(\eta^2-2\eta+2\leftthreetimes-1)+4\leftthreetimes(\leftthreetimes-\eta-1)]}}$$

and

$$|b_3| \leq \frac{2\phi}{3\leftthreetimes-2\eta-1}+\frac{4\phi^2}{(2\leftthreetimes-\eta-1)^2}.$$

Corollary 3.8. *Let $f(z)$ given by (1.1) be in the class $\Lambda_{\Sigma}(\eta,\leftthreetimes,\delta)$, $0\leq\delta<1$. Then,*

$$|b_2| \leq \sqrt{\frac{2(1-\delta)}{\eta^2+2\leftthreetimes^2-2\leftthreetimes\eta-\leftthreetimes}}$$

and

$$|b_3| \leq \frac{4(1-\delta)^2}{(2\leftthreetimes -\eta-1)^2} + \frac{2(1-\delta)}{3\leftthreetimes -2\eta-1}.$$

Choosing $\eta = 0$ into Corollaries 3.9 and 3.10, we have:

Corollary 3.9. *[9] Let $f(z)$ given by (1.1) be in the class $\Lambda_\Sigma(\leftthreetimes, \phi)$, $0 < \phi \leq 1$. Then*

$$|b_2| \leq \frac{2\phi}{\sqrt{1+\phi(2\leftthreetimes -1)+4\leftthreetimes(\leftthreetimes -1)}}$$

and

$$|b_3| \leq \frac{2\phi}{3\leftthreetimes -1} + \frac{4\phi^2}{(2\leftthreetimes -1)^2}.$$

Corollary 3.10. *[9] Let $f(z)$ given by (1.1) be in the class $\Lambda_\Sigma(\leftthreetimes, \delta)$, $0 \leq \delta < 1$. Then,*

$$|b_2| \leq \sqrt{\frac{2(1-\delta)}{2\leftthreetimes^2 -\leftthreetimes}}$$

and

$$|b_3| \leq \frac{4(1-\delta)^2}{(2\leftthreetimes -1)^2} + \frac{2(1-\delta)}{3\leftthreetimes -1}.$$

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